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Fine compactified Jacobians

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We study Esteves's fine compactified Jacobians for nodal curves. We give a proof of the fact that, for a one-parameter regular local smoothing of a nodal curve X, the relative smooth locus of a relative fine compactified Jacobian is isomorphic to the Néron model of the Jacobian of the general fiber, and thus it provides a modular compactification of it. We show that each fine compactified Jacobian of X admits a stratification in terms of certain fine compactified Jacobians of partial normalizations of X and, moreover, that it can be realized as a quotient of the smooth locus of a suitable fine compactified Jacobian of the total blowup of X. Finally, we determine when a fine compactified Jacobian is isomorphic to the corresponding Oda-Seshadri's coarse compactified Jacobian.

1 Introduction

1.1 Motivation

The Jacobian variety of a smooth curve is an abelian variety that carries important information about the curve itself. Its properties have been widely studied along the decades, giving rise to a significant amount of beautiful mathematics.

However, for singular (reduced) curves, the situation is more involved since the generalized Jacobian variety is not anymore an abelian variety, once it is, in general, not compact. The problem of compactifying it is, of course, very natural, and it is considered to go back to the work of Igusa in [24] and Mayer-Mumford in [32] in the 50's–60's. Since then, several solutions appeared, differing from one another in various aspects as the generality of the construction, the modular description of the boundary and the functorial properties.

For families of irreducible curves, after the important work of D'Souza in [18], a very satisfactory solution has been found by Altman and Kleiman in [3]: their relative compactification is a fine moduli space, i.e., it admits a universal, or Poincaré, sheaf after an étale base change.

For reducible curves, the problem of compactifying the generalized Jacobian variety is much more intricate from a combinatorial and also functorial point of view. The case of a single curve over an algebraically closed field was dealt with by Oda-Seshdari in [35] in the nodal case and by Seshadri in [39] in the general case. For families of reducible curves, a relative compactification is provided by the work of Simpson in [40], which in great generality deals with coherent sheaves on families of projective varieties. A different approach is that of considering the universal Picard scheme over the moduli space of smooth curves and compactify it over the moduli space of stable curves. This point of view was the one considered by Caporaso in [9] and by Pandharipande in [37] (the later holds more generally for bundles of any rank) and by Jarvis in [26]. A common feature of these compactifications is that they are constructed using geometric invariant theory (GIT), hence they only give coarse

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moduli spaces for their corresponding moduli functors. We refer to [1] and [16] for an account on the way the different coarse compactified Jacobians for nodal curves relate to one another.

The problem of constructing fine compactified jacobians for reducible curves remained open until the work of Esteves in [19]. Given a family $f : \mathcal{X} \to S$ of reduced curves endowed with a vector bundle $\mathcal{E}$ of integral slope, called polarization, and with a section σ, Esteves constructs an algebraic space $J_{\mathcal{E}}^{\sigma}$ over S, which is a fine moduli space for simple torsion-free sheaves on the family satisfying a certain stability condition with respect to $\mathcal{E}$ and σ (see Section 2.6). The algebraic space $J_{\mathcal{E}}^{\sigma}$ is always proper over S and, in the case of a single curve X defined over an algebraically closed field, it is indeed a projective scheme (see [20, Theorem 2.4]).

However, not much is known on the geometry of Esteves's fine compactified Jacobians, for example how do they vary with the polarization and the choice of a section or how do they relate to the coarse compactified Jacobians. This last problem started to be investigated by Esteves in [20], where a sufficient condition ensuring that a fine compactified Jacobian is isomorphic to the corresponding coarse compactified Jacobian (in the sense of Section 2.6) is found for curves with locally planar singularities.

1.2 Results

The aim of the present work is to study the geometry of Esteves's fine compactified Jacobians for a nodal curve X over an algebraically closed field k. We introduce the notation $J_X^P\left(\underline{q}\right)$ for the fine compactified Jacobians of X, where P is a smooth point of X and $\underline{q} = \left\{\underline{q}_{C_i}\right\}$ is a collection of rational numbers, one for each irreducible component C_i of X, summing up to an integer number

$$|\underline{q}| := \sum_{C_i} \underline{q}_{C_i} \in \mathbb{Z}$$

(which corresponds to the choice of a polarization, see Section 2.6).

Our first result is Theorem 4.1, where we show that fine compactified Jacobians $J_X^P\left(\underline{q}\right)$ provide a geometrically meaningful compactification of Néron models of nodal curves or, according to the terminology of [11, Definition 2.3.5] and [13, Definition 1.4 and Proposition 1.6], that they are of Néron-type. Explicitly, this means the following: given a one-parameter regular local smoothing $f : \mathcal{X} \to S = \operatorname{Spec}(R)$ of X with a section σ such that $P = \sigma(\operatorname{Spec}(k))$ (see Section 2.3), where R is a Henselian DVR with algebraically closed residue field k and quotient field K, consider the relative fine compactified Jacobian $J_f^{\sigma}\left(\underline{q}\right)$, having special fiber isomorphic to $J_X^P\left(\underline{q}\right)$ and general fiber isomorphic to $\operatorname{Pic}^{|\underline{q}|}(\mathcal{X}_K)$. Then the S-smooth locus of $J_f^{\sigma}\left(\underline{q}\right)$, which consists of the sheaves on $\mathcal{X}$ whose restriction to $X = \mathcal{X}_k$ is locally free (see Fact 2.13), is naturally isomorphic to the Néron model $N\left(\operatorname{Pic}^{|\underline{q}|} \mathcal{X}_K\right)$ of the degree $\left|\underline{q}\right|$ Jacobian of the general fiber $\mathcal{X}_K$ of f. In particular, one gets that, independently of the choice of the polarization $\underline{q}$ and of the smooth point $P \in X_{\text{sm}}$, the number of irreducible components of the fine compactified Jacobians $J_X^P\left(\underline{q}\right)$ is always equal to the complexity $c(\Gamma_X)$ of the dual graph Γ_X of the curve X, or equivalently to the cardinality of the degree class group Δ_X (see Section 2.2). A different proof of this result already appears in the (unpublished) PhD thesis of Busonero [6].

Next, we show in Theorem 5.1 that the fine compactified Jacobians $J_X^P\left(\underline{q}\right)$ of X admit a canonical stratification

$$J_X^P\left(\underline{q}\right) = \coprod_{\emptyset \subseteq S \subseteq X_{\text{sing}}} J_{X,S}^P\left(\underline{q}\right)$$

where $J_{X,S}^P\left(\underline{q}\right)$ is the locally closed subset consisting of sheaves $\mathcal{I} \in J_X^P\left(\underline{q}\right)$ that are not free exactly at $S \subseteq X_{\text{sing}}$ and $J_{X,S}^P\left(\underline{q}\right)$ is not empty if and only if the partial normalization X_S of X at S is connected. We show that the closure of $J_{X,S}^P\left(\underline{q}\right)$ in $J_X^P\left(\underline{q}\right)$ is equal to the union of the strata $J_{X,S'}^P\left(\underline{q}\right)$ such that $S \subseteq S'$ and that it is canonically isomorphic to a fine compactified Jacobian $J_{X_S}^P\left(\underline{q^S}\right)$ for a suitable polarization $\underline{q^S}$ of X_S (see Section 2.4). In particular, each stratum $J_{X,S}^P\left(\underline{q}\right)$ is a disjoint union of $c(\Gamma_{X_S})$ copies of the generalized Jacobian $J(X_S)$ of X_S. Combined with the previous result, this implies that fine compactified Jacobians of a nodal curve X yield compactifications of Néron models of X such that the boundary is made of Néron models of certain partial normalizations of X.

In Theorem 6.1, we describe $J_X^P\left(\underline{q}\right)$ as a quotient of the smooth locus of a fine compactified Jacobian $J_{\widehat{X}}^P\left(\widehat{\underline{q}}\right)$ for a suitable polarization $\widehat{\underline{q}}$ on the total blowup $\widehat{X}$ of X (see Section 2.4). In Theorem 6.2, we show that a similar relation holds for the relative fine compactified Jacobians of suitable one-parameter regular local smoothings of X and $\widehat{X}$. In particular, the fine compactified Jacobian $J_X^P\left(\underline{q}\right)$ is a quotient of the special fiber of the Néron model of $\widehat{X}$ in degree $|\underline{q}|$.

Note that the above results were proved by Caporaso in [10] and [13] for the coarse canonical degree-d compactified Jacobians $\overline{P_X^d}$ (see Remark 2.12(v)) for a special class of stable curves X, called d-general (see Remark 7.5). Our results can be seen as a generalization of her results to arbitrary nodal curves and to any polarization.

Finally, in Theorem 7.1, we determine for which polarizations $\underline{q}$ and points $P \in X_{\text{sm}}$, the natural map (see Section 2.6)

$$\Phi : J_X^P\left(\underline{q}\right) \longrightarrow U_X\left(\underline{q}\right)$$

from Esteves's fine compactified Jacobians to the corresponding Oda-Seshadri's coarse compactified Jacobian is an isomorphism. In particular, we prove that this problem depends only on $\underline{q}$ and not on P and that the sufficient conditions on $\underline{q}$ found by Esteves in [20] are also necessary.

1.3 Outline of the paper

The paper is organized as follows. In Section 1, we collect all the notations and basic properties about nodal curves and their combinatorial invariants (dual graph, degree class group, polarizations) that we are going to use in the sequel. Moreover, we review the theory of Néron models for Jacobians and the main properties of Esteves's fine compactified Jacobians as well as Oda-Seshadri's, Seshadri's, Caporaso's and Simpson's coarse compactified Jacobians for nodal curves. We also compare these constructions among each others and we establish formulae linking the different notations.

Section 2 is entirely devoted to the proof of a technical result in graph theory, that is a key ingredient for the results in the subsequent sections.

In Section 3 we prove that fine compactified Jacobians are of Néron type.

In Section 4 we describe a stratification of $J_X^P\left(\underline{q}\right)$ in terms of fine compactifed Jacobians of partial normalizations of X.

Section 5 is devoted to show how to realize fine compactified Jacobians of X as quotients of the Néron model of the total blowup $\widehat{X}$ of X.

In Section 6 we characterize those polarizations for which Esteves's fine compactified Jacobians are isomorphic to Oda-Seshadri's coarse compactified Jacobians.

1.4 Further questions and future work

In the present paper we deal with nodal curves mainly because of the combinatorial tools that we use to prove our results, e.g., the dual graph associated to a nodal curve. It is likely, however, that some of our results could be extended to more general singular curves, e.g., curves with locally planar singularities (see [2] for the relevance of locally planar singularities in the context of compactified Jacobians of singular curves).

The results of this paper show that the fine compactified Jacobians $J_X^P\left(\underline{q}\right)$ of a nodal curve X share very similar properties regardless of the polarization $\underline{q}$ and the choice of the smooth point $P \in X_{\text{sm}}$. The following question arises naturally

Question 1.1 *For a given nodal curve X, how do the fine compactified Jacobians $J_X^P\left(\underline{q}\right)$ change as the polarization $\underline{q}$ and the smooth point $P \in X_{sm}$ vary?*

Note also that, by our comparison's result between fine compactified Jacobians and coarse compactified Jacobians (see Theorem 7.1), the above problem is also closely related to the problem of studying the variation of GIT in the Oda-Seshadri's construction of coarse compactified Jacobians of X. In turn, this problem seems to be related to wall-crossing phenomena for double Hurwitz numbers (see [17] and [23]). We plan to explore this fascinating connection in the future.

Recently, compactified Jacobians of integral curves have played an important role in the celebrated proof of the Fundamental Lemma, since they appear naturally as fibers of the Hitchin's fibration in the case where the spectral curve is integral (see [29], [30], [34]). We plan to extend this description to nodal (reducible) spectral curves using fine compactified Jacobians. We expect that the results on the geometry of fine compactified Jacobians described here can give important insights on the singularities of the fibers of the Hitchin map in the case where the spectral curve is reducible.

After this preprint was posted on arXiv, Jesse Kass posted the preprint [28] (based on his PhD thesis [27]), where he extends our Theorem 4.1 to a larger class of singular curves. Moreover, he pointed out to us that our stratification of the fine compactified Jacobians of nodal curves (see Section 5) is similar to the stratification by local type that the author describes in [27, Section 5.3].

