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Infinitesimal deformations of restricted simple Lie algebras II☆

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ABSTRACT

We compute the infinitesimal deformations of two families of restricted simple modular Lie algebras of Cartan-type: the Contact and the Hamiltonian Lie algebras.

1. Introduction

Simple Lie algebras over an algebraically closed field of positive characteristic different from 2 and 3 were classified by Wilson–Block (see [1]) in the restricted case and by Strade (see [2]) and Premet–Strade (see [3]) in the general case. The classification remains still open in characteristic 2 and 3 (see [4, page 209]).

According to this classification, *simple modular* (that is over a field of positive characteristic) Lie algebras are divided into two big families, called classical-type and Cartan-type algebras. The algebras of classical-type are obtained by the simple Lie algebras in characteristic zero by first taking a model over the integers (via Chevalley bases) and then reducing modulo p (see [5]). The algebras of Cartan-type were constructed by Kostrikin–Shafarevich in 1966 (see [6]) as finite-dimensional analogues of the infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups, and are divided into four families, called Witt–Jacobson, Special, Hamiltonian and Contact algebras. The Witt–Jacobson Lie algebras are derivation algebras of truncated divided power algebras and the remaining three families are the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. Moreover in characteristic 5 there is one exceptional simple modular Lie algebra called the Melikian algebra (introduced in [7]).

A particular important class of simple modular Lie algebras are the ones which are *restricted*. These can be characterized as those modular Lie algebras such that the p-power of an inner derivation (which in characteristic p is a derivation) is still inner (see [8] or [4]). Important examples of restricted Lie algebras are the ones coming from group schemes. Indeed, there is a bijection between restricted Lie algebras over k and infinitesimal k-group schemes of height one (see [9, Chap. 2]).

This paper is devoted to the study of the *infinitesimal deformations* of the restricted simple Lie algebras. The simple Lie algebras of classical-type are known to be rigid over a field of characteristic different from 2 and 3 (see [10]), in analogy of what happens in characteristic zero. In the papers [11,12], the author computed the infinitesimal deformations of the Witt–Jacobson, Special and Melikian restricted simple Lie algebras. In this paper, we compute the infinitesimal deformations of the Contact algebras $K(n)$ and the Hamiltonian algebras $H(n)$ over a field F of characteristic different from 2 and 3.

By standard facts of deformation theory, the infinitesimal deformations of a Lie algebra are parametrized by the second cohomology of the Lie algebra with values in the adjoint representation (see for example [14]).

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Before stating the main results of this paper, we recall that there is a canonical way to produce 2-cocycles in $Z^2(\mathfrak{g},\mathfrak{g})$ for a modular Lie algebra $\mathfrak{g}$ over a field of characteristic $p>0$, namely the squaring operation (see [14]). Given an element $\gamma\in\mathfrak{g}$, one defines the squaring of γ to be

$$\mathrm{Sq}(\gamma)(x,y)=\sum_{i=1}^{p-1}\frac{[\mathrm{ad}(\gamma)^i(x),\mathrm{ad}(\gamma)^{p-i}(y)]}{i!(p-i)!}\in Z^2(\mathfrak{g},\mathfrak{g}) \tag{1.1}$$

where $\mathrm{ad}(\gamma)^i$ is the i-iteration of the inner derivation $\mathrm{ad}(\gamma)$.

Assuming the (standard) notations from Sections 2.1 and 3.1 about the Contact algebras $K(n)$ and the Hamiltonian algebras $H(n)$, we can state the main results of this paper.

Theorem 1.1. *Let $n=2m+1\geq 3$. Then*

$$H^2(K(n),K(n))=\bigoplus_{i=1}^{2m}\langle \mathrm{Sq}(x_i)\rangle_F\oplus\langle \mathrm{Sq}(1)\rangle_F.$$

Theorem 1.2. *Let $n=2m\geq 2$. Then if $n\geq 4$ we have that*

$$H^2(H(n),H(n))=\bigoplus_{i=1}^{n}\langle \mathrm{Sq}(x_i)\rangle_F\bigoplus_{\substack{i<j\\ j\neq i'}}\langle \Pi_{ij}\rangle_F\bigoplus_{i=1}^{m}\langle \Pi_i\rangle_F\bigoplus\langle\Phi\rangle_F,$$

where the above cocycles are defined (and vanish outside) by

$$\begin{cases}\Pi_{ij}(x^a,x^b)=x_{i'}^{p-1}x_{j'}^{p-1}[D_i(x^a)D_j(x^b)-D_i(x^b)D_j(x^a)] & \text{for } j\neq i,i',\\ \Pi_i(x_ix^a,x_{i'}x^b)=x^{a+b+(p-1)\epsilon_i+(p-1)\epsilon_{i'}} & \text{if } a+b<\sigma^i,\\ \Pi_i(x_k,x^{\sigma^i})=-\sigma(k)x^{\sigma-\epsilon_{k'}} & \text{for any } 1\leq k\leq n,\\ \Phi(x^a,x^b)=\displaystyle\sum_{\substack{\underline{0}<\delta\leq a,\widehat{b}\\ |\delta|=3}}\binom{a}{\delta}\binom{b}{\widehat{\delta}}\sigma(\delta)\delta!x^{a+\widehat{b}-\delta-\widehat{\delta}}. \end{cases}$$

If $n=2$ then

$$H^2(H(2),H(2))=\bigoplus_{i=1}^{2}\langle \mathrm{Sq}(x_i)\rangle_F\bigoplus\langle\Phi\rangle_F.$$

In two forthcoming papers [15,13], we use the above computations to determine the restricted infinitesimal deformations of the restricted simple Lie algebras and the infinitesimal deformations of their associated simple finite group schemes.

2. Contact algebra

2.1. Definition and basic properties

We first introduce some notations about the set $\mathbb{N}^n$ of n-tuple of natural numbers. We consider the order relation defined by $a=(a_1,\ldots,a_n)<b=(b_1,\ldots,b_n)$ if $a_i<b_i$ for every $i=1,\ldots,n$. We define the degree of $a\in\mathbb{N}^n$ as $|a|=\sum_{i=1}^n a_i$ and the factorial as $a!=\prod_{i=1}^n a_i!$. For two multi-indices $a,b\in\mathbb{N}^n$ such that $b\leq a$, we set $\binom{a}{b}:=\prod_{i=1}^n\binom{a_i}{b_i}=\frac{a!}{b!(a-b)!}$. For every integer $j\in\{1,\ldots,n\}$ we call ϵ_j the n-tuple having 1 at the j-th entry and 0 outside.

Throughout this section we fix a field F of characteristic $p\neq 2,3$ and an odd integer $n=2m+1\geq 3$. For any $j\in\{1,\ldots,2m\}$, we define the sign $\sigma(j)$ and the conjugate j' of j as follows:

$$\sigma(j)=\begin{cases}1 & \text{if } 1\leq j\leq m,\\ -1 & \text{if } m<j\leq 2m,\end{cases}\quad\text{and}\quad j'=\begin{cases}j+m & \text{if } 1\leq j\leq m,\\ j-m & \text{if } m<j\leq 2m.\end{cases}$$

Given a multi-index $a=(a_1,\ldots,a_{2m})\in\mathbb{N}^{2m}$, we define the sign of a as $\sigma(a)=\prod\sigma(i)^{a_i}$ and the conjugate of a as the multi-index $\hat{a}$ such that $\hat{a}_i=a_{i'}$ for every $1\leq i\leq 2m$. We are going to use often the following special n-tuples: $\underline{0}:=(0,\ldots,0)$, $\tau:=(p-1,\ldots,p-1)$ and $\sigma:=(p-1,\ldots,p-1,0)$.

Let $A(n)=F[x_1,\ldots,x_n]/(x_1^p,\ldots,x_n^p)$ be the ring of p-truncated polynomials in n-variables. Note that $A(n)$ is a finite F-algebra of dimension p^n with a basis given by the monomials $\{x^a=x_1^{a_1}\cdots x_n^{a_n}\mid a\in\mathbb{N}^n, a\leq\tau\}$.

Consider the operator $D_H: A(n)\to W(n):=\mathrm{Der}_F A(n)$ defined as

$$D_H(f)=\sum_{j=1}^{2m}\sigma(j)D_j(f)D_{j'}=\sum_{i=1}^{m}\left[D_i(f)D_{i+m}-D_{i+m}(f)D_i\right],$$

where, as usual, $D_i:=\frac{\partial}{\partial x_i}\in W(n)$.

We denote with $K'(n)$ the graded Lie algebra over F whose underlying F-vector space is $A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, endowed with the grading defined by $\deg(x^a) = |a| + a_n - 2$ and with the Lie bracket defined by

$$[x^a, x^b] = D_H(x^a)(x^b) + \left[a_n \deg(x^b) - b_n \deg(x^a)\right] x^{a+b-\epsilon_n}.$$

Definition 2.1. The Contact algebra is the derived subalgebra of $K'(n)$:

$$K(n) := K'(n)^{(1)} = [K'(n), K'(n)].$$

We need the following characterization of $K(n)$ (see [8, Chap. 4, Theo. 5.5]).

Proposition 2.2. *Denote with $K'(n)_{<\tau}$ the sub-vector space of $K'(n)$ generated over F by the monomials x^a such that $a < \tau$. Then*

$$K(n) = \begin{cases} K'(n) & \text{if } p \not| (m+2), \\ K'(n)_{<\tau} & \text{if } p \mid (m+2). \end{cases}$$

We can describe explicitly the low degree terms of $K(n)$ together with their adjoint action. The negative graded pieces of $K(n)$ are $K(n)_{-2} = \langle 1 \rangle_F$ whose adjoint action is like the action of $2D_n$ on $A(n)$ and $K(n)_{-1} = \oplus_{i=1}^{2m} \langle x_i \rangle_F$ where the adjoint action of x_i is like $\sigma(i)D_{i'} + x_i D_n$. The piece $K(n)_0$ of degree 0 is generated by the central element x_n whose adjoint action is given by $[x_n, x^a] = \deg(x^a)x^a$ and by $x_i x_j$ (with $1 \le i, j \le 2m$) whose adjoint action is like $\sigma(i)x_j D_{i'} + \sigma(j)x_i D_{j'}$. Hence $K(n)_0 \cong \mathfrak{sp}(2m, F) \oplus \langle x_n \rangle_F$.

The algebra $K(n)$ admits a root space decomposition with respect to a canonical Cartan subalgebra.

Proposition 2.3. (a) *$T_K := \oplus_{i=1}^m \langle x_i x_{i'} \rangle_F \oplus \langle x_n \rangle_F$ is a maximal torus of $K(n)$ (called the canonical maximal torus).*

(b) *The centralizer of T_K inside $K(n)$ is the subalgebra $C_K = \{x^a \mid a_i = a_{i'} \text{ and } \deg(x^a) \equiv 0 \mod p\}$, which is hence a Cartan subalgebra (called the canonical Cartan subalgebra). The dimension of C_K is p^m if $p \not| (m+2)$ and $p^m - 1$ otherwise.*

(c) *Let $\Phi_K := \mathrm{Hom}_{\mathbb{F}_p}(\oplus_{i=1}^n \langle x_i x_{i'} \rangle_{\mathbb{F}_p} \oplus \langle x_n \rangle_{\mathbb{F}_p}, \mathbb{F}_p)$, where $\mathbb{F}_p$ is the prime field of F. We have a Cartan decomposition $K(n) = C_K \oplus_{\phi \in \Phi_K - \underline{0}} K(n)_\phi$, where $K(n)_\phi = \{x^a \mid a_{i+m} - a_i \equiv \phi(x_i x_{i'}) \forall i = 1, \ldots, m \text{ and } \deg(x^a) \equiv \phi(x_n)\}$. The dimension of every $K(n)_\phi$, with $\phi \in \Phi_K - \underline{0}$, is p^m.*

Proof. See [8, Chap. 4, Theo. 5.6 and 5.7]. □

2.2. Proof of the Main Theorem 1.1

In this section, assuming the results of the next section, we give a proof of the Main Theorem 1.1.

Proof of the Main Theorem 1.1. It is easy to see that the cochains appearing in Theorem 1.1 are cocycles and that they are independent in $H^2(K(n), K(n))$. Therefore, we are left with showing that $\dim_F H^2(K(n), K(n)) = n$. We divide the proof in three steps.

STEP I: It is enough to show that $\dim_F H^2(K(n), K'(n)) = n$ since there is an inclusion

$$H^2(K(n), K(n)) \hookrightarrow H^2(K(n), K'(n)).$$

Indeed, if p does not divide $m + 2$ then $K'(n) = K(n)$ and we get the equality. Otherwise there is an exact sequence of $K(n)$-modules

$$0 \to K(n) \to K'(n) \to \langle x^\tau \rangle_F \to 0 \tag{2.1}$$

where $\langle x^\tau \rangle_F \cong F$ is the trivial $K(n)$-module. We get the desired inclusion since $H^1(K(n), F) = 0$, which follows from the fact that $[K(n), K(n)] = K(n)$.

STEP II: We have that

$$H^2(K(n), K'(n)) = H^2(K(n)_{\ge 0}, F_{\lambda-\sigma}),$$

where $F_{\lambda-\sigma}$ is the one-dimensional representation of $K(n)_{\ge 0}$ on which x_n acts as -2 and all the others elements act trivially.

This follows from the general results of [16]. Indeed, it is easily seen that $K'(n)$ is the restricted $K(n)$-module induced from the restricted $K(n)_{\ge 0}$-submodule $\langle x^\tau \rangle_F \subset K'(n)$. In the notation of [16], the $K(n)_{\ge 0}$-module $\langle x^\tau \rangle_F$ is isomorphic to F_λ, where F_λ is the one-dimensional $K(n)_{\ge 0}$-module corresponding to the Lie algebra homomorphism $\lambda : K(n)_{\ge 0} \to F$ whose only non-zero value is $\lambda(x_n) = -2m - 4 \equiv \deg x^\tau = (2m+2)(p-1) - 2 \mod p$.

Moreover, consider the Lie algebra homomorphism $\sigma : K(n)_{\ge 0} \to F$ given by $\sigma(x) := \mathrm{tr}(\mathrm{ad}_{K(n)/K(n)_{\ge 0}} x)$ for $x \in K(n)_{\ge 0}$ (see [16, Pag. 155]). It is easily seen that the only non-zero value of σ is given by $\sigma(x_n) = -2m - 2$.

Therefore we have that $\gamma - \sigma : K(n)_{\ge 0} \to F$ is the Lie algebra homomorphism sending x_n to -2 and vanishing on the other elements. Moreover, using Lemma 2.5, it is straightforward to check that, in the notation of [16], we have the equality

$$\{[x, y] - (\lambda - \sigma)(x)y + (\lambda - \sigma)(y)x \mid x, y \in K(n)_{\ge 0}\} := (K(n)_{\ge 0})^{(1)}_{\lambda-\sigma} = I := \ker(\lambda - \sigma).$$

We conclude using [16, Thm. 3.6(1)].

STEP III: In Section 2.3, we prove that

$$\dim_F H^2(K(n)_{\geq 0}, F_{\lambda-\sigma}) = \dim_F H^2(K(n)_{\geq 1}, F_{\lambda-\sigma})^{K(n)_0} = n. \quad \square$$

2.3. Computation of $H^2(K(n)_{\geq 0}, F_{\lambda-\sigma})$

This section is devoted to complete the third step of the proof of the Main Theorem as outlined in Section 2.2, that is the computation of $H^2(K(n)_{\geq 0}, F_{\lambda-\sigma})$. This is done in Propositions 2.4 and 2.6.

Recall that $F_{\lambda-\sigma}$ is the one-dimensional representation of $K(n)_{\geq 0}$ on which x_n acts as -2 and all the other elements act trivially. This action becomes homogeneous with respect to the weight decomposition of $K(n)_{\geq 0}$ if we give the weight $-2\epsilon_n$ to the generator of $F_{\lambda-\sigma}$. As remarked in [11, Sec. 2.1], in this situation we have that

$$H^2(K(n)_{\geq 0}, F_{\lambda-\sigma}) = H^2(K(n)_{\geq 0}, F_{\lambda-\sigma})_{\underline{0}},$$

where the subscript $\underline{0}$ means that we consider only homogeneous cochains with respect to the natural action of the maximal torus T_K (see Proposition 2.3).

Proposition 2.4. *We have that*

$$H^2(K(n)_{\geq 0}, F_{\lambda-\sigma}) = H^2(K(n)_{\geq 1}, F_{\lambda-\sigma})^{K(n)_0}.$$

Proof. Consider the Hochschild–Serre spectral sequence (see [17]) associated to the ideal $K(n)_{\geq 1} \lhd K(n)_{\geq 0}$:

$$E_2^{r,s} = H^r(K(n)_0, H^s(K(n)_{\geq 1}, F_{\lambda-\sigma})) \Rightarrow H^{r+s}(K(n)_{\geq 0}, F_{\lambda-\sigma}).$$

We are going to prove that the first two lines of the above spectral sequence vanish, which clearly imply the Proposition.

The first line $E_2^{*,0} = H^*(K(n)_0, F_{\lambda-\sigma})$ vanish for homogeneity reasons. Indeed, the weight of $F_{\lambda-\sigma}$ is $-2\epsilon_n \neq 0$, while the weights occurring on $K(n)_0$ are $\{\pm\epsilon_i \pm \epsilon_j, 1 \leq i, j \leq 2m\}$ and hence the weights that occur on $K(n)_0^{\otimes k}$ cannot contain ϵ_n with a non-trivial coefficient.

On the other hand, since $F_{\lambda-\sigma}$ is a trivial $K(n)_{\geq 1}$-module and $[K(n)_{\geq 1}, K(n)_{\geq 1}] = K(n)_{\geq 2}$ by Lemma 2.5 below, we have that

$$H^1(K(n)_{\geq 1}, F_{\lambda-\sigma}) = C^1(K(n)_1, F_{\lambda-\sigma}).$$

From this equality, we deduce that the second line $E_2^{*,1} = H^*(K(n)_0, H^1(W(n)_{\geq 1}, F_{\lambda-\sigma}))$ vanish again for homogeneity reasons. Indeed the n-component of the weights appearing in $H^1(K(n)_{\geq 1}, F_{\lambda-\sigma}) = C^1(K(n)_1, F_{\lambda-\sigma})$ is $-3\epsilon_n \neq 0$ (because $p \geq 5$) while the weights appearing in $K(n)_0^{\otimes k}$ have trivial n-component. $\square$

Lemma 2.5. *Let d be an integer greater than or equal to -2. Then*

$$[K(n)_1, K(n)_d] = K(n)_{d+1}.$$

Proof. The inclusion $[K(n)_1, K(n)_d] \subset K(n)_{d+1}$ is clear. In order to prove the other inclusion, we consider an element $x^a \in K(n)_{d+1}$ and we have to show that it belongs to the commutators $[K(n)_1, K(n)_d]$.

The elements of $K(n)_1$ are of the form $x_ix_jx_k$ or x_ix_n (for some $1 \leq i, j, k \leq 2m$). The former ones act, via adjoint action, as $D_H(x_ix_jx_k) - x_ix_jx_kD_n$ while the latter ones act as $\sigma(i)x_nD_{i'} + x_i \deg - x_ix_nD_n$. Consider the decomposition $K'(n) = \oplus_{k=0}^{p-1} A(2m)x_n^k$. The proof is by induction on the coefficient a_n, which in what follows is called the x_n-degree of x^a.

First of all consider the case of x_n-degree equal to 0, that is the case $x^a \in A(2m)$. If there exists an index $1 \leq i \leq 2m$ such that $a_i \geq 2$ and $a_{i'} < p-1$, then we conclude by means of the following formula

$$[x_i^3, x^{a-2\epsilon_i+\epsilon_{i'}}] = 3\sigma(i)(a_{i'}+1)x^a.$$

Therefore it remains to consider the elements x^a for which $a_i = a_{i'} = p-1$ or $0 \leq a_i, a_{i'} \leq 1$ for every $1 \leq i \leq 2m$. If there exists a couple $(a_i, a_{i'}) = (1, 1)$, we are done by the formula

$$[x_i^2x_{i'}, x^{a-\epsilon_i}] = \sigma(i)(2a_{i'} - a_i + 1)x^a = 2\sigma(i)x^a.$$

If there exists a couple $(a_i, a_{i'}) = (1, 0)$, then there are two possibilities: either $x^a = x_i$ or there exists an index $j \neq i, i'$ such that $a_j \geq 1$. In the first case we use $[x_ix_n, 1] = -2x_i$ while in the second we conclude by means of the following formula

$$[x_i^2x_j, x^{a-\epsilon_i+\epsilon_{i'}-\epsilon_j}] = 2\sigma(i)x^a + \sigma(j)a_{j'}x^{a+\epsilon_i+\epsilon_{i'}-\epsilon_j-\epsilon_{j'}},$$

together with the fact that the second element on the right-hand side belongs to $[K(n)_1, K(n)_d]$ by what proved above. Hence we are left with considering the elements x^a for which every couple of conjugated coefficients $(a_i, a_{i'})$ is equal to $(0, 0)$ or $(p-1, p-1)$. If there are two indices $1 \leq i \neq j \leq m$ such that $(a_i, a_{i'}) = (0, 0)$ and $(a_j, a_{j'}) = (p-1, p-1)$ we use the formula

$$[x_j^2x_i, x^{a+\epsilon_{i'}-2\epsilon_j}] = 2(p-1)x^{a+\epsilon_i+\epsilon_{i'}-\epsilon_j-\epsilon_{j'}} + x^a,$$

together with the fact that the first term on the right-hand side belongs to $[K(n)_1, K(n)_d]$ by what proved above. Since the case $x^a = 1$ is excluded by the hypothesis $d + 1 \geq -1$, it remains to consider the element $x^a = x^\sigma$ for which we can take an appropriate linear combination of the two equations (with $k = 0$):

$$[x_i^3, x^{\sigma-3\epsilon_i} x_n^{k+1}] = -3\sigma(i) x^{\sigma-\epsilon_i-\epsilon_{i'}} x_n^{k+1} - (k+1) x^\sigma x_n^k, \tag{2.2}$$

$$[x_i^2 x_{i'}, x^{\sigma-2\epsilon_i-\epsilon_{i'}} x_n^{k+1}] = -\sigma(i) x^{\sigma-\epsilon_i-\epsilon_{i'}} x_n^{k+1} - (k+1) x^\sigma x_n^k. \tag{2.3}$$

For the inductive step, suppose that $a_n = k \geq 1$ and that we have already proved the desired inclusion for the elements of x_n-degree less than or equal to $k - 1$. If there exists an index i such that $a_i < p - 1$, then the formula

$$[x_{i'} x_n, x^{a+\epsilon_i-\epsilon_n}] = \sigma(i')(a_i + 1)x^a + (d - a_n + 1)x^{a+\epsilon_i+\epsilon_{i'}-\epsilon_n},$$

together with the induction hypothesis, gives the conclusion. Otherwise our element is equal to $x^\sigma x_n^k$. If $k < p - 1$, then one concludes by taking an appropriate linear combination of the above formulas (2.2) and (2.3). Finally for the element $x^\sigma x_n^{p-1} = x^\tau$ (which can occur only if $p \not| m + 2$), the conclusion follows from the formula (for an arbitrarily chosen $1 \leq i \leq 2m$)

$$[x_i x_n, x^{\sigma-\epsilon_i} x_n^{p-1}] = -2(m+2)x^\tau. \quad \square$$

For the remaining part of this subsection, we identify the $K(n)_0$-module $F_{\lambda-\sigma}$ with the $K(n)_0$-module $F \cong \langle 1 \rangle_F = K(n)_{-2}$. On both these modules, $K(n)_{\geq 1}$ acts trivially.

Proposition 2.6. *We have that*

$$H^2(K(n)_{\geq 1}, F)^{K(n)_0} = \bigoplus_{i=1}^{2m} \langle \overline{\mathrm{Sq}(x_i)} \rangle_F \oplus \langle \overline{\mathrm{Sq}(1)} \rangle_F,$$

where $\overline{\mathrm{Sq}(x_i)}$ *is the projection of* $\mathrm{Sq}(x_i)$ *onto* $\langle 1 \rangle_F \cong F$ *(analogously for* $\overline{\mathrm{Sq}(1)}$*).*

Proof. It is easy to check that the above cocycles are independent modulo coboundaries, so we have to prove that they generate the whole cohomology group.

The strategy of the proof is exactly the same as that of the proposition [11, Prop. 3.10], that is to compute, step-by-step as d increases, the truncated invariant cohomology groups

$$H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}, F\right)^{K(n)_0}.$$

Observe that if d is big enough (at least $2(m+1)(p-1)-1$) then $K(n)_{\geq d+1} = 0$ and hence we get the cohomology we are interested in. On the other hand, by homogeneity, we get that

$$H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq 2}}, F\right)^{K(n)_0} = C^2(K(n)_1, F)^{K(n)_0} = 0.$$

By taking the Hochschild–Serre spectral sequence (see [17]) associated to the ideal $K(n)_d = \frac{K(n)_{\geq d}}{K(n)_{\geq d+1}} \lhd \frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}$ (for $d \geq 2$):

$$E_2^{r,s} = H^r\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, H^s(K(n)_d, F)\right) \Rightarrow H^{r+s}\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}, F\right), \tag{2.4}$$

we get the same diagram as in [11, Prop. 3.10] (the vanishing of $E_2^{0,2}$ and the injectivity of the map α are proved in exactly the same way).

By taking the cohomology with respect to $K(n)_0$ and using the Lemmas 2.8, 2.9 and 2.11, we see that the only cocycles that contribute to the required cohomology group are $\{\overline{\mathrm{Sq}(x_1)}, \ldots, \overline{\mathrm{Sq}(x_{2m})}, \overline{\mathrm{Sq}(1)}\}$ since the cocycle $\mathrm{inv} \circ [-,-] \in (E_\infty^{1,1})^{K(n)_0}$ in degree $2m(p-1) + 2\mu - 3$ is annihilated by $\mathrm{inv} \in C^1(K(n)_{2m(p-1)+2\mu-2}, F)^{K(n)_0}$. $\square$

The remaining part of this section is devoted to the proof of the lemmas that were used in the proof of Proposition 2.6.

In the next two lemmas, we are going to compute the $K(n)_0$-invariant terms $(E_2^{1,1})^{K(n)_0}$ and $(E_\infty^{1,1})^{K(n)_0}$. Observe that, since $K(n)_d$ is in the center of $\frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}$, we have that $H^s(K(n)_d, F) = C^s(K(n)_d, F)$ and $\frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}$ acts trivially on it. Therefore, using Lemma 2.5, we deduce that $E_2^{1,1} = C^1(K(n)_1 \times K(n)_d, F)$.

Lemma 2.7. *Let* $0 \leq \mu, \nu \leq p - 1$ *such that* $\mu \equiv m \bmod p$ *and* $\nu \equiv (m+1) \bmod p$*. Then we have that*

$$\bigoplus_{d \geq 2} C^1(K(n)_1 \times K(n)_d, F)^{K(n)_0} = \langle \Phi_1 \rangle_F \oplus \langle \Phi_2 \rangle_F \oplus \langle \Psi_1 \rangle_F \oplus \langle \Psi_2 \rangle_F,$$

where the above cochains are defined by

$$\begin{cases} \Phi_1(x_i x_n, x_{i'} x_n^{p-1}) = \sigma(i), \\ \Phi_2(x^a, x^{\hat{a}} x_n^{p-2}) = \sigma(a) a! \quad \textit{if } a \in \mathbb{N}^{2m} \textit{ and } |a| = 3, \\ \Psi_1(x_i x_n, x^{\sigma-\epsilon_i} x_n^{\mu}) = 1, \\ \Psi_2(x^a, x^{\sigma-a} x_n^{\nu}) = 1 \quad \textit{if } a \in \mathbb{N}^{2m} \textit{ and } |a| = 3. \end{cases}$$

Proof. An easy verification shows that the four cochains of above are $K(n)_0$-invariants and linearly independent. We will conclude by showing that the dimension over the base field F of the space $\bigoplus_{d\geq 2} C^1\,(K(n)_1 \times K(n)_d, F)_{\underline{0}}^{K(n)_0}$ of all invariant homogeneous cochains is less than or equal to 4.

The space $K(n)_1$ admits the decomposition $K(n)_1 = A(2m)_{-1} \cdot x_n \oplus A(2m)_1$ which is invariant under the adjoint action of $K(n)_0 = A(2m)_0 \oplus \langle x_n \rangle_F$. Moreover the action of $K(n)_0$ is transitive in both the summands $A(2m)_{-1} \cdot x_n$ and $A(2m)_1$. Therefore a $K(n)_0$-invariant homogeneous cochain $g \in C^1\left(K(n)_1 \times K(n)_{\geq 2}, F\right)_{\underline{0}}^{K(n)_0}$ is determined by the values on any two elements of $A(2m)_{-1} \cdot x_n$ and $A(2m)_1$, let us say $x_1 x_n$ and x_1^3.

Consider an element $x^a \in K(n)_{\geq 2}$ such that $g(x_1 x_n, x^a) \neq 0$. By homogeneity the element x^a must satisfy $a_{1'} \equiv a_1 + 1 \mod p$, $a_{j'} = a_j$ for every $j \notin \{1, 1'\}$ and $\deg(x^a) \equiv -3 \mod p$. If the couple $(a_1, a_{1'})$ would be different from $(0, 1)$ or $(p-2, p-1)$ then the following invariance condition

$$0 = (x_1^2 \circ g)(x_1 x_n, x^{a-\epsilon_1+\epsilon_{1'}}) = -2(a_{1'} + 1) g(x_1 x_n, x^a)$$

would contradict the hypothesis of non-vanishing. Therefore we can assume that $(a_1, a_{1'}) = (0, 1)$ or $(p-2, p-1)$. If the first case holds, then necessarily $(a_j, a_{j'}) = (0, 0)$ for every $j \notin \{1, 1'\}$. Indeed if this is not the case, then we get a contradiction with the non-vanishing hypothesis by means of the following invariance condition

$$0 = (x_1 x_j \circ g)(x_1 x_n, x^{a+\epsilon_{1'}-\epsilon_j}) = -2 g(x_1 x_n, x^a).$$

Analogously, if $(a_1, a_{1'}) = (p-2, p-1)$ then $(a_j, a_{j'})$ for every $j \notin \{1, 1'\}$ because of the following invariance condition

$$0 = (x_1 x_j \circ g)(x_1 x_n, x^{a-\epsilon_1+\epsilon_{j'}}) = -\sigma(j)(a_{j'} + 1) g(x_1 x_n, x^a).$$

Taking into account the homogeneity condition $\deg(x^a) \equiv -3 \mod p$, we get that the only non-zero values of $g(x_1 x_n, -)$ can be $g(x_1 x_n, x_{1'} x_n^{p-1})$ and $g(x_1 x_n, x^{\sigma-\epsilon_1} x_n^{\mu})$.