2 Preliminaries and notations

Throughout this paper, R will be a Henselian (e.g., complete) discrete valuation ring (a DVR) with algebraically closed residue field k and quotient field K. We set $B = \mathrm{Spec}(R)$.

2.1 Nodal curves

By a genus g nodal curve X we mean a projective and reduced curve of arithmetic genus $g := 1 - \chi(\mathcal{O}_X)$ over k having only nodes as singularities. We will denote by ω_X the canonical or dualizing sheaf of X. We denote by γ_X (or simply γ) the number of irreducible components of X and by $C_1, \ldots, C_\gamma$ its irreducible components.

A *subcurve* $Y \subset X$ is a closed subscheme of X that is a curve, or in other words Y is the union of some irreducible components of X. We say that Y is a proper subcurve, and we write $Y \subsetneq X$, if Y is a subcurve of X and $Y \neq X$. For any proper subcurve $Y \subsetneq X$, we set $Y^c := \overline{X \setminus Y}$ and we call it the complementary subcurve of Y. For a subcurve $Y \subset X$, we denote by g_Y its arithmetic genus and by $\delta_Y := |Y \cap Y^c|$ the number of nodes where Y intersects the complementary curve Y^c. Then, the adjunction formula gives

$$w_Y := \deg(\omega_X)_{|Y} = 2g_Y - 2 + \delta_Y.$$

We denote by X_{sm} the smooth locus of X and by X_{sing} the set of nodes of X. We set $\delta = \delta_X := |X_{\text{sing}}|$. The set of nodes X_{sing} admits a partition

$$X_{\text{sing}} = X_{\text{ext}} \coprod X_{\text{int}},$$

where X_{ext} is the subset of X_{sing} consisting of the nodes at which two different irreducible components of X meet (we call these external nodes), and X_{int} is the subset of X_{sing} consisting of the nodes which are self-intersection of an irreducible component of X (we call these internal nodes).

We denote by Γ_X the *dual graph* of X. With a slight abuse of notation, we identify the edges $E(\Gamma_X)$ of Γ_X with the nodes X_{sing} of X and the vertices $V(\Gamma_X)$ of Γ_X with the irreducible components of X. Note that the subcurves of X correspond to the subsets of $V(\Gamma_X)$ via the following bijection: we associate to a set of vertices $W \subseteq V(\Gamma_X)$ the subcurve $X[W]$ of X given by the union of the irreducible components corresponding to the vertices which belong to W. Given a smooth point $P \in X_{\text{sm}}$, we denote by v_P the vertex corresponding to the unique irreducible component of X on which P lies.

A node $N \in X_{\text{ext}}$ is called a *separating node* if $X - N$ is not connected. Since X is itself connected, $X - N$ would have two connected components. Their closures are called the *tails* attached to N. We denote by $X_{\text{sep}} \subset X_{\text{ext}}$ the set of separating nodes of X. Following [20, Section 3.1], we say that a subcurve Y of X is a *spine* if $Y \cap Y^c \subset X_{\text{sep}}$. Note that the union of spines is again a spine and the connected components of a spine are spines. A tail (attached to some separating node $N \in X_{\text{sep}}$) is a spine Y such that Y and Y^c are connected and conversely.

Given a subset $S \subset X_{\text{sing}}$, we denote by X_S the *partial normalization* of X at S and by $\widehat{X_S}$ the *partial blowup* of X at S, where (with a slight abuse of terminology) by blowup of X at S we mean the nodal curve $\widehat{X_S}$ obtained from X_S by inserting a $\mathbb{P}^1$ attached at every pair of points of X_S that are preimages of a node $n \in S$. We call such a $\mathbb{P}^1 \subset \widehat{X_S}$ the exceptional component lying above $n \in S$ and we denote by $E_S \subset \widehat{X_S}$ the union of all the exceptional components. Note that we have a commutative diagram:

$$\begin{tikzcd}
X_S \arrow[rr, hook, "i_S"] \arrow[dr, "\nu_S"'] & & \widehat{X_S} \arrow[dl, "\pi_S"] \\
& X &
\end{tikzcd} \tag{2.1}$$

Here ν_S is the partial normalization map, π_S contracts to $p \in S$ the exceptional component lying above p and the inclusion i_S realizes X_S as the complementary subcurve of $E_S \subset \widehat{X_S}$. We denote the total blowup of X by $\widehat{X}$ and the natural map to X by $\pi : \widehat{X} \to X$.

For a given subcurve Y of X denote by $Y_S \subset X_S$ the preimage of Y under ν_S. Note that Y_S is the partial normalization of Y at $S \cap Y$ and that every subcurve $Z \subset X_S$ is of the form Y_S for some uniquely determined subcurve $Y \subset X$, namely $Y = \nu_S(Z)$.

The dual graph Γ_{X_S} of X_S is equal to the graph $\Gamma_X \setminus S$ obtained from Γ_X by deleting all the edges belonging to S. The dual graph $\Gamma_{\widehat{X_S}}$ of $\widehat{X_S}$ is equal to the graph $\widehat{(\Gamma_X)_S}$ obtained from Γ_X by adding a new vertex in the middle of every edge belonging to S.

2.2 Degree class group

We call the elements $\underline{d} = (d_1, \dots, d_\gamma)$ of $\mathbb{Z}^\gamma$ *multidegrees*. We set $|\underline{d}| := \sum_1^\gamma d_i$ and call it the total degree of $\underline{d}$. For a line bundle $L \in \operatorname{Pic} X$ its multidegree is $\underline{\deg}\, L := \big(\deg_{C_1} L, \dots, \deg_{C_\gamma} L\big)$ and its (total) degree is $\deg L := \deg_{C_1} L + \cdots + \deg_{C_\gamma} L$.

Given $\underline{d} \in \mathbb{Z}^\gamma$ we set $\operatorname{Pic}^{\underline{d}} X := \big\{L \in \operatorname{Pic} X : \underline{\deg}\, L = \underline{d}\big\}$. Note that $\operatorname{Pic}^{\underline{0}} X := \big\{L \in \operatorname{Pic} X : \underline{\deg}\, L = (0, \dots, 0)\big\}$ is a group (called the *generalized Jacobian* of X and denoted by $J(X)$) with respect to the tensor product of line bundles and each $\operatorname{Pic}^{\underline{d}}(X)$ is a torsor under $\operatorname{Pic}^{\underline{0}}(X)$. We set $\operatorname{Pic}^d X := \{L \in \operatorname{Pic} X : \deg L = d\} = \coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}} X$.

For every component C_i of X denote

$$\delta_{i,j} := \begin{cases} |C_i \cap C_j| & \text{if} \quad i \neq j, \\ -\delta_{C_i} & \text{if} \quad i = j. \end{cases}$$

For every $i = 1, \dots, \gamma$ set $\underline{c_i} := (\delta_{1,i}, \dots, \delta_{\gamma,i}) \in \mathbb{Z}^\gamma$. Then $|\underline{c_i}| = 0$ for all $i = 1, \dots, \gamma$ and the matrix M_X whose columns are the $\underline{c_i}$ can be viewed as an *intersection matrix* for X. Consider the sublattice Λ_X of $\mathbb{Z}^\gamma$ of rank $\gamma - 1$ spanned by the $\underline{c_i}$

$$\Lambda_X := \langle \underline{c_1}, \dots, \underline{c_\gamma} \rangle.$$

Definition 2.1 We say that two multidegrees $\underline{d}$ and $\underline{d}'$ are equivalent, and write $\underline{d} \equiv \underline{d}'$, if and only if $\underline{d} - \underline{d}' \in \Lambda_X$. The equivalence classes of multidegrees that sum up to d are denoted by

$$\Delta_X^d := \big\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d\big\}/_{\equiv}.$$

Note that $\Delta_X := \Delta_X^0$ is a finite group and that each Δ_X^d is a torsor under Δ_X. The group Δ_X is known in the literature under many different names (see [8] and the references therein); we will follow the terminology introduced in [9] and call it the *degree class group* of X.

We shall denote the elements in Δ_X^d by lowercase greek letters δ and write $\underline{d} \in \delta$ meaning that the class $[\underline{d}]$ of $\underline{d}$ is δ.

A well-known theorem in graph theory, namely Kirchhoff's Matrix Tree Theorem (see e.g., [7, Theorem 1.6] and the references therein), asserts that, if X is connected, the cardinality of Δ_X $\big($and hence of each $\Delta_X^d\big)$ is equal to the *complexity* $c(\Gamma_X)$ of the dual graph Γ_X of X, that is the number of spanning trees of Γ_X. Note that $c(\Gamma_X) > 0$ if and only if X is connected.

In the sequel, we will use the following result which gives a formula for the complexity of $\widehat{\Gamma_S}$ (see the notation in Section 2.1):

Fact 2.2 [7, Theorem 3.4] *For any $S \subset E(\Gamma)$, we have that*

$$c(\widehat{\Gamma_S}) = \sum_{\emptyset \subseteq S' \subseteq S} c(\Gamma \backslash S').$$

2.3 Néron models of Jacobians

A *one-parameter regular local smoothing* of X is a morphism $f : \mathcal{X} \to B$ where $\mathcal{X}$ is a regular surface, such that the special fiber $\mathcal{X}_k$ is isomorphic to X and the generic fiber $\mathcal{X}_K$ is a smooth curve.

Fix $f : \mathcal{X} \to B$ a one-parameter regular local smoothing of X. Let Pic_f denote the *relative Picard functor* of f $\big($often denoted $\operatorname{Pic}_{\mathcal{X}/B}$ in the literature, see [5, Chapter 8] for the general theory$\big)$. Pic_f^d is the subfunctor of line bundles of relative degree d. Pic_f $\big($resp. $\operatorname{Pic}_f^d\big)$ is represented by a scheme Pic_f $\big($resp. $\operatorname{Pic}_f^d\big)$ over B, see [5, Theorem 8.2]. Note that Pic_f and Pic_f^d are not separated over B if X is reducible.

For each multidegree $\underline{d} \in \mathbb{Z}^{\gamma}$, there exists a separated closed subscheme $\operatorname{Pic}_f^{\underline{d}} \subset \operatorname{Pic}_f^d$ parametrizing line bundles of relative degree d whose restriction to the closed fiber has multidegree $\underline{d}$. In other words, the special fiber of $\operatorname{Pic}_f^{\underline{d}}$ is isomorphic to $\operatorname{Pic}^{\underline{d}}(X)$ while, clearly, the general fiber is isomorphic to $\operatorname{Pic}^d(\mathcal{X}_K)$. Note that $\operatorname{Pic}_f^{\underline{0}}$ is a group scheme over B and that the $\operatorname{Pic}_f^{\underline{d}}$'s are torsors under $\operatorname{Pic}_f^{\underline{0}}$. It is well-known (see [10, Section 3.9]) that if $\underline{d} \equiv \underline{d}'$ then there is a canonical isomorphism (depending only on f)

$$\iota_f(\underline{d}, \underline{d}') : \operatorname{Pic}_f^{\underline{d}} \longrightarrow \operatorname{Pic}_f^{\underline{d}'}$$

which restricts to the identity on the generic fiber. The isomorphism $\iota_f(\underline{d}, \underline{d}')$ is given by tensoring with a line bundle on $\mathcal{X}$ of the form $\mathcal{O}_{\mathcal{X}}\big(\sum_i n_i C_i\big)$, for suitably chosen integers $n_i \in \mathbb{Z}$ such that $\sum_i n_i = 0$. We shall therefore identify $\operatorname{Pic}_f^{\underline{d}}$ with $\operatorname{Pic}_f^{\underline{d}'}$ for all pairs of equivalent multidegrees $\underline{d}$ and $\underline{d}'$. Thus for every $\delta \in \Delta_X^d$ we define

$$\operatorname{Pic}_f^{\delta} := \operatorname{Pic}_f^{\underline{d}} \tag{2.2}$$

for every $\underline{d} \in \delta$.

For any integer d, denote by $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ the *Néron model* over B of the degree-d Picard variety $\operatorname{Pic}^d \mathcal{X}_K$ of the generic fiber $\mathcal{X}_K$. Recall that $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ is smooth and separated over B, the generic fiber $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)_K$ is isomorphic to $\operatorname{Pic}^d \mathcal{X}_K$ and $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ is uniquely characterized by the following universal property (the Néron mapping property, cf. [5, Definition 1]): every K-morphism $u_K : Z_K \to \mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)_K = \operatorname{Pic}^d \mathcal{X}_K$ defined on the generic fiber of some scheme Z smooth over B admits a unique extension to a B-morphism $u : Z \to \mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$. Moreover, $\mathrm{N}\left(\operatorname{Pic}^0 \mathcal{X}_K\right)$ is a B-group scheme while, for every $d \in \mathbb{Z}$, $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ is a torsor under $\mathrm{N}\left(\operatorname{Pic}^0 \mathcal{X}_K\right)$.

The Néron models $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ can be described as the biggest separated quotient of Pic_f^d ([38, Section 4.8]). Indeed, since Pic_f^d is smooth over B and its general fiber is isomorphic to $\operatorname{Pic}^d(\mathcal{X}_K)$, the Néron mapping property yields a map

$$q : \operatorname{Pic}_f^d \longrightarrow \mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big). \tag{2.3}$$

The scheme Pic_f^d can be described as

$$\operatorname{Pic}_f^d \cong \frac{\coprod_{\underline{d} \in \mathbb{Z}^{\gamma}\,:\,|\underline{d}| = d} \operatorname{Pic}_f^{\underline{d}}}{\sim_K},$$

where $\sim_K$ denotes the gluing of the schemes $\operatorname{Pic}_f^{\underline{d}}$ along their general fibers, which are isomorphic to $\operatorname{Pic}^d(\mathcal{X}_K)$. On the other hand, the Néron model $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$ can be explicitly described as follows

Fact 2.3 [10, Lemma 3.10] *We have a canonical B-isomorphism*

$$\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big) \cong \frac{\coprod_{\delta \in \Delta_X^d} \operatorname{Pic}_f^{\delta}}{\sim_K}. \tag{2.4}$$

Therefore, the above map q sends each $\operatorname{Pic}^{\underline{d}}_{f}$ isomorphically into $\operatorname{Pic}^{[\underline{d}]}_{f}$ and identifies $\operatorname{Pic}^{\underline{d}}_{f}$ with $\operatorname{Pic}^{\underline{d}'}_{f}$ if and only if $\underline{d} \equiv \underline{d}'$.

Note that, from Fact 2.3, it follows that the special fiber of the Néron model $\mathrm{N}\big(\operatorname{Pic}^d \mathcal{X}_K\big)$, which we will denote by N_X^d, is isomorphic to a disjoint union of $c(\Gamma_X)$'s copies of the generalized Jacobian $J(X)$ of X.

2.4 Polarizations

Definition 2.4 A *polarization* on X is a γ-tuple of rational numbers $\underline{q} = \big\{\underline{q}_{C_i}\big\}$, one for each irreducible component C_i of X, such that $\big|\underline{q}\big| := \sum_i \underline{q}_{C_i} \in \mathbb{Z}$.

Given a subcurve $Y \subset X$, we set $\underline{q}_Y := \sum_j \underline{q}_{C_j}$ where the sum runs over all the irreducible components C_j of Y. Note that giving a polarization $\underline{q}$ is the same as giving an assignment $(Y \subset X) \mapsto \underline{q}_Y$ which is additive on Y, i.e., such that if $Y_1, Y_2 \subset X$ are two subcurves of X without common irreducible components then $\underline{q}_{Y_1 \cup Y_2} = \underline{q}_{Y_1} + \underline{q}_{Y_2}$ and such that $\underline{q}_X \in \mathbb{Z}$.

If $Y \subset X$ is a subcurve of X such that $\underline{q}_Y - \frac{\delta_Y}{2} \in \mathbb{Z}$, then we define the *restriction* of the polarization $\underline{q}$ to Y as the polarization $\underline{q}_{|Y}$ on Y such that

$$\big(\underline{q}_{|Y}\big)_Z = \underline{q}_Z - \frac{|Z \cap Y^c|}{2}, \tag{2.5}$$

for any subcurve $Z \subset Y$.

Given a subset $S \subset X_{\text{sing}}$ and a polarization $\underline{q}$ on X, we define a polarization $\underline{q^S}$ $\big($resp. $\underline{\widehat{q^S}}\big)$ on the partial normalization X_S $\big($resp. the partial blowup $\widehat{X_S}\big)$ of X at S (see the notation in Section 2.1).

Lemma-Definition 2.5 *The formula*

$$\underline{q^S}_{Y_S} := \underline{q}_Y - \frac{\big|S_e^Y\big|}{2} - \big|S_i^Y\big|,$$

for any subcurve $Y_S \subset X_S$, where $S_e^Y := S \cap Y \cap Y^c$ and $S_i^Y := S \cap (Y \setminus Y^c)$, defines a polarization on X_S.

Proof. We have to show that $\underline{q^S}$ is additive, i.e., that for any two subcurves Y_S and Z_S of X_S without common components it holds $\underline{q^S}_{Y_S \cup Z_S} = \underline{q^S}_{Y_S} + \underline{q^S}_{Z_S}$. This follows from the additivity of $\underline{q}$ and the easily checked formulas:

$$\begin{cases} \big|S_i^{Y \cup Z}\big| = \big|S_i^Y\big| + \big|S_i^Z\big| + |S \cap Y \cap Z|, \\ \big|S_e^{Y \cup Z}\big| = \big|S_e^Y\big| + \big|S_e^Z\big| - 2|S \cap Y \cap Z|. \end{cases} \tag{2.6}$$

We conclude by observing that $\underline{q^S}_{X_S} = \underline{q}_X - |S| \in \mathbb{Z}$. □

The proof of the following Lemma-Definition is trivial.

Lemma-Definition 2.6 *The formula*

$$\underline{\widehat{q^S}}_Z = \begin{cases} 0 & \text{if } Z \subseteq E_S, \\ \underline{q}_{\pi_S(Z)} & \text{if } Z \not\subseteq E_S, \end{cases}$$

for any subcurve $Z \subset \widehat{X_S}$, defines a polarization on $\widehat{X_S}$.

In the special case of the total blowup $\widehat{X} = \widehat{X_{X_{\text{sing}}}}$, we set $\widehat{\underline{q}} := \underline{\widehat{q^{X_{\text{sing}}}}}$.

In the last part of the paper, we will need the concept of generic and non-degenerate polarizations. First, imitating [20, Definition 3.4], we give the following

Definition 2.7 A polarization $\underline{q}$ is called *integral* at a subcurve $Y \subset X$ if $\underline{q}_Z - \frac{\delta_Z}{2} \in \mathbb{Z}$ for any connected component Z of Y and of Y^c.

Using the above definition, we can give the following

Definition 2.8

(i) A polarization $\underline{q}$ is called *general* if it is not integral at any proper subcurve $Y \subsetneq X$.

(ii) A polarization $\underline{q}$ is called *non-degenerate* if it is not integral at any proper subcurve $Y \subsetneq X$ which is not a spine of X.

2.5 Semistable, torsion-free, rank 1 sheaves

Let X be a connected nodal curve of genus g. Let $\mathcal{I}$ be a coherent sheaf on X. We say that $\mathcal{I}$ is *torsion-free* (or *depth* 1 or *of pure dimension* or *admissible*) if its associated points are generic points of X. Clearly, a torsion-free sheaf $\mathcal{I}$ can be not free only at the nodes of X; we denote by $NF(\mathcal{I}) \subset X_{\rm sing}$ the subset of the nodes of X where $\mathcal{I}$ is not free (NF stands for not free). We say that $\mathcal{I}$ is of *rank* 1 if $\mathcal{I}$ is invertible on a dense open subset of X. We say that $\mathcal{I}$ is *simple* if $\operatorname{End}(\mathcal{I}) = k$. Each line bundle on X is torsion-free of rank 1 and simple.

For each subcurve Y of X, let $\mathcal{I}_Y$ be the restriction $\mathcal{I}_{|Y}$ of $\mathcal{I}$ to Y modulo torsion. If $\mathcal{I}$ is a torsion-free (resp. rank 1) sheaf on X, so is $\mathcal{I}_Y$ on Y. We let $\deg_Y(\mathcal{I})$ denote the degree of $\mathcal{I}_Y$, that is, $\deg_Y(\mathcal{I}) := \chi(\mathcal{I}_Y) - \chi(\mathcal{O}_Y)$.

It is a well-known result of Seshadri (see [39]) that torsion-free, rank 1 sheaves on X can be described either via line bundles on partial normalizations of X or via certain line bundles on partial blowups of X. The precise statement is the following

Proposition 2.9

(i) *For any $S \subset X_{\rm sing}$, the commutative diagram* (2.1) *induces a commutative diagram*

$$\begin{tikzcd}
\operatorname{Pic}(X_S) \arrow[dr, "(\nu_S)_*"', "\cong"] & & \operatorname{Pic}\left(\widehat{X_S}\right)_{\rm prim} \arrow[ll, two heads, "i_S^*"'] \arrow[dl, "(\pi_S)_*"] \\
& \operatorname{Tors}_S(X) &
\end{tikzcd} \tag{2.7}$$

where $\operatorname{Pic}\left(\widehat{X_S}\right)_{\rm prim}$ denotes the line bundles on $\widehat{X_S}$ that have degree -1 on each exceptional component of the morphism π_S and $\operatorname{Tors}_S(X)$ denotes the set of torsion-free, rank 1 sheaves $\mathcal{I}$ on X such that $\operatorname{NF}(\mathcal{I}) = \mathrm{S}$. Moreover we have that

(a) *The maps i_S^* and $(\pi_S)_*$ are surjective*;

(b) *The map $(\nu_S)_*$ is bijective with inverse given by sending a sheaf $\mathcal{I} \in \operatorname{Tors}_S(X)$ to the line bundle on X_S obtained as the quotient of $(\nu_S)^*(\mathcal{I})$ by its torsion subsheaf.*

(ii) *The above diagram* (2.7) *is equivariant with respect to the natural actions of the generalized Jacobians of X_S, $\widehat{X_S}$ and X and the natural morphisms*:

$$\begin{tikzcd}
J(X_S) & & J\left(\widehat{X_S}\right) \arrow[ll, two heads, "i_S^*"'] \\
& J(X) \arrow[ul, two heads, "\nu_S^*"] \arrow[ur, "\pi_S^*"', "\cong"] &
\end{tikzcd} \tag{2.8}$$

Explicitly, for any $L \in \operatorname{Pic}\left(\widehat{X_S}\right)_{\rm prim}$, $M \in \operatorname{Pic}(X_S)$, $\alpha \in J(X)$ and $\beta \in J\left(\widehat{X_S}\right)$, we have that

$$\begin{cases} i_S^*(\beta \otimes L) = i_S^*(\beta) \otimes i_S^*(L), \\ (\pi_S)_*\left(\pi_S^*\alpha \otimes L\right) = \alpha \otimes (\pi_S)_*(L), \\ (\nu_S)_*\left(\nu_S^*\alpha \otimes M\right) = \alpha \otimes (\nu_S)_*(M). \end{cases} \tag{2.9}$$

In particular, the action of $J(X)$ on $\operatorname{Tors}_S(X)$ factors through the map $\nu_S^* : J(X) \twoheadrightarrow J(X_S)$.

(iii) *For any subcurve $Y \subset X$ and any $M \in \mathrm{Pic}(X_S)$, it holds*

$$\deg_Y(\nu_S)_*(M) = \deg_{Y_S} M + \left|S_i^Y\right|,$$

where $S_i^Y := S \cap (Y \setminus Y^c)$ (as in Lemma-Definition 2.5*).*

Proof. Part (i) is a reformulation of [1, Lemma 1.5(i) and Lemma 1.9]).

Part (ii) follows from the multiplicativity of pull-back map i_S^* and the projection formula applied to the morphisms ν_S and π_S.