In exactly the same way, one proves that the only non-zero values of $g(x_1^3, -)$ can be $g(x_1^3, x_{1'}^3 x_n^{p-2})$ and $g(x_1^3, x^{\sigma-3\epsilon_1} x_n^{\nu})$ and therefore we get

$$\dim_F C^1\left(K(n)_1 \times K(n)_{\geq 2}, F\right)_{\underline{0}}^{K(n)_0} \leq 4. \quad \square$$

Lemma 2.8. *In the above spectral sequence (2.4), we have that*

$$(E_\infty^{1,1})^{K(n)_0} = \begin{cases} \langle \overline{\mathrm{Sq}(1)} \rangle_F & \textit{if } d = 2p-3, \\ \langle \mathrm{inv} \circ [-, -] \rangle_F & \textit{if } d = 2m(p-1) + 2\mu - 3, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\mathrm{inv} \in C^1(K(n)_{2m(p-1)+2\mu-2}, F)^{K(n)_0}$ *is defined in Lemma 2.9 and* $\overline{\mathrm{Sq}(1)}$ *is the restriction of* $\mathrm{Sq}(1)$ *to* $K(n)_1 \times K(n)_{2p-3}$.

Proof. $(E_\infty^{1,1})^{K(n)_0}$ is the subspace of $(E_2^{1,1})^{K(n)_0} = C^1(K(n)_1 \times K(n)_d, F)^{K(n)_0}$ consisting of cocycles that can be lifted to $Z^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d+1}}, F\right)$. A direct computation shows that, with the notations of Lemma 2.7, $\overline{\mathrm{Sq}(1)} = 2\overline{\mathrm{Sq}(D_n)} = 2\Phi_1$ and $\mathrm{inv} \circ [-, -] = -(\nu + 2)\Psi_1 - \nu\Psi_2$ and clearly these two cocycles can be lifted. We want to show that any other liftable cocycle is a linear combination of them.

First of all we show that the cocycle Φ_2 is not liftable. By absurd, suppose that a lifting exists and call it again Φ_2. We get a contradiction by means of the following cocycle conditions

$$\begin{cases} 0 = \mathrm{d}\Phi_2(x_i^3, x_{i'}^3, x_n^{p-1}) = -9\sigma(i)\Phi_2(x_i^2 x_{i'}^2, x_n^{p-1}) + \Phi_2(x_i^3 x_n^{p-2}, x_{i'}^3) - \Phi_2(x_{i'}^3 x_n^{p-2}, x_i^3) \\ \quad = -9\sigma(i)\Phi_2(x_i^2 x_{i'}^2, x_n^{p-1}) + 12\sigma(i), \\ 0 = \mathrm{d}\Phi_2(x_i^2 x_{i'}, x_{i'}^2 x_i, x_n^{p-1}) = -3\sigma(i)\Phi_2(x_i^2 x_{i'}^2, x_n^{p-1}) + \Phi_2(x_i^2 x_{i'} x_n^{p-2}, x_{i'}^2 x_i) - \Phi_2(x_{i'}^2 x_i x_n^{p-2}, x_i^2 x_{i'}) \\ \quad = -3\sigma(i)\Phi_2(x_i^2 x_{i'}^2, x_n^{p-1}) - 4\sigma(i). \end{cases}$$

Finally, consider the cocycle $\Psi \in \langle \Psi_1, \Psi_2 \rangle_F$ defined in degree $2m(p-1) + 2\nu - 5$ by

$$\Psi = \begin{cases} a\Psi_1 + b\Psi_2 & \text{if } \nu \neq 0, \\ b\Psi_2 & \text{if } \nu = 0, \end{cases}$$

for certain $a, b \in F$. We will show that Ψ can be lifted to $Z^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq 2m(p-1)+2\nu-4}}, F\right)$ if and only if $b(\nu + 2) \equiv a\nu \mod p$ and this will conclude our proof. Indeed this implies that if $\nu \neq 0$ then Ψ is liftable if and only if it is a multiple of $\mathrm{inv} \circ [-, -]$, while if $\nu = 0$ it implies that Ψ_2 is not liftable and hence again that $\mathrm{inv} \circ [-, -] = -2\Psi_1$ is the only liftable cocycle in the span of Ψ_1 and Ψ_2.

So suppose that a lift exists and call it again Ψ. From the following cocycle condition

$$0 = \mathrm{d}\Psi(x_i^3, x_{i'}^3, x^{\sigma-2\epsilon_{i'}-2\epsilon_i}x_n^\nu) = -9\sigma(i)\Psi(x_i^2x_{i'}^2, x^{\sigma-2\epsilon_{i'}-2\epsilon_i}x_n^\nu) - 9\sigma(i)\Psi(x^{\sigma-3\epsilon_{i'}}x_n^\nu, x_{i'}^3) + 9\sigma(i')\Psi(x^{\sigma-3\epsilon_i}x_n^\nu, x_i^3),$$

we deduce that $\Psi(x_i^2x_{i'}^2, x^{\sigma-2\epsilon_{i'}-2\epsilon_i}x_n^\nu) = 2b$. Using this, we get the following

$$\begin{aligned} 0 &= d\Psi(x_ix_n, x_ix_{i'}^2, x^{\sigma-2\epsilon_i-2\epsilon_{i'}}x_n^\nu) \\ &= -2\sigma(i)\Psi(x_ix_{i'}x_n + x_i^2x_{i'}^2, x^{\sigma-2\epsilon_i-2\epsilon_{i'}}x_n^\nu) + (-4-\nu)\Psi(x^{\sigma-\epsilon_i-2\epsilon_{i'}}x_n^\nu, x_ix_{i'}^2) + \nu\Psi(x^{\sigma-\epsilon_i}x_n^\mu, x_ix_n) \\ &= -2\sigma(i)\Psi(x_ix_{i'}x_n, x^{\sigma-2\epsilon_i-2\epsilon_{i'}}x_n^\nu) + (\nu+2)b - \nu a. \end{aligned}$$

Exchanging i with i' and summing the two expressions, we obtain the required congruence $(\nu+2)b \equiv \nu a \bmod p$. □

In the next lemma, we compute the $K(n)_0$-invariants of the term $E_2^{0,1} = C^1(K(n)_d, F)$ of the above spectral sequence (2.4).

Lemma 2.9. *Let μ be the integer defined in Lemma 2.7. We have that*

$$C^1(K(n)_d, F)^{K(n)_0} = \begin{cases} \langle \mathrm{inv} \rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{inv} \in C^1(K(n)_{2m(p-1)+2\mu-2}, F)$ *sends $x^\sigma x_n^\mu$ into 1 and vanish on the other elements.*

Proof. First of all observe that if p divide $(m+2)$, then $\mu = p-2$ and hence $x^\sigma x_n^\mu \in K(n)$. The (well-defined) cochain inv is $K(n)_0$-invariant. Indeed it is homogeneous and the invariance with respect to an element $x_ix_j \in K(n)_0 \setminus T_K$ (hence with $j \neq i'$) follows from the fact that $[x_ix_j, x^\sigma x_n^\mu] = [x_ix_j, 1] = 0$ together with the fact that $x^\sigma x_n^\mu \notin [x_ix_j, K(n)]$.

Consider next an invariant cochain $f \in C^1(K(n)_d, F)^{K(n)_0}$ and let $x^a \in K(n)_d$ be an element such that $f(x^a) \neq 0$. Then by homogeneity it must hold that $a_i = a_{i'}$ for every $1 \leq i \leq m$ and $\deg(x^a) \equiv -2 \bmod p$. Using the invariance with respect to x_i^2 or $x_{i'}^2$, we obtain that $a_i = a_{i'} = 0$ or $p-1$. Otherwise, assuming, up to interchanging i with i', that $a_i > 0$ and $a_{i'} < p-1$, one gets the vanishing as follows

$$0 = (x_i^2 \circ f)(x^{a-\epsilon_i+\epsilon_{i'}}) = -2\sigma(i)(a_{i'}+1)f(x^a).$$

Moreover, if there are two pairs verifying $(a_i, a_{i'}) = (0,0)$ and $(a_j, a_{j'}) = (p-1, p-1)$ (for $j \neq i, i'$), then we obtain the vanishing by means of the following

$$0 = (x_ix_j \circ f)(x^{a+\epsilon_{i'}-\epsilon_j}) = -\sigma(i)f(x^a).$$

Finally, by imposing $\deg(x^a) \equiv -2 \bmod p$, we deduce that $x^a = 1$ (which we can exclude since $\deg(x^a) = d \geq 2$) or $x^a = x^\sigma x_n^\mu$. □

In the next lemma, we compute the first cohomology group with respect to $K(n)_0$ of the term $E_2^{0,1} = C^1(K(n)_d, F)$ of the above spectral sequence (2.4).

Lemma 2.10. *Let μ be the integer defined in Lemma 2.7. We have that*

$$H^1\left(K(n)_0, C^1(K(n)_d, F)\right) = \begin{cases} \bigoplus_{i=1}^{2m} \langle \overline{\mathrm{Sq}(x_i)} \rangle_F & \text{if } d = p-2, \\ \bigoplus_{j=1}^{2m} \langle \omega_j \rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2 - p, \\ \langle x_n \mapsto \mathrm{inv} \rangle_F & \text{if } d = 2m(p-1) + 2\mu - 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{Sq}(x_i)}$ denotes the restriction of $\mathrm{Sq}(x_i)$ to $K(n)_0 \times K(n)_{p-2}$ and the cocycle ω_i is defined by (with $j \neq i, i'$)

$$\begin{cases} \omega_i(x_i^2, x^{\sigma-\epsilon_i-(p-1)\epsilon_{i'}}x_n^\mu) = 2, \\ \omega_i(x_ix_j, x^{\sigma-(p-1)\epsilon_{i'}-\epsilon_j}x_n^\mu) = 1. \end{cases}$$

Proof. By homogeneity we can restrict to the case $d \equiv -2 \bmod p$. First of all we claim that $f_{x_ix_{i'}} = 0$ for every $1 \leq i \leq n$ and f_{x_n} takes a non-zero value only on the element $x^\sigma x_n^\mu$. Indeed, by the homogeneity assumption, we get for an element $\gamma \in T_K$ that $0 = df_{|\gamma} = \gamma \circ f - d(f_{|\gamma}) = -d(f_{|\gamma})$, that is $f_\gamma \in C^1(K(n)_d, F)^{K(n)_0} = \langle \mathrm{inv} \rangle_F$ (see Lemma 2.9). Moreover the cocycle condition

$$0 = df_{(x_i^2, x_{i'}^2)}(x^\sigma x_n^\mu) = -4\sigma(i)f_{x_ix_{i'}}(x^\sigma x_n^\mu)$$

gives $f_{x_ix_{i'}} = 0$ for every $1 \leq i \leq n$, while the fact that $x_n \notin [K(n)_0, K(n)_0]$ implies that $f_{x_n}(x^\sigma x_n^\mu)$ can be different from zero.

Now we split the proof into two parts according to the cases $d = (2r+1)p-2$ or $d = 2rp-2$ for some integer r.

I CASE : $d = (2r+1)p-2$.

Note that in this case, there are no coboundary elements since, by reasons of parity, $C^1(K(n)_d, F)_{\underline{0}} = 0$. Moreover, for a homogeneous cocycle $f \in C^1(K(n)_0, C^1(K(n)_d, F))_{\underline{0}}$, the value $f_{x_i^2}(x^a)$ can be different from 0 only if one of the following possibilities occur

$$(a_i, a_{i'}) = (p-1, 1) \quad \text{and} \quad a_j = a_{j'} \quad \text{for every } j \neq i, i', \tag{A}$$

$$(a_i, a_{i'}) = (p-2, 0) \quad \text{and} \quad a_j = a_{j'} \quad \text{for every } j \neq i, i'. \tag{B}$$

Analogously, if $j \neq i, i'$, then $f_{x_i x_j}(x^a)$ can be different from 0 only if (up to interchanging i and j)

$$(a_i, a_{i'}) = (p-1, 0), \quad 1 \leq a_{j'} = a_j + 1 \leq p-1 \quad \text{and} \quad a_k = a_{k'} \quad \text{for } k \neq i, i' j, j'. \tag{C}$$

The values of types (C) are determined by the values of types (A) and (B) by means of the following cochain condition (where a is a multi-index as in (C))

$$\begin{aligned} 0 &= df_{(x_i x_j, x_i^2)}(x^{a-\epsilon_i+\epsilon_{i'}}) \\ &= -\sigma(j) a_{j'} f_{x_i^2}(x^{a+\epsilon_{i'}-\epsilon_{j'}}) - \sigma(i) f_{x_i^2}(x^{a-\epsilon_i+\epsilon_j}) + 2\sigma(i) f_{x_i x_j}(x^a) \end{aligned}$$

where in the last equation the first term is of type (A) (or vanish) and the second is of type (B) (or vanish).

The values of type (A) vanish if there exists an index $j \neq i, i'$ such that $a_j = a_{j'} \neq 0$, because of the following condition (where a satisfies the conditions in (A))

$$0 = df_{(x_i x_j, x_i^2)}(x^{a+\epsilon_{i'}-\epsilon_j}) = -2\sigma(i) f_{x_i^2}(x^a).$$

On the other hand, the values of type (B) vanish if there exists a $j \neq i, i'$ such that $a_j = a_{j'} \neq p-1$ because the following cocycle condition (where a satisfies the conditions of (B))

$$0 = df_{(x_i x_j, x_i^2)}(x^{a-\epsilon_i+\epsilon_{j'}}) = -\sigma(j)(a_{j'}+1) f_{x_i^2}(x^a).$$

Therefore a cochain f is completely determined by the values $f_{x_i^2}(x_i^{p-1} x_{i'})$ and $f_{x_i^2}(x^{\sigma-\epsilon_i-(p-1)\epsilon_{i'}} x_n^{\mu})$, whose values determine also the cocycles $\overline{\text{Sq}(x_{i'})}$ and ω_i (respectively), and hence f is a linear combination of $\overline{\text{Sq}(x_{i'})}$ or ω_i.

II CASE : $d = 2rp-2$.

In this case we will prove that f vanish (up to adding a coboundary dg) except for the value $f_{x_n}(x^\sigma x_n^\mu)$ (which can be non-zero as seen before). We have already seen that $f_{x_i x_{i'}}$ vanish for every $1 \leq i \leq n$.