Part (iii): First of all observe that the restriction $((\nu_S)_*M)_{|Y}$ is equal to the pushforward via $(\nu_S)_{|Y_S} : Y_S \to Y$ of the restriction $M_{|Y_S} = M_{Y_S}$. Since $(\nu_S)_{|Y_S}$ is a finite map, we get the equality $\chi\big(((\nu_S)_*M)_{|Y}\big) = \chi(M_{Y_S})$ which, combined with Riemann-Roch, gives that

$$\deg((\nu_S)_*M)_{|Y} + 1 - g(Y) = \chi\big(((\nu_S)_*M)_{|Y}\big) = \chi\big(M_{Y_S}\big) = \deg_{Y_S} M + 1 - g(Y_S). \tag{*}$$

Since Y_S is the normalization of Y at $S \cap Y$, we have that $g(Y_S) = g(Y) - |S \cap Y|$ which, combined with (*), gives that

$$\deg((\nu_S)_*M)_{|Y} = \deg_{Y_S} M + |S \cap Y|. \tag{**}$$

Clearly, the torsion subsheaf of $((\nu_S)_*M)_{|Y}$ is equal to $\bigoplus_{n \in S \cap Y \cap Y^c} \underline{k}_n$, where $\underline{k}_n$ is the skyscraper sheaf supported on n and with stalk equal to the base field k. Therefore

$$\deg_Y((\nu_S)_*M) = \deg((\nu_S)_*M)_Y = \deg((\nu_S)_*M)_{|Y} - |S \cap Y \cap Y^c|. \tag{***}$$

We conclude by putting together (**) and (***). □

Later, we will need the concepts of semistability, P-quasistability and stability of a torsion-free, rank 1 sheaf on X with respect to a polarization on X and to a smooth point $P \in X_{\mathrm{sm}}$. Here are the relevant definitions.

Definition 2.10 Let $\underline{q}$ be a polarization on X and let $P \in X_{\mathrm{sm}}$ be a smooth point of X. Let $\mathcal{I}$ be a torsion-free, rank-1 sheaf on X of degree $d = |\underline{q}|$.

(i) We say that $\mathcal{I}$ is *semistable* with respect to $\underline{q}$ $\big($or $\underline{q}$-semistable$\big)$ if for every proper subcurve Y of X, we have that

$$\deg_Y(\mathcal{I}) \geq \underline{q}_Y - \frac{\delta_Y}{2} \tag{2.10}$$

(ii) We say that $\mathcal{I}$ is *P-quasistable* with respect to $\underline{q}$ $\big($or $\underline{q}$-P-quasistable$\big)$ if it is semistable with respect to $\underline{q}$ and if the inequality (2.10) above is strict when $P \in Y$.

(iii) We say that $\mathcal{I}$ is *stable* with respect to $\underline{q}$ $\big($or $\underline{q}$-stable$\big)$ if it is semistable with respect to $\underline{q}$ and if the inequality (2.10) is always strict.

In what follows we compare our notation with the other notations used in the literature.

Remark 2.11

(i) Given a vector bundle E on X, we define the polarization $\underline{q}^E$ on X by setting

$$\underline{q}^E_Y = -\frac{\deg\left(E_{|Y}\right)}{\mathrm{rk}(E)} + \frac{\deg_Y(\omega_X)}{2},$$

for each subcurve Y (or equivalently for each irreducible component C_i) of X. Then it is easily checked that the above notions of semistability (resp. P-quasistability, resp. stability) with respect to $\underline{q}^E$ agree with the notions of semistability (resp. P-quasistability, resp. stability) with respect to E in the sense of [19, Section 1.2]. Note that, for any subcurve $Y \subset X$ such that $\underline{q}_Y - \frac{\delta_Y}{2} \in \mathbb{Z}$, we have that $\left(\underline{q}^E\right)_{|Y} = \underline{q}^{E_{|Y}}$.

(ii) In the particular case where

$$\underline{q}_Y = d \cdot \frac{\deg_Y(\omega_X)}{2g-2}, \tag{2.11}$$

for a certain integer $d \in \mathbb{Z}$, the inequality (2.10) reduces to the well-known basic inequality of Gieseker-Caporaso (see [9]). In this case, $\underline{q}$ will be called the *canonical polarization of degree d*.

Given a sheaf $\mathcal{I}$ semistable with respect to a polarization $\underline{q}$, there are connected subcurves $Y_1, \ldots, Y_q$ covering X and a filtration

$$0 = \mathcal{I}_0 \subsetneqq \mathcal{I}_1 \subsetneqq \ldots, \subsetneqq \mathcal{I}_{q-1} \subsetneqq \mathcal{I}_q = \mathcal{I}$$

such that the quotient $\mathcal{I}_j/\mathcal{I}_{j+1}$ is a stable sheaf on Y_j with respect to $\underline{q}_{|Y_j}$ for each $j = 1, \ldots, q$. The above filtration is called a *Jordan-Hölder filtration*. The sheaf $\mathcal{I}$ may have many Jordan-Hölder filtrations but the collection of subcurves $\mathfrak{S}(\mathcal{I}) := \{Y_1, \ldots, Y_q\}$ and the isomorphism class of the sheaf

$$\mathrm{Gr}(\mathcal{I}) := \mathcal{I}_1/\mathcal{I}_0 \oplus \mathcal{I}_2/\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_q/\mathcal{I}_{q-1}$$

depend only on $\mathcal{I}$, by the Jordan-Hölder theorem. Notice that $\mathrm{Gr}(\mathcal{I})$ is also $\underline{q}$-semistable and that

$$\mathrm{Gr}(\mathcal{I}) \cong \bigoplus_{Z \in \mathfrak{S}(\mathcal{I})} \mathrm{Gr}(\mathcal{I})_Z .$$

A $\underline{q}$-semistable sheaf $\mathcal{I}$ is called *polystable* if $\mathcal{I} \cong \mathrm{Gr}(\mathcal{I})$.

We say that two $\underline{q}$-semistable sheaves $\mathcal{I}$ and $\mathcal{I}'$ on X are *S-equivalent* if $\mathfrak{S}(\mathcal{I}) = \mathfrak{S}(\mathcal{I}')$ and $\mathrm{Gr}(\mathcal{I}) \cong \mathrm{Gr}(\mathcal{I}')$. Note that in each S-equivalence class of $\underline{q}$-semistable sheaves, there is exactly one $\underline{q}$-polystable sheaf.

2.6 Fine and coarse compactified Jacobians

For any smooth point $P \in X$ and polarization $\underline{q}$ on X, there is a k-projective variety $J_X^P(\underline{q})$, which we call *fine compactified Jacobian*, parametrizing $\underline{q}$-P-quasistable sheaves on the curve X (see [19, Theorem A, p. 3047] and [20, Theorem 2.4]). More precisely, $J_X^P(\underline{q})$ represents the functor that associates to each scheme T the set of T-flat coherent sheaves $\mathcal{I}$ on $X \times T$ such that $\mathcal{I}|_{X \times t}$ is $\underline{q}$-P-quasistable for each $t \in T$, modulo the following equivalence relation $\sim$. We say that two such sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent, and denote $\mathcal{I}_1 \sim \mathcal{I}_2$, if there is an invertible sheaf $\mathcal{N}$ on T such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p_2^*\mathcal{N}$, where $p_2 : X \times T \to T$ is the projection map.

There are other two varieties closely related to $J_X^P(\underline{q})$ (see [19, Section 4]): the variety $J_X^s(\underline{q})$ parametrizing $\underline{q}$-stable sheaves and the variety $J_X^{ss}(\underline{q})$ parametrizing $\underline{q}$-semistable simple sheaves. We have open inclusions

$$J_X^s(\underline{q}) \subset J_X^P(\underline{q}) \subset J_X^{\mathrm{ss}}(\underline{q}),$$

where the last inclusion follows from the fact that a $\underline{q}$-P-quasistable sheaf is simple, as it follows easily from [19, Proposition 1]. It turns out that $J_X^s(\underline{q})$ is separated but, in general, not universally closed, while $J_X^{ss}(\underline{q})$ is universally closed but, in general, not separated (see [19, Theorem A]).

According to [39, Theorem 15, p. 155], there exists a projective variety $U_X(\underline{q})$, which we call *coarse compactified Jacobian*, coarsely representing the functor $\mathbf{U}$ that associates to each scheme T the set of T-flat coherent sheaves $\mathcal{I}$ on $X \times T$ such that $\mathcal{I}_{X \times t}$ is $\underline{q}$-semistable for each $t \in T$. More precisely, there is a map $\mathbf{U} \to U_X(\underline{q})$ such that, for any other k-scheme Z, each map $\mathbf{U} \to Z$ is induced by composition with a unique map $U_X(\underline{q}) \to Z$. Moreover, the k-points on $U_X(\underline{q})$ are in one-to-one correspondence with the S-equivalence classes of $\underline{q}$-semistable sheaves on X, or equivalently with $\underline{q}$-polystable sheaves on X since in each S-equivalence class of $\underline{q}$-semistable sheaves there exists exactly one $\underline{q}$-polystable sheaf. By convention, when we write $\mathcal{I} \in U_X(\underline{q})$, we implicitly assume that $\mathcal{I}$ is polystable. We denote by

$$U_X^s(\underline{q}) \subset U_X(\underline{q})$$

the open subset parametrizing $\underline{q}$-stable sheaves.

Since $J_X^P\left(\underline{q}\right)$ represents a functor, there exists a universal $\underline{q}$-P-quasistable sheaf on $X \times J_X^P\left(\underline{q}\right)$ $\big($uniquely determined up to tensoring with the pull-back of a line bundle on $J_X^P\left(\underline{q}\right)\big)$, and hence a well-defined induced map

$$\Phi : J_X^P\left(\underline{q}\right) \longrightarrow U_X\left(\underline{q}\right). \tag{2.12}$$

This map is surjective (by [19, Theorem 7]) and its fibers parametrize S-equivalence classes of $\underline{q}$-P-quasistable sheaves (see also [20, p. 178]). The map Φ fits in the following diagram

$$\begin{tikzcd}
J_X^s\left(\underline{q}\right) \arrow[r, hook] \arrow[d, "\Phi^s", "\cong"'] & J_X^P\left(\underline{q}\right) \arrow[r, hook] \arrow[d, "\Phi"] & J_X^{ss}\left(\underline{q}\right) \arrow[dl, "\Phi^{ss}"] \\
U_X^s\left(\underline{q}\right) \arrow[r, hook] & U_X\left(\underline{q}\right) &
\end{tikzcd} \tag{2.13}$$

To compare our notations with the others used in the literature, we observe the following

Remark 2.12

(i) Given a vector bundle E on X and a smooth point $P \in X_{\rm sm}$, the variety $J_X^P\left(\underline{q}^E\right)$ coincides with the variety J_E^P in Esteves's notation (see [19]). Similarly, the variety $J_X^s\left(\underline{q}^E\right)$ $\big($resp. $J_X^{ss}\left(\underline{q}^E\right)\big)$ coincides with J_E^s $\left(\text{resp. } J_E^{ss}\right)$ in Esteves's notation.

(ii) Let ϕ be an element of $\partial C_1(\Gamma_X, \mathbb{Q}) \subset C_0(\Gamma_X, \mathbb{Q})$ (see [1, Section 1]), i.e., a collection of rational numbers $\{\phi_v\}$ for any vertex v of Γ_X such that $\sum_{v \in V(\Gamma_X)} \phi_v = 0$. We can associate to ϕ a polarization $\underline{\phi}$ such that $|\underline{\phi}| = 0$ by putting

$$\underline{\phi}_{C_v} = \phi_v \tag{2.14}$$

if C_v is the irreducible component of X corresponding to the vertex v of Γ_X. Then the Oda-Seshadri's compactified Jacobian $\mathrm{Jac}(X)_\phi$ is isomorphic to $U_X\left(\underline{\phi}\right)$ (see [35] and [1]).
Conversely, given a polarization $\underline{q}$, consider a polarization $\underline{d}$ such that $\left|\underline{q}\right| = |\underline{d}|$ and such that $\underline{d}$ is integral, i.e., $\underline{d}_Y \in \mathbb{Z}$ for any subcurve $Y \subseteq X$. Define a new polarization $\underline{\phi}$ by $\underline{\phi}_Y := \underline{q}_Y - \underline{d}_Y$ for any subcurve $Y \subseteq X$. In particular, we have that $|\underline{\phi}| = 0$. Define an element $\phi \in \partial C_1(\Gamma_X, \mathbb{Q}) \subset C_0(\Gamma_X, \mathbb{Q})$ by the equation (2.14). Then the variety $U_X\left(\underline{q}\right)$ is isomorphic to $\mathrm{Jac}(X)_\phi$. Note that this is independent of the choice of the auxiliary integral polarization $\underline{d}$ because we have an isomorphism $\mathrm{Jac}(X)_\phi \cong \mathrm{Jac}(X)_{\phi+\psi}$ for any $\psi \in \partial C_1(\Gamma_X, \mathbb{Z}) \subset C_0(\Gamma_X, \mathbb{Z})$.