We first prove that, by adding coboundaries, we can modify the cochain f (without changing its cohomological class) in such a way that it satisfies $f_{x_i^2} = 0$ for every $1 \leq i \leq m$. The proof is by induction on i. So suppose that for a certain k, we have that $f_{x_i^2} = 0$ for every $i < k$. We want to prove that, by adding coboundaries, we can modify f in such a way that it verifies $f_{x_k^2} = 0$.

First of all note that, by homogeneity and parity condition on d, $f_{x_k^2}(x^a)$ can be different from 0 only if $2 \leq a_{k'} = a_k + 2 \leq p-1$ and $a_{h'} = a_h$ for $h \neq k, k'$. Moreover if there exists an index $1 \leq h < k \leq m$ such that $a_h = a_{h'} \neq 0, (p-1)$, then $f_{x_k^2}(x^a) = 0$ because of the following cocycle condition

$$0 = df_{(x_h^2, x_k^2)}(x^{a-\epsilon_h+\epsilon_{h'}}) = -2(a_{h'}+1) f_{x_k^2}(x^a)$$

where we used that $f_{x_h^2} = 0$ by induction. Therefore we can suppose that for $1 \leq h < k \leq m$, $a_h = a_{h'} = 0$ or $(p-1)$. Fix one of these elements x^a. Define an element $g \in C^1(K(n)_d, F)_{\underline{0}}$ as follows:

$$\begin{cases} g(x^{a+\epsilon_k-\epsilon_{k'}}) = \dfrac{f_{x_k^2}(x^a)}{2a_{k'}}, \\ g(x^b) = 0 \quad \text{if } b \neq a + \epsilon_k - \epsilon_{k'}. \end{cases}$$

By construction, if $1 \leq h < k \leq m$ then $(x_h^2 \circ g) = 0$ while $(x_k^2 \circ g)(x^a) = -f_{x_k^2}(x^a)$. Therefore the new cocycle $\widetilde{f} := f + dg$ satisfies the same inductive hypothesis as before and moreover it verifies $\widetilde{f}_{x_k^2}(x^a) = 0$. Repeating these modifications for all the elements x^a as before, eventually we obtain a new cochain homologous to the old one (which, by an abuse of notation, we continue to call f) and which satisfies $f_{x_i^2} = 0$ for every $1 \leq i \leq k$, as required.

Using the above conditions, we want to show that the cochain f must satisfy also $f_{x_{i'}^2} = 0$ for every $1 \leq i \leq m$ (and hence that $f_{x_j^2} = 0$ for every j). Indeed, as before, we have that $f_{x_{i'}^2}(x^a)$ can be different from 0 only if $2 \leq a_i = a_{i'} + 2 \leq p-1$ and

$a_j = a_{j'}$ for every $j \neq i, i'$. Hence the required vanishing follows from the following cocycle condition

$$0 = df_{(x_i^2, x_{i'}^2)}(x^{a-\epsilon_i+\epsilon_{i'}}) = -2(a_{i'} + 1)f_{x_{i'}^2}(x^a).$$

Finally we have to show that we can modify once more (by adding coboundaries) the cocycle f in such a way that the previous vanishings $f_{x_i^2} = 0$ are still satisfied and moreover also $f_{x_i x_j}$ vanish for every $j \neq i, i'$.

First of all, note that using cocycle conditions of type $0 = df_{(x_h^2, x_i x_j)}$ with $h \neq i', j'$ and the fact that $f_{x_h^2} = 0$, we obtain the vanishing of $f_{x_i x_j}(x^a)$ for all the elements $x^a \in [x_h^2, K(n)_d] \cap K(n)_{-\epsilon_i-\epsilon_j}$ (for $h \neq i', j'$), that is for all the elements of $x^a \in K(n)_d$ with the exception of the ones that verify

$$(a_k, a_{k'}) = \begin{cases} (0,1) \text{ or } (p-2, p-1) & \text{if } k = i \text{ or } j, \\ (0,0) \text{ or } (p-1, p-1) & \text{otherwise.} \end{cases}$$

Therefore, we can assume that our x^a verifies these conditions. For the rest of the proof, we introduce the following definitions. We say that a couple $(a_k, a_{k'})$ is small if it is equal to $(0, 0)$ or $(0, 1)$ or $(1, 0)$ according to the conditions above, while we say that it is big if it is equal to $(p-1, p-1)$ or $(p-2, p-1)$ or $(p-1, p-2)$. Moreover we say that x^a has an ascending jump in position k (with $1 \leq k \leq m-1$) if $(a_k, a_{k'})$ is small and $(a_{k+1}, a_{(k+1)'})$ is big, while we say that it has a descending jump in position k if $(a_k, a_{k'})$ is big and $(a_{k+1}, a_{(k+1)'})$ is small.

We want to modify our cocycle f, by adding coboundaries, in such a way that $f_{x_i x_j}(x^a)$ vanish if x^a has a jump.

We prove this for the elements $f_{x_i x_{i+1}}$ with $1 \leq i \leq m-1$. It is enough to prove that $f_{x_i x_j}(x^a) = 0$ if there is a jump in a position less than or equal to i. Indeed if the jump on x^a occurs for $h > i$, then one obtains the vanishing using the cocycle condition $0 = df_{(x_i x_{i+1}, x_h x_{h+1})}$. Hence, by induction on i, suppose that we have already proved this for the elements $i \leq k-1$ and we want to prove it for $f_{x_k x_{k+1}}$. If there is a jump in the element x^a occurring in a position $h < k$ then the vanishing follows from a cocycle condition of type $0 = df_{(x_h x_{h+1}, x_k x_{k+1})}$ plus the induction hypothesis. If the first jump occurring in x^a is in the k-th position, then we define an element $g \in C^1(K(n)_d, F)_{\underline{0}}$ as follows:

$$\begin{cases} g(x^{a-\epsilon_{k'}+\epsilon_{k+1}}) = f_{x_k x_{k+1}}(x^a) & \text{if the jump is ascending,} \\ g(x^{a-\epsilon_{(k+1)'}+\epsilon_k}) = f_{x_k x_{k+1}}(x^a) & \text{if the jump is descending,} \\ g(x^b) = 0 & \text{otherwise.} \end{cases}$$

By construction (and the hypothesis on x^a), for every $1 \leq j \leq 2m$ we have that $(x_j^2 \circ g) = 0$ and if $1 \leq h < k \leq m-1$ then $(x_h x_{h+1} \circ g) = 0$ while $(x_k x_{k+1} \circ g)(x^a) = -f_{x_k x_{k+1}}(x^a)$. Therefore the new cocycle $\widetilde{f} = f + dg$ satisfies the same vanishing conditions of f (namely $\widetilde{f}_{x_j^2} = 0$ for every j and $\widetilde{f}_{x_h x_{h+1}} = 0$ for $1 \leq h < k$) plus the new one $\widetilde{f}_{x_k x_{k+1}}(x^a) = 0$. Repeating these modifications for all the elements x^a as above, we find a new cocycle (which, by an abuse of notation, we will still call f) that satisfies $f_{x_k x_{k+1}} = 0$, concluding thus the inductive step.

From the previous special cases, it follows also the vanishing of $f_{x_i x_j}(x^a)$ (always under the presence of a jump) if $1 \leq i, j \leq m$. Indeed, if an element x^a as before has a jump in position k then the coboundary condition

$$0 = df_{(x_i x_j, x_k x_{k+1})}(x^{a+\epsilon_{k'}-\epsilon_{k+1}}) = -\sigma(k)(a_{k'} + 1)f_{x_i x_j}(x^a),$$

in the case of an ascending jump, and

$$0 = df_{(x_i x_j, x_k x_{k+1})}(x^{a+\epsilon_{(k+1)'}-\epsilon_k}) = -\sigma(k+1)(a_{(k+1)'} + 1)f_{x_i x_j}(x^a),$$

in the case of a descending jump, gives the required vanishing.

Finally, the general case (in which i and j can vary from 1 to $2m$) follows from cocycle conditions of type $0 = df_{(x_i x_j, x_{i'}^2)} = -(x_{i'}^2 \circ f_{x_i x_j}) - 2\sigma(i)f_{x_{i'} x_j}$.

So it remains to consider only the elements x^a without jumps or, in other words, it remains to prove the vanishing of the following values of f: $f_{x_i x_j}(x_{i'} x_{j'} x_n^{p-1}) = \alpha_{ij} \cdot 1$ and $f_{x_i x_j}(x^{\sigma-\epsilon_i-\epsilon_j} x_n^{\nu}) = \beta_{ij} \cdot 1$, where $\nu \equiv m+1 \mod p$ and $0 \leq \nu \leq p-1$. The first ones vanish because of the following two cocycle conditions

$$\begin{cases} 0 = df_{(x_{i'}^2, x_i x_j)}(x_i x_{j'} x_n^{p-1}) = -2\sigma(i')\alpha_{ij} - 2\sigma(i')\alpha_{i'j}, \\ 0 = df_{(x_i x_j, x_{i'} x_j)}(x_{j'}^2 x_n^{p-1}) = -2\sigma(j)\alpha_{i'j} + 2\sigma(j)\alpha_{ij}. \end{cases}$$

The second ones vanish because of the following two cocycle conditions

$$\begin{cases} 0 = df_{(x_{i'}^2, x_i x_j)}(x^{\sigma-\epsilon_{i'}-\epsilon_j} x_n^{\nu}) = -2\sigma(i')(p-1)\beta_{ij} - 2\sigma(i')\beta_{i'j}, \\ 0 = df_{(x_i x_j, x_{i'} x_j)}(x^{\sigma-2\epsilon_j} x_n^{\nu}) = -\sigma(i)(p-1)\beta_{i'j} + \sigma(i')(p-1)\beta_{ij}. \quad \square \end{cases}$$

In the next (and last) lemma, we consider the differential map

$$\mathrm{d} : E_2^{0,1} = C^1(K(n)_d, F) \to E_2^{2,0} = H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right), \tag{2.5}$$

induced by the above spectral sequence (2.4). We compute the kernel of the induced map on the first cohomology group with respect to $K(n)_0$.

Lemma 2.11. *Consider the map*

$$\mathrm{d}^{(1)} : H^1(K(n)_0, C^1(K(n)_d, F)) \longrightarrow H^1\left(K(n)_0, H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right)\right)$$

induced by the differential map (2.5). The kernel of $\mathrm{d}^{(1)}$ *is given by*

$$\mathrm{Ker}(\mathrm{d}) = \begin{cases} \bigoplus_{i=1}^{2m} \langle \overline{\mathrm{Sq}(x_i)} \rangle_F & \text{if } d = p-2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{Sq}(x_i)}$ *denotes the restriction of* $\mathrm{Sq}(x_i)$ *to* $K(n)_0 \times K(n)_{p-2}$.

Proof. Clearly the cocycles $\overline{\mathrm{Sq}(x_i)}$, being the restriction of global cocycles, belong to the kernel of d. We want to show that the other generators of $H^1(K(n)_0, C^1(K(n)_d, F))$ (see Lemma 2.10) does not belong to $\mathrm{Ker}(\mathrm{d}^{(1)})$. First of all we have that

$$\mathrm{d}^{(1)}\langle x_n \mapsto \mathrm{inv}\rangle_F = \langle x_n \mapsto \mathrm{inv} \circ [-,-]\rangle,$$

and this last cocycle is not a coboundary since $\mathrm{inv} \circ [-,-] \in H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right)_{\underline{0}}$ and x_n acts trivially on this space.

Consider the cocycles ω_i for $1 \leq i \leq 2m$. At least one of the following values is non-zero (depending on μ):

$$\begin{cases} (\mathrm{d}\omega_i)_{x_i^2}(x_i x_n, x^{\sigma - 2\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = (\omega_i)_{x_i^2}([x_i x_n, x^{\sigma - 2\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}]) \\ \quad = (-3-\mu)(\omega_i)_{x_i^2}(x^{\sigma - \epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = 2(-3-\mu), \\ (\mathrm{d}\omega_i)_{x_i^2}(x_i^3, x^{\sigma - 4\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = (\omega_i)_{x_i^2}([x_i^3, x^{\sigma - 4\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}]) \\ \quad = -\mu(\omega_i)_{x_i^2}(x^{\sigma - \epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = -2\mu. \end{cases}$$

On the other hand, for every $g \in H^2\left(\frac{K(n)_{\geq 1}}{K(n)_{\geq d}}, F\right)$, it holds that

$$\begin{cases} (x_i^2 \circ g)(x_i x_n, x^{\sigma - 2\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = 0, \\ (x_i^2 \circ g)(x_i^3, x^{\sigma - 4\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}) = 0 \end{cases}$$

since $[x_i^2, x_i x_n] = [x_i^2, x^{\sigma - 2\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}] = [x_i^2, x_i^3] = [x_i^2, x^{\sigma - 4\epsilon_i - (p-1)\epsilon_{i'}} x_n^{\mu}] = 0$. □

3. Hamiltonian algebra

3.1. Definition and basic properties

Throughout this section we fix a field F of characteristic $p \neq 2, 3$ and an even integer $n = 2m \geq 2$.

We are going to use all the notations about multi-indices introduced at the beginning of Section 2.1. We are going to use often the following special n-tuples: $\underline{0} := (0, \ldots, 0)$, $\sigma := (p-1, \ldots, p-1)$ and $\sigma^i := \sigma - (p-1)\epsilon_i - (p-1)\epsilon_{i'}$.

The vector space $A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, endowed with the grading defined by $\deg(x^a) = |a| - 2$, becomes a graded Lie algebra by means of

$$[x^a, x^b] = D_H(x^a)(x^b),$$

where $D_H : A(n) \to W(n) = \mathrm{Der}_F A(n)$ is defined by

$$D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_{j'} = \sum_{i=1}^{m} \left[D_i(f) D_{i+m} - D_{i+m}(f) D_i\right].$$

We denote with $H'(n)$ the quotient of $A(n)$ by the central element $1 = x^{\underline{0}}$ so that there is an exact sequence of $H'(n)$-modules

$$0 \to \langle 1 \rangle_F \to A(n) \to H'(n) \to 0, \tag{3.1}$$

where $\langle 1 \rangle_F \cong F$ is the trivial $H'(n)$-module.

Definition 3.1. The Hamiltonian algebra is the derived subalgebra of $H'(n)$:

$$H(n) := H'(n)^{(1)} = [H'(n), H'(n)].$$

There is an exact sequence of $H(n)$-modules (see [8, Chap. 4, Prop. 4.4]):

$$0 \to H(n) \to H'(n) \to \langle x^{\sigma}\rangle_F \to 0, \tag{3.2}$$

where $\langle x^{\sigma}\rangle_F \cong F$ is the trivial $H(n)$-module.

Note that the unique term of negative degree is $H(n)_{-1} = \oplus_{i=1}^{n}\langle x_i\rangle_F$ where x_i acts, via the adjoint action, as $D_H(x_i) = \sigma(i)D_{i'}$. The term of degree 0 is $H(n)_0 = \oplus_{1\le i,j\le n}\langle x_i x_j\rangle_F$ and its adjoint action on $H(n)_{-1}$ induces an isomorphism $H(2m)_0 \cong \mathfrak{sp}(2m, F)$.

The algebra $H(n)$ admits a root space decomposition with respect to a canonical Cartan subalgebra.

Proposition 3.2. (a) *$T_H := \oplus_{i=1}^{m}\langle x_i x_{i'}\rangle_F$ is a maximal torus of $H(n)$ (called the canonical maximal torus).*
(b) *The centralizer of T_H inside $H(n)$ is the subalgebra $C_H = \{x^a \mid a_{i'} = a_i\}$, which is hence a Cartan subalgebra (called the canonical Cartan subalgebra). The dimension of C_H is $p^m - 2$.*
(c) *Let $\Phi_H := \mathrm{Hom}_{\mathbb{F}_p}(\oplus_{i=1}^{n}\langle x_i x_{i'}\rangle_{\mathbb{F}_p}, \mathbb{F}_p)$, where $\mathbb{F}_p$ is the prime field of F. We have a Cartan decomposition $H(n) = C_H \oplus_{\phi\in\Phi_H-\underline{0}} H(n)_{\phi}$, where $H(n)_{\phi} = \{x^a \mid a_{i+m} - a_i \equiv \phi(x_i x_{i'}) \bmod p\}$. The dimension of every $H(n)_{\phi}$, with $\phi \in \Phi_H - \underline{0}$, is p^m.*

Proof. See [8, Chap. 4, Theo. 4.5 and 4.6]. □

3.2. Proof of the Main Theorem 1.2

In this section, assuming the results of the next two sections, we give a proof of Main Theorem 1.2. As a first step towards the proof, we compute the cohomology group of the second cohomology group of the $H(n)$-module $H'(n)$.

Proposition 3.3. *The second cohomology group of $H'(n)$ is given by*

$$H^2(H(n), H'(n)) = \bigoplus_{i=1}^{n}\langle \mathrm{Sq}(x_i)\rangle_F \bigoplus_{i<j}\langle \Pi_{ij}\rangle_F \bigoplus \langle \Phi\rangle_F,$$

where Π_{ij} and Φ are the cocycles appearing in Theorem 1.2.

Proof. From the exact sequence (3.1) and using Propositions 3.4 and 3.10, we get the exact sequence

$$0 \to \bigoplus_{i=1}^{n}\langle \mathrm{Sq}(x_i)\rangle_F \bigoplus_{i<j}\langle \Pi_{ij}\rangle_F \bigoplus \langle \Phi\rangle_F \to H^2(H(n), H'(n)) \xrightarrow{\partial}$$
$$\to H^3(H(n), \langle 1\rangle_F) \to H^3(H(n), A(n)).$$

We have to verify that the coboundary map ∂ is equal to zero, or in other words that the cocycles which generate $H^3(H(n), \langle 1\rangle_F)$ (see Proposition 3.5) do not become zero in the group $H^3(H(n), A(n))$.

The cocycle Γ_{ij} (for certain $i < j, j \neq i'$) cannot be the coboundary of an element $h \in C^2(H(n), A(n))$. Indeed we have that $\Gamma_{ij}(x_i^2, x_j^2, x^{\sigma-(p-1)(\epsilon_{i'}+\epsilon_{j'})-\epsilon_i-\epsilon_j}) = 4$ while the element $dh(x_i^2, x_j^2, x^{\sigma-(p-1)(\epsilon_{i'}+\epsilon_{j'})-\epsilon_i-\epsilon_j})$ cannot contain the monomial 1 since the bracket of any two of the above elements vanish and all the three elements have degree greater than or equal to 0.

Assume now that $n \equiv -4 \mod p$ and suppose, by absurd, that the cocycle Ξ is the coboundary of a cochain $f \in C^2(H(n), A(n))$. For a multi-index $\underline{0} \le a \le \sigma$, call $\phi(x^a) = -\phi(x^{\sigma-a})$ the coefficient of 1 in the element $f(x^a, x^{\sigma-a})$. Consider a triple of elements $(x^a, x^b, x^c) \in H(n) \times H(n) \times H(n)$ such that $a + b + c = \sigma + \epsilon_k + \epsilon_{k'}$ (for a certain k) and $\deg(x^a) = \deg(x^b) = \deg(x^c) \ge 0$. By taking the coefficient of 1 in the equality $\Xi(x^a, x^b, x^c) = df(x^a, x^b, x^c)$ and using the relations $c_k \equiv -a_k - b_k \mod p$ and $c_{k'} = -a_{k'} - b_{k'} \mod p$, we get that

$$\phi(x^a) + \phi(x^b) = \phi(x^{a+b-\epsilon_k-\epsilon_{k'}}) + 1 \quad \text{if } a_k b_{k'} - a_{k'} b_k \not\equiv 0 \bmod p.$$

By considering triples as above with $\deg(x^a) = \deg(x^b) = 0$, we get the relations $2\phi(x_i x_j) = \phi(x_i^2) + \phi(x_j^2)$ and $2 = \phi(x_i^2) + \phi(x_{i'}^2)$, from which we deduce that the restriction of ϕ to $H(n)_0$ is determined by the values $\phi(x_i^2)$ for $1 \le i \le m$. Analogously, by taking $\deg(x^a) = 0$ and $\deg(x^b) = 1$, one gets that the restriction of ϕ to $H(n)_1$ is determined by the value $\phi(x_1^3)$ together with the restriction of ϕ to $H(n)_0$. Finally, by taking $\deg(x^a) = 1$ and $1 \le \deg(x^b) = d \le n(p-1) - 5$, one gets that the values of ϕ on $H(n)_{d+1}$ are determined by the values of ϕ on $H(n)_1$ and on $H(n)_d$. Therefore the values of ϕ on the elements having degree $0 \le d \le n(p-1) - 4$ is determined by the values $\phi(x_i^2)$ for $1 \le i \le m$ and $\phi(x_1^3)$. Explicitly, for an element $x^a \in H(n)$ such that $0 \le \deg(x^a) \le n(p-1) - 4$, one gets the following formula

$$\phi(x^a) = \left(\sum_{i=1}^{n} a_i - 2\right)\phi(x_1^3) + \left(-a_1 - 2a_{1'} - \sum_{j\neq 1,1'} \frac{3a_j}{2} + 3\right)\phi(x_1^2) + \sum_{k=2}^{m} \frac{a_k - a_{k'}}{2}\phi(x_k^2) + \sum_{h=m+1}^{n} a_h.$$

Imposing the antisymmetric relation $\phi(x^{\sigma-a}) = -\phi(x^a)$, we get the relation

$$-(n+4)\phi(x_1^3) + \frac{3(n+4)}{2}\phi(x_1^2) - \frac{n}{2} = 0,$$

which is impossible by the hypothesis $n \equiv -4 \bmod p$ (and $p \neq 2$).

Finally, the cocycles belonging to $H^3(H(n), H(n)_{-1};\langle 1\rangle_F)$ are not in the image of the coboundary map ∂ of above. Indeed, consider a cohomology class of $H^3(H(n), \langle 1\rangle_F)$ coming from $H^2(H(n), H'(n))$ and choose a representative $f \in Z^3(H(n), \langle 1\rangle_F)$ such that $f = \partial g$ where $g \in Z^2(H(n), H'(n))$. Since g takes values in $H'(n) = A(n)_{\geq 0}$, then the cocycle f vanish on the 3-tuples of elements having non-negative degree. On the other hand, if f belongs to $Z^3(H(n), H(n)_{-1};\langle 1\rangle_F)$, then by definition it must vanish on the 3-tuples of elements such that at least one has negative degree. Putting together these two vanishings, we deduce that $f = 0$. □

Now, using the above Proposition, we can prove the Main Theorem 1.2.

Proof of Theorem 1.2. From the exact sequence (3.2) and using that $H^1(H(n), \langle x^\sigma\rangle_F) = 0$, we get the exact sequence

$$0 \to H^2(H(n), H(n)) \to H^2(H(n), H'(n)) \to H^2(H(n), \langle x^\sigma\rangle_F),$$

so that we have to check which of the cocycles of the above Proposition 3.3 go to 0 under the projection onto $H^2(H(n), \langle x^\sigma\rangle_F)$. Clearly the cocycles $\mathrm{Sq}(x_i)$ take values in $H(n) = [H'(n), H'(n)]$ by definition.

Consider the cocycles $\Pi_{ij} \in H^2(H(n), H'(n))$. If $j \neq i, i'$ then the projection of Π_{ij} onto $H^2(H(n), \langle x^\sigma\rangle_F)$ is 0. Indeed $\Pi_{ij}(x^a, x^b) \subset \langle x^\sigma\rangle_F$ if and only if $a + b = \sigma - (p-1)\epsilon_i - (p-1)\epsilon_j + \epsilon_{i'} + \epsilon_{j'}$ but, for these pairs of elements, it is easily checked that $D_i(x^a)D_j(x^b) - D_j(x^a)D_i(x^b) = (a_ib_j - a_jb_i)x^{a+b-\epsilon_i-\epsilon_j} = 0$. On the other hand, if $j = i'$, then the only non-zero values of $\Pi_{ii'}$ are given by

$$\Pi_{ii'}(x_ix^a, x_{i'}x^b) = x^{a+b+(p-1)\epsilon_i+(p-1)\epsilon_{i'}} \quad \text{for } a + b \leq \sigma^i.$$

Therefore, if $n = 2$, the cocycle Π_{12} satisfy $\Pi_{12}(x_1, x_2) = x^\sigma$ and hence it cannot be lifted to $H^2(H(n), H(n))$. On the other hand, for $n \geq 4$, if we define $g_i \in C^1(H(n), H'(n))$ by $g_i(x^{\sigma^i}) = x^\sigma$, then the only non-zero values of the coboundary $\mathrm{d}g_i$ (for $1 \leq i \leq m$) can be

$$\begin{cases} \mathrm{d}g_i(x_ix^a, x_{i'}x^b) = -g_i([x_ix^a, x_{i'}x^b]) = -x^\sigma & \text{if } a + b = \sigma^i, \\ \mathrm{d}g_i(x_k, x^{\sigma^i}) = [x_k, g_i(x^{\sigma^i})] = -\sigma(k)x^{\sigma-\epsilon_{k'}} & \text{for any } 1 \leq k \leq n. \end{cases}$$

Therefore $\Pi_i = \Pi_{ii'} + \mathrm{d}g_i$ and clearly $\Pi_i \in H^2(H(n), H(n))$ since it vanish on the pairs $(x_ix^a, x_{i'}x^b)$ such that $a + b = \sigma^i$.

Consider now the cocycle Φ. We want to prove that its projection onto $H^2(H(n), \langle x^\sigma\rangle_F)$ vanish. From the explicit description of Φ, it follows that its projection onto $\langle x^\sigma\rangle_F$ is given by

$$\Phi(x^a, x^b) = \sum_{\substack{|\delta|=3, \delta+\widehat{\delta}<a \\ \delta+\widehat{\delta}=a+b-\delta}} \binom{a}{\delta}\binom{b}{\widehat{\delta}}\sigma(\delta)\delta! x^\sigma,$$

where the above sum is set equal to 0 if there are no elements δ verifying the hypothesis. Each element δ verifying the above hypothesis contributes to the summation with the coefficient

$$\sigma(\delta)\delta!\binom{a}{\delta}\binom{b}{\widehat{\delta}} = -\sigma(\delta)\delta!\binom{a}{\delta}\binom{a-\delta}{\widehat{\delta}} = -\frac{\sigma(\delta)}{\widehat{\delta}!}\frac{a!}{(a-\delta-\widehat{\delta})!},$$

where in the first equality we substitute $b = \sigma - a + \delta + \widehat{\delta}$ and we use the relation $\binom{\sigma-c}{d} = (-1)^{|d|}\binom{c+d}{d}$ which follows from the congruence $k!(p-1-k)! \equiv (-1)^{k+1} \mod p$ (for $0 \leq k \leq p-1$). Now note that if a certain δ appears in the above summation, then its conjugate also appears $\widehat{\delta}$ and we have that $\delta \neq \widehat{\delta}$ because of the oddness of the degree $|\delta|$. Using the easy relations $\delta! = \widehat{\delta}!$ and $\sigma(\delta) = (-1)^{|\delta|}\sigma(\widehat{\delta}) = -\sigma(\widehat{\delta})$, it follows that the contributions of δ and $\widehat{\delta}$ are opposite and therefore the sum vanish. □

3.3. Cohomology of the trivial module

In this section we compute the second and third cohomology group of $H(n)$ with coefficients in the trivial module F.

Proposition 3.4. *The second cohomology group of the trivial module is equal to*

$$H^2(H(n), F) = \begin{cases} \bigoplus_{i=1}^n \langle \Omega_i\rangle_F \bigoplus \langle \Sigma\rangle_F & \text{if } n \not\equiv -4 \mod p, \\ \bigoplus_{i=1}^n \langle \Omega_i\rangle_F \bigoplus \langle \Sigma\rangle_F \bigoplus \langle \Delta\rangle_F & \text{otherwise,} \end{cases}$$

where the only non-zero values of the above cocycles are

$$\begin{cases} \Omega_i(x^a, x^b) = a_i & \text{if } a + b = \sigma + \epsilon_i - (p-1)\epsilon_{i'}, \\ \Sigma(x_k, x_{k'}) = \sigma(k), \\ \Delta(x^a, x^b) = \deg(x^a) & \text{if } a + b = \sigma. \end{cases}$$

Proof. Note that the cochain Δ is antisymmetric if and only if $n \equiv -4 \mod p$, because if $a+b=\sigma$ then $\deg(x^a)+\deg(x^b) = n(p-1)-4 \equiv -n-4 \mod p$.

The verification that the above cochains are cocycles and are independent in $H^2(H(n),F)$ is straightforward and is left to the reader. We conclude by [18, Thm 2.4], which gives that

$$\dim_F H^2(H(n),F) = \begin{cases} n+1 & \text{if } n \not\equiv -4 \mod p, \\ n+2 & \text{otherwise.} \end{cases} \quad \square$$

In order to compute $H^3(H(n),F)$, we use the Hochschild–Serre spectral sequence (see [17]) relative to the subalgebra $H(n)_{-1} < H(n)$:

$$E_1^{r,s} = H^s(H(n)_{-1}, C^r(H(n)/H(n)_{-1},F)) \Rightarrow H^{r+s}(H(n),F). \tag{3.3}$$

For the first line of the second page of the above spectral sequence, we have the equality

$$E_2^{r,0} = H^r(H(n),H(n)_{-1};F), \tag{3.4}$$

where $H^*(H(n),H(n)_{-1};F)$ are the relative cohomology groups of $H(n)$ with respect to the subalgebra $H(n)_{-1}$ with coefficients in the trivial module F (as defined in [19]).

Moreover, as remarked in [11, Sec. 2.1], we can restrict ourselves to consider homogeneous cohomology with respect to the maximal torus $T_H \subset H(n)$ (see Proposition 3.2).