(iii) Given a pair $(\mathfrak{a}, \chi)$, where $\chi \in \mathbb{Z}$ and $\mathfrak{a} = \{\mathfrak{a}_{C_i}\}$ is a polarization such that $|\mathfrak{a}| = 1$, consider the polarization $\underline{q}$ defined by

$$\underline{q}_Y = \mathfrak{a}_Y \chi + \frac{\deg_Y(\omega_X)}{2},$$

for every subcurve $Y \subset X$. Then the variety $U_X\left(\underline{q}\right)$ coincides with the variety $U_X(\mathfrak{a}, \chi)$ in Seshadri's notation (see [39]).

(iv) Given an ample line bundle L on X and an integer $d \in \mathbb{Z}$, consider the polarization $\underline{q}$ defined by

$$\underline{q}_Y = \frac{\deg_Y(\omega_X)}{2} + \frac{\deg_Y(L)}{\deg(L)}(d - g + 1),$$

for every subcurve $Y \subseteq X$. Then the Simpson's moduli space (see [40]) $\mathrm{Jac}(X)_{d,L}$ of S-equivalence classes of torsion-free, rank one sheaves of degree d that are slope-semistable with respect to L is isomorphic to $U_X\left(\underline{q}\right)$ (see [1]). However, note that, contrary to what asserted in [1, Section 2.1], it is not true that every $U_X\left(\underline{q}\right)$ with $\left|\underline{q}\right| = d$ is isomorphic to $\mathrm{Jac}(X)_{d,L}$ for some ample line bundle L on X. For instance, if $d = g - 1$ then it follows easily from the above equation that all the Simpson's compactified Jacobians $\mathrm{Jac}(X)_{g-1,L}$ are isomorphic among them regardless of the chosen L (as observed also in [1, Lemma 3.1]), while there many compactified Jacobians of the form $U_X\left(\underline{q}\right)$ with $\left|\underline{q}\right| = g - 1$, just as in every other degree d!

(v) In the particular case where $\underline{q}$ is the canonical polarization of degree d (see Remark 2.11(ii)), the variety $U_X\left(\underline{q}\right)$ coincides with the variety $\overline{P_X^d}$ in Caporaso's notation (see [9]) and it will be called the *coarse canonical* degree d compactified Jacobian of X. Moreover, we set $J_X^{d,P} := J_X^P(\underline{q})$ and call it the *fine canonical* degree d compactified Jacobian of X with respect to P. This notation agrees with the one introduced in [14, Section 2.4]. In particular, we have a surjective map $J_X^{d,P} \twoheadrightarrow \overline{P_X^d}$.

In what follows, we will need the following well-known results concerning the smooth loci of $J_X^P\left(\underline{q}\right)$ $\big($or $J_X^s\left(\underline{q}\right)$ or $J_X^{ss}\left(\underline{q}\right)\big)$ and $U_X\left(\underline{q}\right)$:

Fact 2.13

(i) *The variety* $J_X^P\left(\underline{q}\right)$ $\big($*resp.* $J_X^s\left(\underline{q}\right)$, *resp.* $J_X^{ss}\left(\underline{q}\right)\big)$ *is smooth at* $\mathcal{I}$ *if and only if* $\mathcal{I}$ *is a line bundle on* X.

(ii) *The variety* $U_X\left(\underline{q}\right)$ *is smooth at a polystable sheaf* $\mathcal{I}$ *if and only if* $\mathcal{I}$ *is locally free at all non-separating nodes of* X.

For the proof of part (i), observe that, since $J_X^P(\underline{q})$ is a fine compactified Jacobian, the completion of the local ring of $J_X^P(\underline{q})$ at $\mathcal{I}$ is isomorphic to the miniversal deformation ring of $\mathcal{I}$. The same thing is true for $(\underline{q})$, resp. $J_X^{ss}(\underline{q})$. The result then follows from [16, Lemma 3.14]. Part (ii) follows from [16, Theorem B(ii)].

Now fix a one-parameter regular local smoothing $f : \mathcal{X} \to B = \operatorname{Spec}(R)$ of X (see Section 2.3).

It follows from [25] that there exists a B-scheme $U_f\left(\underline{q}\right)$ whose special fiber is isomorphic to $U_X\left(\underline{q}\right)$ and whose general fiber is isomorphic to $\operatorname{Pic}^{|\underline{q}|}(\mathcal{X}_K)$. Denote by $U_f^s\left(\underline{q}\right)$ the open subset of $U_f\left(\underline{q}\right)$ whose special fiber is isomorphic to $U_X^s\left(\underline{q}\right) \subset U_X\left(\underline{q}\right)$ and whose general fiber is isomorphic to $\operatorname{Pic}^{|\underline{q}|}(\mathcal{X}_K)$.

Note that, since R is assumed to be Henselian, for any $P \in X_{\text{sm}}$ there exists a section $\sigma : B \to \mathcal{X}$ of f such that $\sigma(\operatorname{Spec} k) = P$ (see e.g., [5, Proposition 14]). Conversely, every section σ of f is such that $\sigma(\operatorname{Spec} k)$ is a smooth point of $\mathcal{X}_k = X$ (see e.g., [31, Chapter 9, Corollary 1.32]). Fix now a section σ of f and let $P := \sigma(\operatorname{Spec} k) \in X_{\text{sm}}$. Then, according to [19, Theorems A and B], there exist B-schemes $J_f^s\left(\underline{q}\right)$, $J_f^\sigma\left(\underline{q}\right)$ and $J_f^{ss}\left(\underline{q}\right)$ together with open inclusions

$$J_f^s\left(\underline{q}\right) \subset J_f^\sigma\left(\underline{q}\right) \subset J_f^{ss}\left(\underline{q}\right),$$

such that the general fibers over B of the above schemes is $\operatorname{Pic}^{|\underline{q}|}(\mathcal{X}_K)$ while the special fibers are isomorphic to, respectively, $J_X^s\left(\underline{q}\right)$, $J_X^P\left(\underline{q}\right)$ and $J_X^{ss}\left(\underline{q}\right)$. The above diagram (2.13) becomes the special fiber of the following diagram of B-schemes

$$\begin{tikzcd}
J_f^s\left(\underline{q}\right) \arrow[r, hook] \arrow[d, "\Phi_f^s", "\cong"'] & J_f^\sigma\left(\underline{q}\right) \arrow[r, hook] \arrow[d, "\Phi_f"] & J_f^{ss}\left(\underline{q}\right) \arrow[dl, "\Phi_f^{ss}"] \\
U_f^s\left(\underline{q}\right) \arrow[r, hook] & U_f\left(\underline{q}\right) &
\end{tikzcd} \tag{2.15}$$

3 Graph-theoretic results

3.1 Notations

Let Γ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We allow loops or multiple edges, although, in what follows, loops will play no role, i.e., we could consider the graph $\widetilde{\Gamma}$ obtained from Γ by removing all the loops and obtain exactly the same answers we get for Γ.

We will be interested in two kinds of *subgraphs* of Γ:

- Given a subset $T \subset E(\Gamma)$, we denote by $\Gamma\backslash T$ the subgraph of Γ obtained from Γ by deleting the edges belonging to T. Thus we have that $V(\Gamma\backslash T) = V(\Gamma)$ and $E(\Gamma\backslash T) = E(\Gamma) \setminus T$. The subgraphs of the form $\Gamma\backslash T$ are called complete subgraphs.
- Given a subset $W \subset V(\Gamma)$, we denote by $\Gamma[W]$ the subgraph whose vertex set is W and whose edges are all the edges of Γ that join two vertices in W. The subgraphs of the form $\Gamma[W]$ are called induced subgraphs and we say that $\Gamma[W]$ is induced from W.

If W_1 and W_2 are two disjoint subsets of $V(\Gamma)$, then we set $\mathrm{val}(W_1, W_2) := |E(\Gamma[W_1], \Gamma[W_2])|$, where $E(\Gamma[W_1], \Gamma[W_2])$ is the subset of $E(\Gamma)$ consisting of all the edges of Γ that join some vertex of W_1 with some vertex of W_2. We call $\mathrm{val}(W_1, W_2)$ the *valence* of the pair (W_1, W_2). For a subset $W \subset V(\Gamma)$, we denote by $W^c := V(\Gamma) \setminus W$ its complementary subset. We set $\mathrm{val}(W) = \mathrm{val}(W^c) := \mathrm{val}(W, W^c)$ and call it the valence of W. In particular $\mathrm{val}(\emptyset) = \mathrm{val}(V(\Gamma)) = 0$. Note that for $w \in V(\Gamma)$, the valence $\mathrm{val}(w)$ is the number of edges joining w with a vertex of Γ different from w, i.e., loops are not taken into account in our definition of valence.

Given a subset $S \subseteq E(\Gamma)$, we define the valence of the pair (W_1, W_2) of disjoint subsets $W_1, W_2 \subset V(\Gamma)$ with respect to S to be $\mathrm{val}_S(W_1, W_2) := |S \cap E(\Gamma[W_1], \Gamma[W_2])|$. Obviously, we always have that $\mathrm{val}_S(W_1, W_2) \leq \mathrm{val}(W_1, W_2)$ with equality if $S = E(\Gamma)$.

Note that the valence is additive: if W_1, W_2, W_3 are pairwise disjoint subsets of $V(\Gamma)$, we have that

$$\mathrm{val}(W_1 \cup W_2, W_3) = \mathrm{val}(W_1, W_3) + \mathrm{val}(W_2, W_3). \tag{3.1}$$

A similar property holds for val_S.

3.2 0-cochains

Given an abelian group A (usually $A = \mathbb{Z}, \mathbb{Q}$), we define the space $C^0(\Gamma, A)$ of 0-cochains with values in A as the free A-module $A^{V(\Gamma)}$ of functions from $V(\Gamma)$ to A. If $\underline{d} \in C^0(\Gamma, A)$, we set

$$\begin{cases} \underline{d}_v := \underline{d}(v) \in A & \text{for any} \quad v \in V(\Gamma), \\ \underline{d}_W := \sum\limits_{w \in W} \underline{d}_w \in A & \text{for any} \quad W \subseteq V(\Gamma), \\ |\underline{d}| := \underline{d}_{V(\Gamma)} \in A. \end{cases}$$

For any element $a \in A$, we set

$$C^0(\Gamma, A)_a := \left\{ \underline{d} \in C^0(\Gamma, A) \ : \ |\underline{d}| = a \right\} \subseteq C^0(\Gamma, A).$$

Given a subset $W \subset V(\Gamma)$, we will denote by $\underline{\chi(W)} \in C^0(\Gamma, \mathbb{Z})$ the characteristic function of W, i.e., the element of $C^0(\Gamma, \mathbb{Z})$ uniquely defined by

$$\underline{\chi(W)}_v = \begin{cases} 1 & \text{if} \quad v \in W, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The space of 0-cochains with values in A is endowed with an endomorphism, called Laplacian and denoted by Δ_0 (see for example [4, p. 169]), defined as

$$\Delta_0(\underline{d})_v := -\underline{d}_v \,\mathrm{val}(v) + \sum_{w \neq v} \underline{d}_w \,\mathrm{val}(v, w). \tag{3.3}$$

It is easy to check that $\mathrm{Im}(\Delta_0) \subset C^0(\Gamma, A)_0$. In the case where $A = \mathbb{Z}$ and Γ is connected, the kernel $\ker(\Delta_0)$ consists of the constant 0-cochains and therefore the quotient

$$\mathrm{Pic}(\Gamma) := \frac{C^0(\Gamma, \mathbb{Z})_0}{\mathrm{Im}(\Delta_0)}$$

is a finite group, called the Jacobian group (see [4]).