Proposition 3.5. *The third cohomology group of the trivial module is equal to*

$$H^3(H(n),F) = \begin{cases} H^3(H(n),H(n)_{-1};F) \bigoplus_{i<j,i\neq j'} \langle \Gamma_{ij}\rangle_F & \text{if } n \not\equiv -4 \mod p, \\ H^3(H(n),H(n)_{-1};F) \bigoplus_{i<j,i\neq j'} \langle \Gamma_{ij}\rangle_F \bigoplus \langle \Xi\rangle_F & \text{otherwise} \end{cases}$$

where, by definition, the only non-zero values of the above cocycles are (for $j \neq i, i'$)

$$\begin{cases} \Gamma_{ij}(x^a,x^b,x^c) = a_ib_j - a_jb_i & \text{if } a+b+c = \sigma-(p-1)\epsilon_{i'}-(p-1)\epsilon_{j'}+\epsilon_i+\epsilon_j, \\ \Xi(x^a,x^b,x^c) = \sigma(k)[a_kb_{k'}-a_{k'}b_k] & \text{if } a+b+c = \sigma+\epsilon_k+\epsilon_{k'} \text{ for some } k. \end{cases}$$

Proof. The verification that the above cochains are cocycles is straightforward and is left to the reader. In order to show that they freely generate the third cohomology group, we divide the proof into four steps according to the spectral sequence (3.3).

STEP I : $(E_1^{0,3})_{\underline{0}} = H^3(H(n),F)_{\underline{0}} = 0$ by homogeneity.

STEP II : $(E_\infty^{1,2})_{\underline{0}} = \bigoplus_{i<j,i\neq j'} \langle \Gamma_{ij}\rangle_F$.

With the notations of Lemma 3.7, consider a cochain $\zeta = \sum_{k=1}^n d_k\zeta_k \in (E_1^{1,2})_{\underline{0}}$ and suppose that it can be lifted to a global cocycle in $Z^3(H(n),F)_{\underline{0}}$ (which we continue to call ζ). Consider the following cocycle condition

$$d\zeta(x_i,x_{i'},x_i^2,x_{i'}^2) = -4\sigma(i)\zeta(x_i,x_{i'},x_ix_{i'}) = -4\sigma(i)[\sigma(i')d_i - \sigma(i)d_{i'}] = 4[d_i+d_{i'}],$$

from which we deduce the relation $d_i = -d_{i'}$. It is easily checked that ζ is the coboundary of the cocycle $f \in (E_1^{0,2})_{\underline{0}} = H^2(H(n)_{-1},F)_{\underline{0}}$ defined by $f(x_i,x_{i'}) = -d_{i'} = d_i$, since we have (for $j \neq i, i'$)

$$\zeta(x_ix_j,x_{i'},x_{j'}) = \sigma(j')d_{i'} - \sigma(i')d_{j'} = d(f)(x_ix_j,x_{i'},x_{j'}).$$

Suppose now that $n \geq 4$. The cocycles $\overline{\Gamma_{ij}}$ with $j \neq i, i'$ appearing in Lemma 3.7 are clearly lifted by the cocycles Γ_{ij}. On the other hand, the cocycles $\overline{\Gamma_{ii'}}$ cannot be lifted to $Z^3(H(n),F)_{\underline{0}}$. Indeed, by absurd, suppose that we can find such a lift and call it $\Gamma_{ij} \in Z^3(H(n),F)_{\underline{0}}$. We can suppose that Γ_{ij} takes its non-zero values on the triples $(x^\alpha,x^\beta,x^\gamma)$ such that $\alpha+\beta+\gamma = \sigma^i+\epsilon_i+\epsilon_{i'}$, where $\sigma^i := \sigma-(p-1)\epsilon_i-(p-1)\epsilon_{i'}$. Consider the following cocycle condition (where a, b, c are multi-indices verifying $a+b+c=\sigma^i$):

$$0 = \sigma(i)\Gamma_{ii'}(x_i^2x^a,x_{i'},x_{i'}x^b,x^c) = -2\Gamma_{ii'}(x_ix^a,x_{i'}x^b,x^c) - 2\Gamma_{ii'}(x_ix^{a+b},x_{i'},x^c).$$

We deduce that the value of $\Gamma_{ii'}(x_ix^a,x_{i'}x^b,x^c)$ depends only on the multi-index c and therefore, for every $\underline{0} \leq c \leq \sigma^i$, we can define $\omega(c) := \Gamma_{ii'}(x_ix^a,x_{i'}x^b,x^c)$ for every pair of indices a, b such that $a+b+c=\sigma^i$. By the fact that $\Gamma_{ij|H(n)_{-1}\times H(n)_{-1}} = \overline{\Gamma_{ij}}$, we get

$$\begin{cases} \omega(\sigma^i) = \Gamma_{ij}(x_i,x_{i'},x^{\sigma^i}) = \overline{\Gamma_{ij}}(x_i,x_{i'},x^{\sigma^i}) = 1, \\ \omega(\epsilon_j) = \Gamma_{ij}(x_ix^{\sigma^i-\epsilon_j},x_{i'},x_j) = \overline{\Gamma_{ij}}(x_ix^{\sigma^i-\epsilon_j},x_{i'},x_j) = 0. \end{cases}$$

Finally consider the following cocycle condition where $j \neq i, i'$ and $\underline{0} \leq d \leq \sigma^i$ is a multi-index such that $d_{j'} > 0$:

$$0 = \sigma(j)\Gamma_{ii'}(x_i,x^{\sigma^i-d+\epsilon_{j'}}x_{i'},x_j,x^d) = d_{j'}[\omega(d)-\omega(d-\epsilon_{j'})],$$

where we used that $\omega(\epsilon_j) = 0$. We deduce that the value $\omega(d)$ does not depend on the coefficient $d_{j'}$ and, by repeating for every index $j \neq i, i'$, we conclude that ω must be constant. But this contradicts with $\omega(\epsilon_j) = 0$ and $\omega(\sigma^i) = 1$.

STEP III : $(E_\infty^{2,1})_{\underline{0}} = \begin{cases} \langle \Xi \rangle_F & \text{if } n \equiv -4 \bmod p, \\ 0 & \text{otherwise.} \end{cases}$

With the notations of Lemma 3.8, consider the cochain $\xi = \sum_{k=1}^{n} e_k \xi_k$ and suppose that it can be lifted to a global cocycle of $Z^3(H(n), F)_{\underline{0}}$ (which, as usual, we continue to call ξ). From the following cocycle condition

$$0 = \mathrm{d}\xi(x_{k'}, x_k^2, x_{k'}^2, x^{\sigma-\epsilon_{k'}}) = \sigma(k')\xi(x_k^2, x_{k'}^2, x^{\sigma-\epsilon_k-\epsilon_{k'}}) + 4e_k$$

together with the analogous one obtained interchanging k with k', we get that $e_k = e_{k'}$.

If $n \equiv -4 \mod p$, then the cochain $\sum_{i=1}^{n} \xi_i$ is lifted by the global cocycle $\Xi \in (E_\infty^{2,1})_{\underline{0}}$. We will show that under the assumption that either $n \not\equiv -4 \mod p$ or $n \equiv -4 \mod p$ and $\sum_{i=1}^{m} e_i = 0$, then ξ belongs to the image of the differential map

$$\mathrm{d} : (E_1^{1,1})_{\underline{0}} \to (E_1^{2,1})_{\underline{0}},$$

coming from the spectral sequence (3.3).

Consider the cochains $\eta_k \in (E_1^{1,1})_{\underline{0}} = H^1(H(n)_{-1}, C^1(H(n)/H(n)_{-1}, F))_{\underline{0}}$ (for $1 \leq k \leq m$), whose only non-zero values are given by

$$\eta_k(x_k, x^{\sigma-\epsilon_k}) = \eta_k(x_{k'}, x^{\sigma-\epsilon_{k'}}) = -1.$$

Form the cochain $\eta := \sum_{i=1}^{m}(\frac{e_i}{2} + \beta)\eta_i$, where $\beta \in F$ is defined as

$$\beta = \begin{cases} -\dfrac{\sum_{i=1}^{m} e_i}{n+4} & \text{if } n \not\equiv -4 \bmod p \\ 0 & \text{if } n \equiv -4 \bmod p \quad \text{and} \quad \displaystyle\sum_{i=1}^{m} e_i = 0. \end{cases}$$

It is straightforward to check that the cocycle $\xi - \mathrm{d}\eta \in C^1(H(n)_{-1}, C^2(H(n)/H(n)_{-1}, F))_{\underline{0}}$ is the coboundary of the cochain $g \in C^2(H(n)/H(n)_{-1}, F)$ defined by (and vanishing elsewhere)

$$g(x^a, x^{\sigma-a}) = \sum_{i=1}^{m}(a_i + a_{i'})\frac{e_i}{2} + \deg_H(x^a)\beta \quad \text{if } |a|, |\sigma - a| \geq 2.$$

This shows that $[\xi] = [\mathrm{d}\eta] \in (E_1^{2,1})_{\underline{0}}$.

Suppose next that the cocycle ρ_{ij} (for certain $i < j$) can be lifted to a global cocycle of $Z^3(H(n), F)$, which we continue to call ρ_{ij}. For $l = 2, \ldots, p-1$, we define $f_l := \rho_{ij}(x_i x_{j'}, x^{\sigma-l\epsilon_{j'}}, x^{\sigma-(p-1)\epsilon_{i'}-(p+1-l)\epsilon_{j'}})$. Consider the following cocycle condition for $1 \leq l \leq p-1$:

$$\begin{aligned} 0 &= \mathrm{d}\rho_{ij}(x_i x_{j'}, x_j, x^{\sigma-l\epsilon_{j'}}, x^{\sigma-(p-1)\epsilon_{i'}-(p-l)\epsilon_{j'}}) \\ &= \frac{-(1+\delta_{ji'})}{l} + \delta_{1l}(1+\delta_{ji'}) - \sigma(j)(l+1)f_{l+1} + \sigma(j)(l-1)f_l. \end{aligned} \tag{$*$}$$

The above Eq. $(*)$ with $l = 1, \ldots, p-2$ gives that

$$f_l = \frac{-(1+\delta_{ji'})(l-2)\sigma(j)}{(l-1)l} \quad \text{for } l = 2, \ldots, p-1.$$

Substituting in the above Eq. $(*)$ with $l = p-1$, we get

$$0 = \frac{-(1+\delta_{ji'})}{p-1} + \sigma(j)(p-2)\frac{-(1+\delta_{ji'})(p-3)\sigma(j)}{(p-2)(p-1)} = (1+\delta_{ji'}) - 3(1+\delta_{ji'}),$$

which is impossible since $p \neq 2$. Therefore the cocycles ρ_{ij} do not belong to $(E_\infty^{2,1})_{\underline{0}}$.

STEP IV : $(E_\infty^{3,0})_{\underline{0}} = (E_2^{3,0})_{\underline{0}} = H^3(H(n), H(n)_{-1}; F)_{\underline{0}}$.

From Proposition 3.4, one can easily deduce that

$$\begin{cases} (E_\infty^{0,2})_{\underline{0}} = (E_2^{0,2})_{\underline{0}} = \langle \Sigma \rangle_F, \\ (E_\infty^{1,1})_{\underline{0}} = (E_2^{1,1})_{\underline{0}} = \begin{cases} \displaystyle\bigoplus_{i=1}^{n} \langle \Omega_i \rangle_F \oplus \langle \Delta \rangle_F & \text{if } n \equiv -4 \bmod p, \\ \displaystyle\bigoplus_{i=1}^{n} \langle \Omega_i \rangle_F & \text{otherwise.} \end{cases} \end{cases}$$

This implies that $(E_\infty^{3,0})_{\underline{0}} = (E_2^{3,0})_{\underline{0}}$ and the result follows from equality (3.4). □

Remark 3.6. It can be proved that $H^3(H(n), H(n)_{-1}; F) = \oplus_{i=1}^n \langle \Upsilon_i \rangle_F$ where the cocycles Υ_i are defined by

$$\Upsilon_i(x^a, x^b, x^c) = \sigma(k)[a_k b_{k'} - a_{k'} b_k] \quad \text{if } a+b+c = \sigma + p\epsilon_i + \epsilon_k + \epsilon_{k'} \quad \text{for some } k.$$

We omit the proof, since we do not need this result to prove Main Theorem 1.2.

Lemma 3.7. *In the above spectral sequence (3.3), we have that*

$$(E_1^{1,2})_{\underline{0}} = \begin{cases} \dfrac{\bigoplus_{k=1}^n \langle \zeta_k \rangle_F}{\left\langle \sum_{k=1}^n \sigma(k)\zeta_k \right\rangle_F} \bigoplus_{i<j} \langle \overline{\Gamma_{ij}} \rangle_F & \text{if } n \geq 4, \\ \dfrac{\bigoplus_{k=1}^n \langle \zeta_k \rangle_F}{\left\langle \sum_{k=1}^n \sigma(k)\zeta_k \right\rangle_F} & \text{if } n = 2, \end{cases}$$

where the only non-zero values of the above cocycles are

$$\begin{cases} \overline{\Gamma_{ij}}(x_i, x_j, x^{\sigma-(p-1)\epsilon_{i'}-(p-1)\epsilon_{j'}}) = 1, \\ \zeta_k(x_k, x_h, x_{k'}x_{h'}) = \sigma(h) \quad \text{for } h = 1, \dots, n. \end{cases}$$

Proof. Consider the exact sequence

$$0 \to C^1(x^\sigma, F) \to C^1(H'(n)/H'(n)_{-1}, F) \to C^1(H(n)/H(n)_{-1}, F) \to 0, \tag{3.5}$$

where $C^1(x^\sigma, F)$ is a trivial $H(n)_{-1}$-module. The coboundary map

$$H^1(H(n)_{-1}, C^1(H(n)/H(n)_{-1}, F))_{\underline{0}} \xrightarrow{\partial^{(2)}} H^2(H(n)_{-1}, C^1(x^\sigma, F))_{\underline{0}}$$

is surjective. Indeed consider the cocycles $\eta_k \in H^1(H(n)_{-1}, C^1(H(n)/H(n)_{-1}, F))_{\underline{0}}$ (with $1 \leq k \leq m$), defined as

$$\eta_k(x_k, x^{\sigma-\epsilon_k}) = \eta_k(x_{k'}, x^{\sigma-\epsilon_{k'}}) = -1.$$

It is easy to check that $\partial^{(2)}$ sends η_k into the cocycles $\{(x_k, x_{k'}, x^\sigma) \mapsto -2\}$ which generate the last group $H^2(H(n)_{-1}, C^1(x^\sigma, F))_{\underline{0}}$.