For any $d \in \mathbb{Z}$, the set $C^0(\Gamma, \mathbb{Z})_d$ is clearly a torsor for the group $C^0(\Gamma, \mathbb{Z})_0$. Therefore, the subgroup $\mathrm{Im}(\Delta_0)$ acts on the sets $C^0(\Gamma, \mathbb{Z})_d$ and

$$|\mathrm{Pic}(\Gamma)| = \left| \frac{C^0(\Gamma, \mathbb{Z})_d}{\mathrm{Im}(\Delta_0)} \right|. \tag{3.4}$$

Remark 3.1 Let X be a connected nodal curve and consider the dual graph of X, Γ_X. Then Γ_X is connected and it is easy to check that $\mathrm{Pic}(\Gamma_X) \cong \Delta_X$ (see Section 2.2). Moreover, for any $d \in \mathbb{Z}$, there is a bijection $\frac{C^0(\Gamma_X,\mathbb{Z})_d}{\mathrm{Im}(\Delta_0)} \leftrightarrow \Delta_X^d$. In particular, we have that:

$$c(\Gamma_X) = \left|\frac{C^0(\Gamma_X,\mathbb{Z})_d}{\mathrm{Im}(\Delta_0)}\right|. \tag{3.5}$$

For later use, we record the following formula (for any $W, V \subseteq V(\Gamma)$):

$$\begin{aligned}
\Delta_0(\underline{\chi(V)})_W &= \sum_{w\in W}\left[-\underline{\chi(V)}_w \,\mathrm{val}(w) + \sum_{v\neq w}\underline{\chi(V)}_v \,\mathrm{val}(v,w)\right]\\
&= \sum_{w\in V\cap W}\left[-\mathrm{val}(w) + \sum_{w\neq v\in V}\mathrm{val}(v,w)\right] + \sum_{w\in W\setminus V}\sum_{v\in V}\mathrm{val}(v,w)\\
&= \sum_{w\in V\cap W}\left[-\sum_{v\in V^c}\mathrm{val}(v,w)\right] + \sum_{w\in W\setminus V}\sum_{v\in V}\mathrm{val}(v,w) \qquad (3.6)\\
&= -\mathrm{val}(V\cap W, V^c) + \mathrm{val}(W\setminus V, V)\\
&= -\mathrm{val}(V\cap W, W\setminus V) - \mathrm{val}(V\cap W, (V\cup W)^c)\\
&\quad +\mathrm{val}(W\setminus V, V\cap W) + \mathrm{val}(W\setminus V, V\setminus W)\\
&= -\mathrm{val}(V\cap W, (V\cup W)^c) + \mathrm{val}(W\setminus V, V\setminus W).
\end{aligned}$$

3.3 Quasistable 0-cochains $B^{v_0}_{\Gamma\setminus S}(\underline{q})$

Throughout this subsection, we fix the following data:

(1) A finite graph Γ;
(2) $v_0 \in V(\Gamma)$;
(3) $S \subset E(\Gamma)$;
(4) $\underline{q} \in C^0(\Gamma,\mathbb{Q})$ such that $q := |\underline{q}| \in \mathbb{Z}$.

Since we will be using two different graphs throughout this section, Γ and $\Gamma\setminus S$, we will adopt the following convention on the notation used. Given two disjoint subsets $W_1, W_2 \subseteq V(\Gamma) = V(\Gamma\setminus S)$, we will be considering three different notions of valence, namely:

$$\begin{cases}
\mathrm{val}(W_1,W_2) := |E(\Gamma[W_1],\Gamma[W_2])|,\\
\mathrm{val}_S(W_1,W_2) := |S\cap E(\Gamma[W_1],\Gamma[W_2])|,\\
\mathrm{val}_{\Gamma\setminus S}(W_1,W_2) := |E((\Gamma\setminus S)[W_1],(\Gamma\setminus S)[W_2])|.
\end{cases}$$

Note that $\mathrm{val}(W_1,W_2) = \mathrm{val}_S(W_1,W_2) + \mathrm{val}_{\Gamma\setminus S}(W_1,W_2)$. As usual, we set $\mathrm{val}(W) := \mathrm{val}(W,W^c)$ and similarly for val_S and $\mathrm{val}_{\Gamma\setminus S}$.

We now introduce the main characters of this subsection.

Definition 3.2

(i) A 0-cochain $\underline{d} \in C^0(\Gamma,\mathbb{Z})$ is said to be semistable on $\Gamma\setminus S$ with respect to $\underline{q}$ if the following two conditions are satisfied:
   (a) $|\underline{d}| = q - |S|$;
   (b) $\underline{d}_W + |S\cap E(\Gamma[W])| \geq \underline{q}_W - \frac{\mathrm{val}(W)}{2}$ for any proper subset $W \subset V(\Gamma)$.
   We denote the set of all such 0-cochains by $B_{\Gamma\setminus S}(\underline{q})$.

(ii) A 0-cochain $\underline{d} \in C^0(\Gamma,\mathbb{Z})$ is said to be v_0-*quasistable* on $\Gamma\setminus S$ with respect to $\underline{q}$ if $\underline{d} \in B_{\Gamma\setminus S}(\underline{q})$ and the inequality in (ib) above is strict when $v_0 \in W$. We denote the set of all such 0-cochains by $B^{v_0}_{\Gamma\setminus S}(\underline{q})$.

Remark 3.3 Let $\underline{d} \in B_{\Gamma\setminus S}(\underline{q})$ and let W be a proper subset of $V(\Gamma)$. By applying the condition (ib) of Definition 3.2 to $W^c \subset V(\Gamma)$ and using (ia), we get that

$$\underline{q}_W - \frac{\mathrm{val}(W)}{2} + \mathrm{val}_{\Gamma\setminus S}(W) = \underline{q}_W + \frac{\mathrm{val}(W)}{2} - \mathrm{val}_S(W) \geq \underline{d}_W + |S \cap E(\Gamma[W])|.$$

If moreover $\underline{d} \in B^{v_0}_{\Gamma\setminus S}(\underline{q})$ then the above inequality is strict if $v_0 \notin W$.

We want to determine the cardinality of the set $B^{v_0}_{\Gamma\setminus S}(\underline{q})$. We begin with the following necessary condition in order that $B^{v_0}_{\Gamma\setminus S}(\underline{q})$ is not empty. Later (see Corollary 3.7), we will see that it is also a sufficient condition.

Lemma 3.4 *If $B^{v_0}_{\Gamma\setminus S}(\underline{q}) \neq \emptyset$ then $\Gamma\setminus S$ is connected.*

Proof. By contradiction, assume that $\Gamma\setminus S$ is not connected and $B^{v_0}_{\Gamma\setminus S}(\underline{q}) \neq \emptyset$. This means that there exist $\underline{d} \in B^{v_0}_{\Gamma\setminus S}(\underline{q})$ and a proper subset $W \subset V(\Gamma)$ such that $\mathrm{val}_{\Gamma\setminus S}(W) = 0$. By the Definition 3.2 and Remark 3.3, we get that

$$\underline{q}_W - \frac{\mathrm{val}(W)}{2} \leq \underline{d}_W + |S \cap E(\Gamma(W))| \leq \underline{q}_W - \frac{\mathrm{val}(W)}{2} + \mathrm{val}_{\Gamma\setminus S}(W) = \underline{q}_W - \frac{\mathrm{val}(W)}{2}.$$

This contradicts the fact that one of the above two inequalities must be strict, according to whether $v_0 \in W$ or $v_0 \in W^c$. ☐

In what follows, we are going to consider the 0-cochains $C^0(\Gamma\setminus S, \mathbb{Z})$ endowed with the Laplacian operator Δ_0 as in (3.3) with respect to $\Gamma\setminus S$. Note that, although $C^0(\Gamma\setminus S, \mathbb{Z}) = C^0(\Gamma, \mathbb{Z})$ is independent of the chosen $S \subset E(\Gamma)$, the Laplacian Δ_0 depends on S.

Proposition 3.5 *If $\Gamma\setminus S$ is connected, then the composed map*

$$\pi : B^{v_0}_{\Gamma\setminus S}(\underline{q}) \subseteq C^0(\Gamma\setminus S, \mathbb{Z})_{q-|S|} \twoheadrightarrow \frac{C^0(\Gamma\setminus S, \mathbb{Z})_{q-|S|}}{\mathrm{Im}(\Delta_0)}$$

is bijective.

Proof. Consider the auxiliary map

$$\overline{\pi} : B_{\Gamma\setminus S}(\underline{q}) \subseteq C^0(\Gamma\setminus S, \mathbb{Z})_{q-|S|} \twoheadrightarrow \frac{C^0(\Gamma\setminus S, \mathbb{Z})_{q-|S|}}{\mathrm{Im}(\Delta_0)}.$$

Clearly we have that $\pi = \overline{\pi}_{|B^{v_0}_{\Gamma\setminus S}(\underline{q})}$. We divide the proof in three steps.

STEP I: π is injective.

By contradiction, assume that there exist $\underline{d} \neq \underline{e} \in B^{v_0}_{\Gamma\setminus S}(\underline{q})$ such that $\pi(\underline{d}) = \pi(\underline{e})$. This is equivalent to the existence of an element $\underline{t} \in C^0(\Gamma\setminus S, \mathbb{Z})$ such that $\Delta_0(\underline{t}) = \underline{d} - \underline{e}$. Since $\underline{d}, \underline{e} \in B^{v_0}_{\Gamma\setminus S}(\underline{q})$, by Definition 3.2 and Remark 3.3, we get that for any proper subset $W \subset V(\Gamma)$:

$$\begin{aligned} \underline{d}_W - \underline{e}_W &< \left(\underline{q}_W + \frac{\mathrm{val}(W)}{2} - \mathrm{val}_S(W)\right) - \left(\underline{q}_W - \frac{\mathrm{val}(W)}{2}\right) \\ &= \mathrm{val}(W) - \mathrm{val}_S(W) = \mathrm{val}_{\Gamma\setminus S}(W), \end{aligned} \tag{3.7}$$

where the inequality is strict since either $v_0 \in W$ or $v_0 \in W^c$.

Consider now the (non-empty) subset

$$V_0 := \left\{v \in V(\Gamma) = V(\Gamma\setminus S) \;:\; \underline{t}_v = \min_{w \in V(\Gamma)} \underline{t}_w := l\right\} \subseteq V(\Gamma) = V(\Gamma\setminus S).$$

If $V_0 = V(\Gamma\setminus S)$ then $\underline{t}$ is a constant 0-cochain in $\Gamma\setminus S$, and therefore $0 = \Delta_0(\underline{t}) = \underline{d} - \underline{e}$, which contradicts the hypothesis that $\underline{d} \neq \underline{e}$. Therefore V_0 is a proper subset of $V(\Gamma\setminus S)$.

From the definition (3.3), using the additivity of $\mathrm{val}_{\Gamma\setminus S}$ and the fact that $\underline{t}_v \geq l$ for any $v \in V(\Gamma\setminus S)$ with equality if $v \in V_0$, we get

$$\begin{aligned}
\Delta_0(\underline{t})_{V_0} &= \sum_{v\in V_0}\left[-l\cdot \mathrm{val}_{\Gamma\setminus S}(v) + \sum_{w\neq v}\underline{t}_w\,\mathrm{val}_{\Gamma\setminus S}(v,w)\right]\\
&= \sum_{v\in V_0}\left[-l\cdot \mathrm{val}_{\Gamma\setminus S}(v) + \sum_{w\in V_0\setminus\{v\}} l\cdot\mathrm{val}_{\Gamma\setminus S}(v,w) + \sum_{w\in V_0^c}\underline{t}_w\,\mathrm{val}_{\Gamma\setminus S}(v,w)\right]\\
&= \sum_{v\in V_0}\left[-l\cdot \mathrm{val}_{\Gamma\setminus S}(v) + l\cdot\mathrm{val}_{\Gamma\setminus S}(v,V_0\setminus\{v\}) + \sum_{w\in V_0^c}\underline{t}_w\,\mathrm{val}_{\Gamma\setminus S}(v,w)\right]\\
&= \sum_{v\in V_0}\left[-l\cdot \mathrm{val}_{\Gamma\setminus S}\left(v,V_0^c\right) + \sum_{w\in V_0^c}\underline{t}_w\,\mathrm{val}_{\Gamma\setminus S}(v,w)\right]\\
&= \sum_{v\in V_0,w\in V_0^c}\left(\underline{t}_w - l\right)\mathrm{val}_{\Gamma\setminus S}(v,w)\\
&\geq \sum_{v\in V_0,w\in V_0^c}\mathrm{val}_{\Gamma\setminus S}(v,w) = \mathrm{val}_{\Gamma\setminus S}\left(V_0,V_0^c\right) = \mathrm{val}_{\Gamma\setminus S}(V_0).
\end{aligned}\tag{3.8}$$

Using the fact that $\Delta_0(\underline{t}) = \underline{d} - \underline{e}$, the above inequality (3.8) contradicts the strict inequality (3.7) for $W = V_0$, which holds since V_0 is a proper subset of $V(\Gamma\setminus S)$.

STEP II: $\overline{\pi}$ is surjective.

We introduce two rational numbers measuring how far is an element $\underline{d} \in C^0(\Gamma\setminus S,\mathbb{Z})_{q-|S|}$ from being in $B_{\Gamma\setminus S}\left(\underline{q}\right)$. For any $\underline{d} \in C^0(\Gamma\setminus S,\mathbb{Z})_{q-|S|}$ and any $W \subseteq V(\Gamma)$ (non necessarily proper), set

$$\begin{cases}\epsilon(\underline{d},W) := \underline{d}_W + |S\cap E(\Gamma[W])| - \underline{q}_W - \dfrac{\mathrm{val}(W)}{2} + \mathrm{val}_S(W),\\ \eta(\underline{d},W) := -\underline{d}_W - |S\cap E(\Gamma[W])| + \underline{q}_W - \dfrac{\mathrm{val}(W)}{2}.\end{cases}\tag{3.9}$$

Using the two relations

$$\begin{cases}\underline{d}_W + \underline{d}_{W^c} + |S| = \underline{q}_W + \underline{q}_{W^c},\\ |S| = |S\cap E(\Gamma[W])| + |S\cap E(\Gamma[W^c])| + \mathrm{val}_S(W),\end{cases}$$

it is easy to check that

$$\epsilon(\underline{d},W) = \eta(\underline{d},W^c).\tag{3.10}$$

We set also for any $\underline{d} \in C^0(\Gamma\setminus S,\mathbb{Z})_{q-|S|}$

$$\begin{cases}\epsilon(\underline{d}) := \max\limits_{W\subseteq V(\Gamma)}\epsilon(\underline{d},W),\\ \eta(\underline{d}) := \max\limits_{W\subseteq V(\Gamma)}\eta(\underline{d},W).\end{cases}\tag{3.11}$$

From Equation (3.10), we get that

$$\epsilon(\underline{d}) = \eta(\underline{d}).\tag{3.12}$$

We will often use in what follows that the invariants ϵ and η satisfy the following additive formula: for any disjoint subsets $W_1, W_2 \subset V(\Gamma)$, we have that

$$\begin{cases}\epsilon(\underline{d},W_1\cup W_2) = \epsilon(\underline{d},W_1) + \epsilon(\underline{d},W_2) + \mathrm{val}_{\Gamma\setminus S}(W_1,W_2),\\ \eta(\underline{d},W_1\cup W_2) = \eta(\underline{d},W_1) + \eta(\underline{d},W_2) + \mathrm{val}_{\Gamma\setminus S}(W_1,W_2).\end{cases}\tag{3.13}$$

Let us prove the second additive formula; the proof of the first one is similar and left to the reader. Using the additivity (3.1) of val and val_S, we compute:

$$\begin{aligned}\eta(\underline{d}, W_1 \cup W_2) &= -\underline{d}_{W_1\cup W_2} - |S \cap E(\Gamma[W_1 \cup W_2])| + \underline{q}_{W_1\cup W_2} - \frac{\mathrm{val}(W_1 \cup W_2)}{2} \\ &= -\underline{d}_{W_1} - \underline{d}_{W_2} - |S \cap E(\Gamma[W_1])| - |S \cap E(\Gamma[W_2])| - \mathrm{val}_S(W_1, W_2) + \underline{q}_{W_1} + \underline{q}_{W_2} \\ &\quad - \frac{\mathrm{val}\, W_1 + \mathrm{val}\, W_2 - 2\,\mathrm{val}(W_1, W_2)}{2} \\ &= \eta(\underline{d}, W_1) + \eta(\underline{d}, W_2) - \mathrm{val}_S(W_1, W_2) + \mathrm{val}(W_1, W_2) \\ &= \eta(\underline{d}, W_1) + \eta(\underline{d}, W_2) + \mathrm{val}_{\Gamma\backslash S}(W_1, W_2).\end{aligned}$$

For an element $\underline{d} \in C^0(\Gamma\backslash S, \mathbb{Z})_{q-|S|}$, consider the following sets:

$$\begin{cases} S_{\underline{d}}^+ := \{W \subseteq V(\Gamma) \,:\, \epsilon(\underline{d}, W) = \epsilon(\underline{d})\}, \\ S_{\underline{d}}^- := \{W \subseteq V(\Gamma) \,:\, \eta(\underline{d}, W) = \eta(\underline{d})\}. \end{cases}$$

From formula (3.10) and the equality $\epsilon(\underline{d}) = \eta(\underline{d})$, it follows easily that

$$W \in S_{\underline{d}}^+ \Longleftrightarrow W^c \in S_{\underline{d}}^-. \tag{3.14}$$

The sets $S_{\underline{d}}^\pm$ are stable under intersection:

$$W_1, W_2 \in S_{\underline{d}}^\pm \Longrightarrow W_1 \cap W_2 \in S_{\underline{d}}^\pm. \tag{3.15}$$

We will prove this for $S_{\underline{d}}^+$; the proof for $S_{\underline{d}}^-$ works exactly the same. Let $\Pi_1 := W_1 \setminus (W_1 \cap W_2)$. Using the additivity formula (3.13) applied to the pair (W_2, Π_1) of disjoint subsets of $V(\Gamma)$ and the fact that $W_2 \in S_{\underline{d}}^+$, we get that

$$0 = \epsilon(\underline{d}) - \epsilon(\underline{d}, W_2) \geq \epsilon(\underline{d}, \Pi_1 \cup W_2) - \epsilon(\underline{d}, W_2) = \epsilon(\underline{d}, \Pi_1) + \mathrm{val}_{\Gamma\backslash S}(\Pi_1, W_2).$$

Using this inequality, the additivity formula (3.13) for the disjoint pair $(W_1 \cap W_2, \Pi_1)$ of subsets of $V(\Gamma)$ and the fact that $W_1 \in S_{\underline{d}}^+$, we get that

$$\begin{aligned}\epsilon(\underline{d}) &= \epsilon(\underline{d}, W_1) \\ &= \epsilon(\underline{d}, (W_1 \cap W_2) \cup \Pi_1) \\ &= \epsilon(\underline{d}, W_1 \cap W_2) + \epsilon(\underline{d}, \Pi_1) + \mathrm{val}_{\Gamma\backslash S}(\Pi_1, W_1 \cap W_2) \\ &\leq \epsilon(\underline{d}, W_1 \cap W_2) + \epsilon(\underline{d}, \Pi_1) + \mathrm{val}_{\Gamma\backslash S}(\Pi_1, W_2) \\ &\leq \epsilon(\underline{d}, W_1 \cap W_2).\end{aligned}$$

By the maximality of $\epsilon(\underline{d})$, we conclude that $\epsilon(\underline{d}) = \epsilon(\underline{d}, W_1 \cap W_2)$, i.e., that $W_1 \cap W_2 \in S_{\underline{d}}^+$.

Since the sets $S_{\underline{d}}^\pm$ are stable under intersection, they admit minimum elements:

$$\Omega^\pm(\underline{d}) := \bigcap_{W \in S_{\underline{d}}^\pm} W \subseteq V(\Gamma). \tag{3.16}$$

Note that (3.14) implies that $\Omega^+(\underline{d})^c \in S_{\underline{d}}^-$. Since $\Omega^-(\underline{d})$ is the minimum element of $S_{\underline{d}}^-$, we get that $\Omega^-(\underline{d}) \subseteq \Omega^+(\underline{d})^c$, or in other words

$$\Omega^+(\underline{d}) \cap \Omega^-(\underline{d}) = \emptyset. \tag{3.17}$$

We set

$$\Omega^0(\underline{d}) := V(\Gamma) \setminus \big(\Omega^+(\underline{d}) \cup \Omega^-(\underline{d})\big),$$

so that $V(\Gamma)$ is the disjoint union of $\Omega^+(\underline{d})$, $\Omega^-(\underline{d})$ and $\Omega^0(\underline{d})$.

From (3.12) and the fact that $\epsilon(\underline{d}, V(\Gamma)) = \eta(\underline{d}, V(\Gamma)) = \epsilon(\underline{d}, \emptyset) = \eta(\underline{d}, \emptyset) = 0$, we get that $\epsilon(\underline{d}) = \eta(\underline{d}) \geq 0$. From Definition 3.2(i) and the definition of $\Omega^{\pm}(\underline{d})$, it follows that

$$\underline{d} \in B_{\Gamma\setminus S}\big(\underline{q}\big) \Longleftrightarrow \epsilon(\underline{d}) \quad \text{or} \quad \eta(\underline{d}) = 0 \Longleftrightarrow \Omega^{+}(\underline{d}) \quad \text{or} \quad \Omega^{-}(\underline{d}) = \emptyset. \tag{3.18}$$

Fix now an element $\underline{d} \in C^0(\Gamma\setminus S, \mathbb{Z})_{q-|S|}$ such that $\underline{d} \notin B_{\Gamma\setminus S}\big(\underline{q}\big)$. Set

$$\underline{e} := \underline{d} + \Delta_0\big(\underline{\chi\big(\Omega^{+}(\underline{d})\big)}\big). \tag{3.19}$$

Claim: The 0-cochain $\underline{e}$ satisfies one of the two following properties:

(i) $\epsilon(\underline{e}) < \epsilon(\underline{d})$,
(ii) $\epsilon(\underline{e}) = \epsilon(\underline{d})$ and $\Omega^{+}(\underline{e}) \supsetneq \Omega^{+}(\underline{d})$.

Note that the Claim concludes the proof of Step II. Indeed, if $\underline{e}$ satisfies condition (ii), we can iterate the substitution (3.19) until we reach an element $\underline{e'}$ which satisfies condition (i), i.e., $\epsilon(\underline{e'}) < \epsilon(\underline{d})$, and such that $\underline{e'} - \underline{d} \in \operatorname{Im}\Delta_0$. Now observe that, if we set N to be equal to two times the least common multiple of all the denominators of the rational numbers $\big\{\underline{q}_v\big\}_{v\in V(\Gamma)}$, then $N \cdot \epsilon(\underline{f}) \in \mathbb{Z}$, for any $\underline{f} \in C^0(\Gamma\setminus S, \mathbb{Z})$. Therefore, by iterating the substitution (3.19), we will finally reach an element $\underline{e''}$ such that $\epsilon(\underline{e''}) = 0$, i.e., $\underline{e''} \in B_{\Gamma\setminus S}\big(\underline{q}\big)$, and such that $\underline{e'} - \underline{d} \in \operatorname{Im}\Delta_0$. This proves that $\overline{\pi}$ is surjective.