Using the above surjectivity, together with the vanishing $H^3(H(n)_{-1}, C^1(x^\sigma, F))_{\underline{0}} = 0$ which follows directly by homogeneity considerations, we get that

$$(E_2^{1,2})_{\underline{0}} = H^2(H(n)_{-1}, C^1(H'(n)/H'(n)_{-1}, F))_{\underline{0}}.$$

Consider now the following exact sequence of $H(n)_{-1}$-modules

$$0 \to C^1(H'(n)/H'(n)_{-1}, F) \to C^1(A(n), F) \to C^1(A(n)_{<0}, F) \to 0, \tag{3.6}$$

obtained from the fact that $H'(n)/H'(n)_{-1} = A(n)/A(n)_{<0}$ (see (3.1)). Using the following isomorphism of $H(n)_{-1}$-modules

$$\begin{aligned} \chi : A(n) &\xrightarrow{\cong} C^1(A(n), F) \\ x^a &\mapsto \chi_{x^a}(x^b) = \begin{cases} 1 & \text{if } b = \sigma - a, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{3.7}$$

together with [11, Prop. 3.4], we get that

$$H^1(H(n)_{-1}, C^1(H'(n)/H'(n)_{-1}, F))_{\underline{0}} = H^1(H(n)_{-1}, C^1(A(n), F))_{\underline{0}} = \oplus_{i=1}^n \langle \overline{\Omega_i} \rangle_F,$$

where the cocycles $\overline{\Omega_i}$ are defined by (and vanish outside) $\overline{\Omega_i}(x_i, x^{\sigma-(p-1)\epsilon_{i'}}) = 1$. Therefore we get the exact sequence

$$0 \to H^1(H(n)_{-1}, C^1(A(n)_{<0}, F))_{\underline{0}} \xrightarrow{\partial^{(2)}} (E_1^{1,2})_{\underline{0}} \to H^2(H(n)_{-1}, C^1(A(n), F))_{\underline{0}}.$$

The first group on the left is generated over F by the cocycles $\widetilde{\zeta_k}$ $(k = 1, \dots, n)$ defined by $\widetilde{\zeta_k}(x_k, x_{k'}) = 1$ and subject to the relation $\sum_{k=1}^n \sigma(k')\widetilde{\zeta_k} = 0$ coming from the element $\langle 1 \mapsto 1 \rangle_F \in C^1(A(n)_{<0}, F)_{\underline{0}}$. It is easily checked that $\partial^{(2)}(\widetilde{\zeta_k}) = \zeta_k$.

Moreover, using the isomorphism (3.7) of $H(n)_{-1}$-modules $A(n) \cong C^1(A(n), F)$ and [11, Prop. 3.4], we get that $H^2(H(n)_{-1}, C^2(A(n), F))_{\underline{0}}$ is freely generated over F by the cocycles $\overline{\Gamma_{ij}}$ for $1 \leq i < j \leq n$. We conclude by observing that $\overline{\Gamma_{ij}}$ can be lifted to $H^2(H(n)_{-1}, C^1(H'(n)/H'(n)_{-1}, F))_{\underline{0}}$ if and only if $n \geq 4$. □

Lemma 3.8. *In the above spectral sequence (3.3), we have that*

$$(E_1^{2,1})_{\underline{0}} = \begin{cases} \bigoplus_{k=1}^{n} \langle \xi_k \rangle_F \bigoplus_{i<j} \langle \rho_{ij} \rangle_F & \text{if } n \geq 4, \\ \bigoplus_{k=1}^{n} \langle \xi_k \rangle_F \oplus & \text{if } n = 2, \end{cases}$$

where the only non-zero values of the above cocycles are

$$\begin{cases} \xi_k(x_h, x_{h'}x_k, x^{\sigma-\epsilon_k}) = \sigma(h)(1+\delta_{h'k}) & \text{for every } h = 1, \ldots, n, \\ \rho_{ij}(x_i, x^{\sigma-l\epsilon_{j'}}, x^{\sigma-(p-1)\epsilon_{i'}-(p-l)\epsilon_{j'}}) = -\dfrac{\sigma(j)}{l} & \text{for every } l = 1, \ldots, p-1, \\ \rho_{ij}(x_j, x^{\sigma-l\epsilon_{i'}}, x^{\sigma-(p-l)\epsilon_{i'}-(p-1)\epsilon_{j'}}) = \dfrac{\sigma(i)}{l} & \text{for every } l = 1, \ldots, p-1. \end{cases}$$

Proof. Consider the following exact sequence of $H(n)_{-1}$-modules

$$0 \to C^2(H'(n)/H'(n)_{-1}, F) \to C^2(A(n), F) \xrightarrow{\text{res}} C^1(A(n)_{<0} \times A(n), F) \to 0, \tag{3.8}$$

It is easy to see that $C^1(A(n)_{<0} \times A(n), F)_{\underline{0}}^{H(n)_{-1}}$ is generated by the cocycle ζ defined by $\zeta(1, x^\sigma) = \zeta(x_i, x^{\sigma-\epsilon_i}) = 1$ (for every $i = 1, \ldots, n$) and that the image of ζ under the first coboundary map is non-zero and equal to $-\sum_{k=1}^n \xi_k$. Therefore, using Lemma 3.9, we get that

$$H^1(H(n)_{-1}, C^2(H'(n)/H'(n)_{-1}, F))_{\underline{0}} = \left\langle \sum_{k=1}^{n} \xi_k \right\rangle_F. \tag{3.9}$$

Consider finally the following exact sequence

$$C^1(H(n)/H(n)_{-1}, F) \overset{\theta}{\hookrightarrow} C^2(H'(n)/H'(n)_{-1}, F) \twoheadrightarrow C^2(H(n)/H(n)_{-1}, F), \tag{3.10}$$

where the map θ sends the cocycle g into the cocycle $\theta(g)$ defined by $\theta(g)(x^\sigma, x^a) = g(x^a)$. By taking cohomology, we get the exact sequence

$$H^1(H(n)_{-1}, C^2(H'(n)/H'(n)_{-1}, F))_{\underline{0}} \to (E_1^{2,1})_{\underline{0}} \xrightarrow{\partial^{(2)}} (E_1^{1,2})_{\underline{0}}.$$

We conclude by using (3.9), Lemma 3.7 and the facts that $\partial^{(2)}(\rho_{ij}) = 2\overline{\Gamma_{ij}}$ and $\partial^{(2)}(\xi_k) = \sigma(k)\zeta_{k'}$. □

Lemma 3.9. *Consider $A(n)$ as a $H(n)_{-1}$-module. Then we have that*

$$H^1(H(n)_{-1}, C^2(A(n), F))_{\underline{0}} = 0.$$

Proof. During this proof, we use the generators $D_i := \sigma(i')x_{i'}$ of $H(n)_{-1}$. Moreover, if $g \in C^2(A(n), 1)$, we set $\widetilde{g}(x^a, x^b) := \frac{g(x^a, x^b)}{a!b!}$ where, as usual, for a multi-index $a = (a_1, \ldots, a_n)$ we set $a! := \prod_i a_i!$. Analogously, if $f \in C^1(H(n)_{-1}, C(A(n), F))$, we set $\widetilde{f_{D_i}}(x^a, x^b) := \frac{f_{D_i}(x^a, x^b)}{a!b!}$ where $f_{D_i} \in C^2(A(n), F)$ denotes, as usual, the value of f on $D_i \in H(n)_{-1}$.

Take a homogeneous cochain $f \in Z^1(H(n)_{-1}, C^2(A(n), F))_{\underline{0}}$ The cocycle conditions for f are $D_i \circ f_{D_j} = D_j \circ f_{D_i}$ for every $1 \leq i, j \leq n$.

STEP I: The cocycle f verifies the following condition

$$0 = \sum_{k=-\alpha_i}^{\beta_i} (-1)^k \widetilde{f_{D_i}}(x^{\alpha+k\epsilon_i}, x^{\beta-k\epsilon_i}) \quad \text{for every } (x^\alpha, x^\beta) \text{ such that } \alpha_i + \beta_i \leq p-1. \tag{$*$}$$

For every pair (x^α, x^β) as above (such that $\alpha_i + \beta_i \leq p-1$), define

$$\phi_i(x^\alpha, x^\beta) := \sum_{k=-\alpha_i}^{\beta_i} (-1)^k \widetilde{f_{D_i}}(x^{\alpha+k\epsilon_i}, x^{\beta-k\epsilon_i}).$$

We have to prove that $\phi_i(x^\alpha, x^\beta) = 0$. Using the cocycle conditions $D_j \circ \widetilde{f_{D_i}} = D_i \circ \widetilde{f_{D_j}}$ and a telescopic sum, it is easy to see that $(D_j \circ \phi_i)(x^\alpha, x^\beta) = 0$ for every $j \neq i$. From these conditions, we get that (for every index $j \neq i$)

$$\phi_i(x^\alpha, x^\beta) = \begin{cases} 0 & \text{if } \alpha_j + \beta_j < p-1 \text{ for some } j \neq i, \\ (-1)^{|\beta|-\beta_i}\phi_i(x^{\alpha+\beta-\sigma+(p-1-\beta_i)\epsilon_i}, x^{\sigma-(p-1-\beta_i)\epsilon_i}) & \text{otherwise.} \end{cases}$$

So assume that we are in the second case, that is $\alpha_j + \beta_j \geq p-1$ for every $j \neq i$. Consider the same formula as above for the couple (x^β, x^α). By using using the antisymmetry of ϕ_i and the property $\phi_i(x^{\alpha+d\epsilon_i}, x^{\beta-d\epsilon_i}) = (-1)^d\phi_i(x^\alpha, x^\beta)$ for $-\alpha_i \leq d \leq p-1-\alpha_i$ and $-\beta_i \leq d \leq p-1-\beta_i$, we obtain

$$\left[(-1)^{|\beta|} + (-1)^{|\alpha|}\right]\phi_i(x^{\alpha+\beta-\sigma+(p-1-\beta_i)\epsilon_i}, x^{\sigma-(p-1-\beta_i)\epsilon_i}) = 0.$$

Now recall that f is homogeneous and therefore we have to consider only the pairs (x^α, x^β) such that the sum of the weights of x^α, x^β and D_i is $\underline{0}$. Using the conditions $\alpha_i + \beta_i \leq p-1$ and $\alpha_j + \beta_j \geq p-1$ for every $j \neq i$, we find the equalities (and not merely the congruences modulo p):

$$\alpha_{i'} + \beta_{i'} = \alpha_i + \beta_i + (p-1) \quad \text{and} \quad \alpha_{j'} + \beta_{j'} = \alpha_j + \beta_j \quad \text{for every } j \neq i, i'.$$

We deduce that $|\alpha| + |\beta|$ is even and, substituting in the expression above, we get the required vanishing.

STEP II: The cocycle f is a coboundary.

We have to find an element $g \in C(A(n), F)_{\underline{0}}$ such that $f_{D_i} = D_i \circ g$. For a homogeneous pair (x^a, x^b) (that is a pair such that the sum of the weights of x^a and x^b is $\underline{0}$), we define

$$\widetilde{g}(x^a, x^b) = \begin{cases} \displaystyle\sum_{k=0}^{a_i} (-1)^k \widetilde{f_{D_i}}(x^{a-k\epsilon_i}, x^{b+(k+1)\epsilon_i}) & \text{if } a_i + b_i < p-1, \\ \displaystyle\sum_{\emptyset \neq I \subset \{1,\ldots,n\}} \sum_{(c,d)\in S_I(a,b)} \frac{(-1)^{|I|-1}\text{sign}(c,d)}{2} \widetilde{f_{D_I}}(x^c, x^d) & \text{if } a + b \geq \sigma, \end{cases}$$

where for a non-empty subset I of $\{1, \ldots, n\}$ (of cardinality $|I|$), we define $S_I(a, b)$ to be the set of pairs (c, d) of multi-indices verifying: $c_i + d_i = a_i + b_i + 1$ and $\min(a_i, b_i) + 1 \leq c_i, d_i \leq \max(a_i, b_i)$ if $i \in I$ and $c_j = a_j$, $d_j = b_j$ if $j \notin I$ (in particular $S_I(a, b) \neq \emptyset$ if and only if $a_i \neq b_i$ for $i \in I$). Moreover, if $(c, d) \in S_I(a, b)$, we put $\text{sign}(c, d) = \prod_{i\in I} \text{sign}_i(c, d)$ and $\text{sign}_i(c, d) = d_i - b_i$ or $c_i - a_i$ according, respectively, to the cases $b_i < a_i$ and $a_i < b_i$. Finally, if $I = \{i_1, \ldots, i_r\}$, we put $f_{D_I} := D_{i_1} \circ \cdots \circ D_{i_{r-1}} \circ f_{D_{i_r}}$ which does not depend upon the order of the elements of I by the cocycle conditions verified by f.

The cochain g is well-defined because if i and j are two indices such that $a_i + b_i < p-1$ and $a_j + b_j < p-1$, then the following expression

$$\widetilde{g}(x^a, x^b) = \sum_{k=0}^{a_i}\sum_{h=0}^{a_j} (-1)^{k+h}(D_i \circ \widetilde{f_{D_j}})(x^{a-k\epsilon_i-h\epsilon_j}, x^{b+(k+1)\epsilon_i+(h+1)\epsilon_j})$$

is symmetric in i and j because of $D_i \circ f_{D_j} = D_j \circ f_{D_i}$ and reduces, via a telescopic sum, to the first expression occurring in the definition of $\widetilde{g}$.

Moreover it is clear from the definition that $\widetilde{g}$ is antisymmetric in the case $a + b \geq \sigma$, while in the case $a_i + b_i < p-1$ (for a certain i) the antisymmetry follows from the condition $(*)$ of above.

Finally we have to check that $(D_i \circ \widetilde{g})(x^\alpha, x^\beta) = \widetilde{f_{D_i}}(x^\alpha, x^\beta)$ for every index i and every pair (x^α, x^β) such that the sum of the weights of x^α, x^β and of D_i is $\underline{0}$. If $\alpha_i = \beta_i = 0$ then $(D_i \circ \widetilde{g})(x^\alpha, x^\beta) = 0$ and $\widetilde{f_{D_i}}(x^\alpha, x^\beta) = 0$ by condition $(*)$ of above. If $\alpha_i = 0$ and $\beta_i < 0$ then we get that $(D_i \circ \widetilde{g})(x^\alpha, x^\beta) = -\widetilde{g}(x^\alpha, x^{\beta-\epsilon_i})$ is equal to $\widetilde{f_{D_i}}(x^\alpha, x^\beta)$ by the first case of the definition of $\widetilde{g}$. The case $\alpha_i > 0$ and $\beta_i = 0$ follows from the preceding one by the antisymmetry of g. Therefore we are left with the case $\alpha_i, \beta_i > 0$.

Suppose first that $\alpha + \beta - \epsilon_i \not\geq \sigma$. Take an index j (may be equal to i) such that $(\alpha + \beta - \epsilon_i)_j < p-1$. Using the first case of the definition of g, we have

$$\begin{aligned}(D_i \circ \widetilde{g})(x^\alpha, x^\beta) &= \sum_{k=0}^{\alpha_j} (-1)^{k+1}(D_i \circ \widetilde{f_{D_j}})(x^{\alpha-k\epsilon_j}, x^{\beta+(k+1)\epsilon_j}) \\ &= \sum_{k=0}^{\alpha_j} (-1)^{k+1}(D_j \circ \widetilde{f_{D_i}})(x^{\alpha-k\epsilon_j}, x^{\beta+(k+1)\epsilon_j}) = \widetilde{f_{D_i}}(x^\alpha, x^\beta),\end{aligned}$$

where in the last equality we used a telescopic summation.

On the other hand, suppose that $\alpha + \beta - \epsilon_i \geq \sigma$. We need two auxiliary facts before proving the required equality in this case. First of all, observe that the hypothesis $\alpha + \beta - \epsilon_i \geq \sigma$ forces the equalities (and not merely the congruences modulo p) $\alpha_i + \beta_i - 1 = \alpha_{i'} + \beta_{i'}$ and $\alpha_j + \beta_j = \alpha_{j'} + \beta_{j'}$ for every $j \neq i, i'$. Therefore the sum of the degrees of the multi-indices $|\alpha| + |\beta|$ must be odd. Moreover, we can re-write the second expression occurring in the definition of g in a way that will be more suitable for our purpose. Fix an index i, a homogeneous pair (x^a, x^b) satisfying $a + b \geq \sigma$ and suppose that $a_i < b_i$. By splitting the summation occurring in the definition of $\widetilde{g}(x^a, x^b)$ according to the cases $I = \{i\}$, $I = \{i\} \cup J$ and $I = J$ with $i \in J \neq \emptyset$, and using a telescopic summation, we get

$$2\widetilde{g}(x^a, x^b) = \sum_{k=1}^{b_i-a_i} (-1)^k \widetilde{f_{D_i}}(x^{a+k\epsilon_i}, x^{b+(1-k)\epsilon_i}) + \sum_{i\notin J\neq\emptyset}\sum_{(c,d)\in S_J(a,b)} (-1)^{|J|+b_i-a_i+1}\text{sign}(c,d)\widetilde{f_{D_J}}(x^{c+(b_i-a_i)\epsilon_i}, x^{d-(b_i-a_i)\epsilon_i}). \quad (**)$$

If $a_i = b_i$ then the above expression is trivially true while if $a_i > b_i$ then we get an analogous expression using the antisymmetry of g.

Finally, in order to prove the required equality $(D_i \circ \widetilde{g})(x^\alpha, x^\beta) = \widetilde{f_{D_i}}(x^\alpha, x^\beta)$, we have to distinguish two cases: $\alpha_i < \beta_i - 1$ and $\alpha_i = \beta_i$ (the case $\alpha_i > \beta_i$ follows by antisymmetry). In the first case $\alpha_i < \beta_i$, consider

$$-2(D_i \circ \widetilde{g})(x^\alpha, x^\beta) = 2\widetilde{g}(x^{\alpha-\epsilon_i}, x^\beta) + 2\widetilde{g}(x^\alpha, x^{\beta-\epsilon_i})$$

and apply formula (∗∗) to the terms $(x^{\alpha-\epsilon_i}, x^\beta)$ and $(x^\alpha, x^{\beta-\epsilon_i})$, which verify the required conditions in virtue of our hypothesis. By summing the first terms in the corresponding expressions (∗∗), we get

$$-\widetilde{f_{D_i}}(x^\alpha, x^\beta) + (-1)^{\delta_i}\widetilde{f_{D_i}}(x^{\alpha+\delta_i\epsilon_i}, x^{\beta-\delta_i\epsilon_i}), \tag{$***1$}$$

where we put $\delta_i := \beta_i - \alpha_i - 1 \geq 0$. By summing the last terms in the corresponding expressions (∗∗) and using that if $i \notin J$ then a pair (c, d) belongs to $S_J(\alpha, \beta)$ if and only if $(c - \epsilon_i, d) \in S_J(\alpha - \epsilon_i, \beta)$ (and, analogously, if and only if $(c, d - \epsilon_i) \in S_J(\alpha, \beta - \epsilon_i)$), we obtain

$$\sum_{i\notin J\neq\emptyset}\ \sum_{(c,d)\in S_J(\alpha,\beta)} (-1)^{\delta_i+|J|}\operatorname{sign}(c,d)(D_i \circ \widetilde{f_{D_J}})(x^{c+\delta_i\epsilon_i}, x^{d-\delta_i\epsilon_i}). \tag{$***2$}$$

Using that $D_i \circ \widetilde{f_{D_J}} = D_J \circ \widetilde{f_{D_i}}$ and iterated telescopic summations, the above expression (∗ ∗ ∗2) reduces to

$$-(-1)^{\delta_i}\widetilde{f_{D_i}}(x^{\alpha+\delta_i\epsilon_i}, x^{\beta-\delta_i\epsilon_i}) - (-1)^{|\beta|-|\alpha|}\widetilde{f_{D_i}}(x^\beta, x^\alpha). \tag{$***2'$}$$

Summing expressions (∗∗∗1) and (∗∗∗2′) and using the fact that $|\beta|+|\alpha|$ is odd, we end up with $-\widetilde{f_{D_i}}(x^\alpha, x^\beta)+\widetilde{f_{D_i}}(x^\beta, x^\alpha) = -2\widetilde{f_{D_i}}(x^\alpha, x^\beta)$. □

The proof in the other case $\alpha_i = \beta_i$ is similar apart from the fact that one has to use both the expression (∗∗) and the analogous one with $a_i > b_i$. We leave the details to the reader. □

3.4. Cohomology of $A(n)$

In this section we compute the second cohomology group of the $H(n)$-module $A(n)$.

Proposition 3.10. *The second cohomology group of $A(n)$ is given by*

$$H^2(H(n), A(n)) = \begin{cases} \bigoplus_{i=1}^n \langle \mathrm{Sq}(x_i)\rangle_F \bigoplus_{i=1}^n \langle \Omega_i\rangle \bigoplus_{i<j} \langle \Pi_{ij}\rangle_F \bigoplus \langle \Phi\rangle_F & \text{if } n \not\equiv -4 \bmod p, \\ \bigoplus_{i=1}^n \langle \mathrm{Sq}(x_i)\rangle_F \bigoplus_{i=1}^n \langle \Omega_i\rangle \bigoplus_{i<j} \langle \Pi_{ij}\rangle_F \bigoplus \langle \Phi\rangle_F \bigoplus \langle \Delta\rangle_F & \text{otherwise.} \end{cases}$$

where Ω_i and Δ are the cocycles of Proposition 3.4, Φ and Π_{ij} (with $j \neq i'$) are the cocycles of Theorem 1.2 and the remaining cocycles $\Pi_{ii'}$ are defined by (and vanish outside):

$$\Pi_{ij}(x_i x^a, x_{i'} x^b) = x^{a+b+(p-1)\epsilon_i+(p-1)\epsilon_{i'}} \quad \text{if } a + b \leq \sigma^i.$$

Proof. It is straightforward to verify that the above cochains are cocycles and that they are independent in $H^2(H(n), A(n))$. Therefore it is enough to prove that

$$\dim_F H^2(H(n), A(n)) = \begin{cases} \binom{n}{2} + 2n + 1 & \text{if } n \not\equiv -4 \bmod p, \\ \binom{n}{2} + 2n + 2 & \text{otherwise.} \end{cases}$$

It is easily seen that $A(n)$ is the restricted $H(n)$-module induced from the restricted trivial $H(n)_{\geq 0}$-submodule $F \cong \langle x^\sigma\rangle_F \subset A(n)$. Moreover, it is also easy to see that the Lie algebra homomorphism $\sigma : H(n)_{\geq 0} \to F$ given by $\sigma(x) := \mathrm{tr}(\mathrm{ad}_{H(n)/H(n)_{\geq 0}} x)$ for $x \in H(n)_{\geq 0}$ (see [16, Pag. 155]) is trivial.

Moreover, using Lemma 3.12, it is straightforward to check that, in the notation of [16], we have the equality

$$[H(n)_{\geq 0}, H(n)_{\geq 0}] := H(n)_{\geq 0}^{(1)} = H(n)_{\geq 0}.$$

Therefore, using [16, Thm. 3.6(2)], we get that

$$H^2(H(n), A(n)) = H^2(H(n)_{\geq 0}, F) \oplus \bigwedge^2 (H(n)/H(n)_{\geq 0}).$$

Since $\dim_F \bigwedge^2 (H(n)/H(n)_{\geq 0}) = \binom{n}{2}$, we conclude using Proposition 3.11. □

Proposition 3.11. *The second cohomology group of* $H(n)_{\geq 0}$ *with coefficients in the trivial module* F *is given by*

$$H^2(H(n)_{\geq 0}, F) = \begin{cases} \bigoplus_{i=1}^{n} \langle \overline{\mathrm{Sq}(x_i)} \rangle_F \bigoplus_{i=1}^{n} \langle \Omega_i \rangle_F \bigoplus \langle \Phi \rangle_F & \text{if } n \not\equiv -4 \bmod p, \\ \bigoplus_{i=1}^{n} \langle \overline{\mathrm{Sq}(x_i)} \rangle_F \bigoplus_{i=1}^{n} \langle \Omega_i \rangle_F \bigoplus \langle \Phi \rangle_F \bigoplus \langle \Delta \rangle_F & \text{otherwise.} \end{cases}$$

where Ω_i *and* Δ *are the cocycles of Proposition 3.4,* Φ *and* Π_{ij} *(with* $j \neq i'$*) are the cocycles of Theorem 1.2 and* $\overline{\mathrm{Sq}(x_i)}$ *is the projection of* $\mathrm{Sq}(x_i)$ *onto* $\langle 1 \rangle_F \cong F$.

Proof. We prove first that

$$H^2(H(n)_{\geq 0}, F) = H^2(H(n)_{\geq 1}, F)^{H(n)_0},$$

where $H(n)_0$ acts on $H(n)_{\geq 1}$ via adjoint action. To this aim, consider the Hochschild–Serre spectral sequence with respect to the ideal $H(n)_{\geq 1} \lhd H(n)_{\geq 0}$:

$$E_2^{r,s} = H^r(H(n)_0, H^s(H(n)_{\geq 1}, F)) \Rightarrow H^{r+s}(H(n)_{\geq 0}, F). \tag{3.11}$$

By Lemma 3.12 and homogeneity, it follows that

$$E_2^{1,1} = H^1(H(n)_0, H^1(H(n)_{\geq 1}, F)) = H^1(H(n)_0, C^1(H(n)_1, F))_{\underline{0}} = 0.$$

Therefore we are left with showing that $H^2(H(n)_{\geq 0}, F) = 0$. First of all, we prove that $Z^1(H(n)_0, F)_{\underline{0}} = 0$. Indeed, a homogeneous element $g \in C^1(H(n)_0, 1)_{\underline{0}}$ can only take the following non-zero values $g(x_i x_{i'}) = \alpha_i \cdot 1$, with $\alpha_i = \alpha_{i'} \in F$. The vanishing of g follows from the following cocycle condition

$$0 = \mathrm{d}g(x_i^2, x_{i'}^2) = -4\sigma(i) g(x_i x_{i'}) = -4\sigma(i)\alpha_i. \tag{$*$}$$

Consider now a homogeneous cochain $f \in C^2(H(n)_0, F)_{\underline{0}}$. By applying the cocycle condition to the elements of T_H and using homogeneity, one gets that $f_{|T_H} \in Z^1(H(n)_0, 1)_{\underline{0}}$ which vanishes as proved above. Moreover, by adding to f a coboundary, we can suppose that $f(x_i^2, x_{i'}^2) = 0$ (see Eq. ($*$) of above). Therefore the only non-zero values of f can be $f(x_i x_j, x_{i'} x_{j'}) = \alpha_{ij} \cdot 1$ (for $j \neq i, i'$) with the obvious relations $\alpha_{ij} = \alpha_{ji}$ and $\alpha_{ij} = -\alpha_{i'j'}$. We conclude by means of the following cocycle condition

$$0 = \mathrm{d}f(x_i^2, x_{i'} x_j, x_{i'} x_{j'}) = -2\sigma(i)\alpha_{ij} + 2\sigma(i)\alpha_{ij'},$$

which gives $\alpha_{ij} = \alpha_{ij'} = \alpha_{i'j'} = -\alpha_{ij}$ and hence $\alpha_{ij} = 0$.

In order to compute $H^2(H(n)_{\geq 1}, F)^{H(n)_0}$, we will use the same strategy of Proposition 2.6, that is to compute, step-by-step as d increases, the truncated invariant cohomology groups

$$H^2\left(\frac{H(n)_{\geq 1}}{H(n)_{\geq d+1}}, F\right)^{H(n)_0}.$$

By using the Hochschild–Serre spectral sequence associated to the ideal

$$H(n)_d = \frac{H(n)_{\geq d}}{H(n)_{\geq d+1}} \lhd \frac{H(n)_{\geq 1}}{H(n)_{\geq d+1}},$$

we obtain the same diagram as in [11, Prop. 3.10] (the vanishing of $E_2^{0,2}$ and the injectivity of the map α are proved in exactly the same way) and then we take the cohomology with respect to $H(n)_0$. An easy inspection of their proof shows that Lemmas 2.7, 2.9 and 2.10 of the preceding section (for the algebra $K(2m+1)$) can be easily adapted to the present case simply by ignoring the variable x_{2m+1}. In particular we get that (for $d \geq 1$)

$$C^1(H(n)_1 \times H(n)_d, F)^{H(n)_0} = \begin{cases} \langle \Phi_2 \rangle_F & \text{if } d = 1, \\ \langle \Psi_2 \rangle_F & \text{if } d = n(p-1) - 5, \end{cases}$$

$$C^1(H(n)_d, F)^{H(n)_0} = 0,$$

$$H^1(H(n)_0, C^1(H(n)_d, F)) = \begin{cases} \oplus \langle \overline{\mathrm{Sq}(x_i)} \rangle_F & \text{if } d = p-2, \\ \oplus_{i=1}^{n} \langle \omega_i \rangle_F & \text{if } d = n(p-1) - p - 2. \end{cases}$$

where Φ_2, Ψ_2 and ω_i are defined as in the case of $K(n)$ but ignoring the part involving the variable $x_{2m+1} = x_n$.

By definition $\overline{\mathrm{Sq}(x_i)}$ is the restriction of $\mathrm{Sq}(x_i)$ and it is easy to see that ω_i is the restriction of Ω_i. Moreover if we extend Φ_2 by 0 outside $H(n)_1 \times H(n)_1$, then it is clear that $\Phi_2 \in H^2(H(n)_{\geq 0}, F) \subset H^2(H(n)_{\geq 1}, F)^{H(n)_0}$ and that Φ_2 is the restriction of the cocycle Φ (see also [11, Prop. 3.7]).

Finally, suppose that there is a lifting of Ψ_2 to a global $H(n)_0$-invariant cocycle of $Z^2(H(n)_{\geq 1}, A(n))$, which we will continue to call Ψ_2. Then using the cocycle condition $0 = \mathrm{d}\Psi_{2|H(n)_1}$ together with Lemma 3.12 and proceeding by induction on the degree, it is easy to see that Ψ_2 must agree with Δ on the couples $(x^a, x^b) \in H(n)_{\geq 1} \times H(n)_{\geq 1}$ such that $a + b = \sigma$ and we know from Proposition 3.4 that Δ is an antisymmetric cocycle if and only if $n \equiv -4 \mod p$. □

Lemma 3.12. *Let d be an integer greater than or equal to −1. Then*

$$[H(n)_1, H(n)_d] = H(n)_{d+1}.$$

Proof. The proof is the same as the first part of Lemma 2.5 (where we consider elements belonging to $A(2m) \subset K(2m+1)$) except for the fact that we do not have to consider the elements x_i because they have degree −1 and the element x^σ which does not belong to $H(n)$. □

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