Let us now prove the Claim. Take any subset $W \subset V(\Gamma)$ and decompose it as a disjoint union

$$W = W^{+} \coprod W^{-} \coprod W^{0},$$

where $W^{\pm} = W \cap \Omega^{\pm}(\underline{d})$ and $W^0 = W \cap \Omega^0(\underline{d})$. Note that

$$\epsilon\big(\underline{d}, W^{+}\big) \leq \epsilon(\underline{d}), \tag{3.20}$$

with equality if and only if $W^{+} = \Omega^{+}(\underline{d})$ because of the minimality property of $\Omega^{+}(\underline{d})$. Applying (3.13) to the disjoint pair $\big(\Omega^{+}(\underline{d}), W^0\big)$, we get

$$\begin{aligned} \epsilon\big(\underline{d}, W^0\big) &= \epsilon\big(\underline{d}, W^0 \cup \Omega^{+}(\underline{d})\big) - \epsilon\big(\underline{d}, \Omega^{+}(\underline{d})\big) - \operatorname{val}_{\Gamma\setminus S}\big(W^0, \Omega^{+}(\underline{d})\big) \\ &\leq -\operatorname{val}_{\Gamma\setminus S}\big(W^0, \Omega^{+}(\underline{d})\big), \end{aligned} \tag{3.21}$$

where we used that $\epsilon\big(\underline{d}, W^0 \cup \Omega^{+}(\underline{d})\big) \leq \epsilon(\underline{d}) = \epsilon(\underline{d}, \Omega^{+}(\underline{d}))$. Applying once more formula (3.13) to the disjoint pair $(W^{-}, \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d}))$, we get

$$\begin{aligned} \epsilon(\underline{d}, W^{-}) &= \epsilon\big(\underline{d}, W^{-} \cup \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big) - \epsilon\big(\underline{d}, \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big) \\ &\quad -\operatorname{val}_{\Gamma\setminus S}\big(W^{-}, \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big) \\ &\leq -\operatorname{val}_{\Gamma\setminus S}\big(W^{-}, \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big), \end{aligned} \tag{3.22}$$

where we used that (see (3.12) and (3.10))

$$\begin{aligned} \epsilon\big(\underline{d}, W^{-} \cup \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big) &\leq \epsilon(\underline{d}) = \eta(\underline{d}) = \eta(\underline{d}, \Omega^{-}(\underline{d})) = \epsilon(\underline{d}, \Omega^{-}(\underline{d})^c) \\ &= \epsilon\big(\underline{d}, \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big). \end{aligned}$$

Moreover, if the equality holds in (3.22), then by (3.10)

$$\eta(\underline{d}) = \epsilon\big(\underline{d}, W^{-} \cup \Omega^{+}(\underline{d}) \cup \Omega^0(\underline{d})\big) = \eta(\underline{d}, \Omega^{-}(\underline{d}) \setminus W^{-}),$$

which implies that $\Omega^{-}(\underline{d})\setminus W^{-} \in S^{-}_{\underline{d}}$ and hence that $W^{-} = \emptyset$ because of the minimality property of $\Omega^{-}(\underline{d})$. Using the formula

$$\epsilon(\underline{e}, W) = \epsilon(\underline{d}, W) + \Delta_0\big(\underline{\chi\big(\Omega^{+}(\underline{d})\big)}\big)_W$$

and (3.6), the above inequalities (3.20), (3.21), (3.22) give:

$$\begin{cases} \epsilon\big(\underline{e}, W^+\big) = \epsilon\big(\underline{d}, W^+\big) - \mathrm{val}_{\Gamma\setminus S}\left(W^+, \Omega^+(\underline{d})^c\right) \leq \epsilon(\underline{d}) - \mathrm{val}_{\Gamma\setminus S}\left(W^+, \Omega^+(\underline{d})^c\right), \\ \epsilon\big(\underline{e}, W^0\big) = \epsilon\big(\underline{d}, W^0\big) + \mathrm{val}_{\Gamma\setminus S}\left(W^0, \Omega^+(\underline{d})\right) \leq 0, \\ \epsilon\big(\underline{e}, W^-\big) = \epsilon\big(\underline{d}, W^-\big) + \mathrm{val}_{\Gamma\setminus S}\left(W^-, \Omega^+(\underline{d})\right) \leq -\,\mathrm{val}_{\Gamma\setminus S}\left(W^-, \Omega^0(\underline{d})\right), \end{cases} \tag{3.23}$$

Using twice the additive formula (3.13) for the disjoint union $W = W^+ \coprod W^0 \coprod W^-$ and the above inequalities (3.23), we compute

$$\begin{aligned} \epsilon(\underline{e}, W) &= \epsilon\big(\underline{e}, W^+\big) + \epsilon\big(\underline{e}, W^0\big) + \epsilon\big(\underline{e}, W^-\big) + \mathrm{val}_{\Gamma\setminus S}\left(W^+, W^0\right) \\ &\quad + \mathrm{val}_{\Gamma\setminus S}\left(W^+, W^-\right) + \mathrm{val}_{\Gamma\setminus S}\left(W^0, W^-\right) \\ &\leq \epsilon(\underline{d}) - \mathrm{val}_{\Gamma\setminus S}\left(W^+, \Omega^0(\underline{d}) \setminus W^0\right) - \mathrm{val}_{\Gamma\setminus S}\left(W^+, \Omega^-(\underline{d}) \setminus W^-\right) \\ &\quad - \mathrm{val}_{\Gamma\setminus S}\left(W^-, \Omega^0(\underline{d}) \setminus W^0\right) \leq \epsilon(\underline{d}). \end{aligned} \tag{3.24}$$

In particular, we have that $\epsilon(\underline{e}) \leq \epsilon(\underline{d})$. If the inequality in (3.24) is attained for some $W \subseteq V(\Gamma)$, i.e., if $\epsilon(\underline{e}) = \epsilon(\underline{d})$, then also the inequalities in (3.20) and (3.22) are attained for W, and we observed before that this implies that

$$\begin{cases} W^+ = \Omega^+(\underline{d}), \\ W^- = \emptyset. \end{cases} \tag{3.25}$$

Moreover, all the inequalities in (3.24) are attained for W and, substituting (3.25), this implies that

$$\begin{cases} \mathrm{val}_{\Gamma\setminus S}\left(\Omega^+(\underline{d}), \Omega^0(\underline{d}) \setminus W^0\right) = 0, \\ \mathrm{val}_{\Gamma\setminus S}\left(\Omega^+(\underline{d}), \Omega^-(\underline{d})\right) = 0. \end{cases} \tag{3.26}$$

Since $\Gamma \setminus S$ is connected by hypothesis and $\Omega^+(\underline{d})$ is a proper subset of $V(\Gamma \setminus S) = V(\Gamma)$ because we fixed $\underline{d} \notin B_{\Gamma\setminus S}\big(\underline{q}\big)$ (see (3.18)), we deduce that (using (3.26)):

$$0 < \mathrm{val}_{\Gamma\setminus S}\left(\Omega^+(\underline{d})\right) = \mathrm{val}_{\Gamma\setminus S}\left(\Omega^+(\underline{d}), \Omega^-(\underline{d}) \cup \Omega^0(\underline{d})\right) = \mathrm{val}_{\Gamma\setminus S}\left(\Omega^+(\underline{d}), W^0\right).$$

This gives that $W^0 \neq \emptyset$, which implies that $W = W^+ \cup W^0 \supsetneq W^+ = \Omega^+(\underline{d})$ by (3.25). Since this holds for all $W \subseteq V(\Gamma)$ such that $\epsilon(\underline{e}, W) = \epsilon(\underline{d}) (= \epsilon(\underline{e}))$, it holds in particular for $\Omega^+(\underline{e})$. Therefore, we get that $\Omega^+(\underline{e}) \supsetneq \Omega^+(\underline{d})$ and the claim is proved.

STEP III: $\mathrm{Im}(\overline{\pi}) = \mathrm{Im}(\pi)$.

Let $\underline{d} \in B_{\Gamma\setminus S}\big(\underline{q}\big)$, which by (3.18) is equivalent to have that $\epsilon(\underline{d}) = \eta(\underline{d}) = 0$. Let

$$S^-_{\underline{d}, v_0} := \{W \subseteq V(\Gamma) \,:\, \eta(\underline{d}, W) = \eta(\underline{d}) = 0 \text{ and } v_0 \in W\}.$$

The same proof as in Step II gives that $S^-_{\underline{d}, v_0}$ is stable for the intersection (see (3.15)). Therefore, the set $S^-_{\underline{d}, v_0}$ admits a minimum element

$$\Omega^-(\underline{d}, v_0) := \bigcap_{W \in S^-_{\underline{d}, v_0}} W \subseteq V(\Gamma).$$

Note that, by the definition 3.2(ii), it follows that

$$\underline{d} \in B^{v_0}_{\Gamma\setminus S}\big(\underline{q}\big) \Longleftrightarrow \underline{d} \in B_{\Gamma\setminus S}\big(\underline{q}\big) \quad \text{and} \quad \Omega^-(\underline{d}, v_0) = V(\Gamma). \tag{3.27}$$

Fix now an element $\underline{d} \in B_{\Gamma\setminus S}\big(\underline{q}\big) \setminus B^{v_0}_{\Gamma\setminus S}\big(\underline{q}\big)$ and consider the element

$$\underline{e} := \underline{d} - \Delta_0\big(\underline{\chi(\Omega^-(\underline{d}, v_0))}\big).$$

Claim: The 0-cochain $\underline{e}$ satisfies the following two properties:

(i) $\eta(\underline{e}) = 0$;

(ii) $\Omega^-(\underline{e}, v_0) \supsetneq \Omega^-(\underline{d}, v_0)$.

The Claim concludes the proof of Step III. Indeed, property (i) says that $\underline{e} \in B_{\Gamma\setminus S}(\underline{q})$ by (3.18) and therefore, by iterating the above construction, we will find an element $\underline{e'} \in B_{\Gamma\setminus S}(\underline{q})$ such that $\underline{d} - \underline{e'} \in \mathrm{Im}(\Delta_0)$ and $\Omega^-(\underline{e}, v_0) = V(\Gamma)$, which implies that $\overline{\pi}(\underline{d}) = \pi(\underline{e'})$ and $\underline{e'} \in B^{v_0}_{\Gamma\setminus S}(\underline{q})$ by (3.27). This shows that $\mathrm{Im}(\overline{\pi}) = \mathrm{Im}(\pi)$, q.e.d.

Let us now prove the Claim. Given any subset $W \subseteq V(\Gamma)$, we decompose it as a disjoint union

$$W = W^- \coprod W^+,$$

where $W^- := W \cap \Omega^-(\underline{d}, v_0)$ and $W^+ := W \setminus \Omega^-(\underline{d}, v_0)$. Applying formula (3.13) to the disjoint pair $\big(W^+, \Omega^-(\underline{d}, v_0)\big)$ and using that $\eta(\underline{d}) = 0$ and $\Omega^-(\underline{d}, v_0) \in S^-_{\underline{d}, v_0}$, we get

$$\begin{aligned} \eta\big(\underline{d}, W^+\big) &= \eta\big(\underline{d}, W^+ \cup \Omega^-(\underline{d}, v_0)\big) - \eta(\underline{d}, \Omega^-(\underline{d}, v_0)) - \mathrm{val}_{\Gamma\setminus S}\big(W^+, \Omega^-(\underline{d}, v_0)\big) \\ &\leq -\mathrm{val}_{\Gamma\setminus S}\big(W^+, \Omega^-(\underline{d}, v_0)\big). \end{aligned} \tag{3.28}$$

Applying again formula (3.13) to the disjoint pair $\big(W^+, W^-\big)$ and using $\eta(\underline{d}) = 0$ and (3.28), we get

$$\begin{aligned} \eta(\underline{d}, W) &= \eta(\underline{d}, W^-) + \eta\big(\underline{d}, W^+\big) + \mathrm{val}_{\Gamma\setminus S}\big(W^+, W^-\big) \\ &\leq 0 - \mathrm{val}_{\Gamma\setminus S}\big(W^+, \Omega^-(\underline{d}, v_0)\big) + \mathrm{val}_{\Gamma\setminus S}\big(W^+, W^-\big) \\ &= -\mathrm{val}_{\Gamma\setminus S}\big(W^+, \Omega^-(\underline{d}, v_0) \setminus W^-\big). \end{aligned} \tag{3.29}$$

Using the formula

$$\eta(\underline{e}, W) = \eta(\underline{d}, W) + \Delta_0(\underline{\chi(\Omega^-(\underline{d}, v_0))})_W \tag{3.30}$$

and (3.6), the above inequality (3.29) gives:

$$\begin{aligned} \eta(\underline{e}, W) &= \eta(\underline{d}, W) - \mathrm{val}_{\Gamma\setminus S}\big(W^-, \big(\Omega^-(\underline{d}, v_0) \cup W^+\big)^c\big) \\ &\quad + \mathrm{val}_{\Gamma\setminus S}\big(W^+, \Omega^-(\underline{d}, v_0) \setminus W^-\big) \\ &\leq -\mathrm{val}_{\Gamma\setminus S}\big(W^-, (\Omega^-(\underline{d}, v_0) \cup W^+)^c\big) \leq 0, \end{aligned} \tag{3.31}$$

which proves part (i) of the Claim. Assume moreover that the inequality in (3.31) is attained for some $W \subseteq V(\Gamma)$ such that $v_0 \in W$. Then all the inequalities must be attained also in (3.29) and in particular $\eta(\underline{d}, W^-) = 0$. Since $v_0 \in W \cap \Omega^-(\underline{d}, v_0) = W^-$, we deduce that $W^- \in S^-_{\underline{d}, v_0}$ and hence, by the minimality of $\Omega^-(\underline{d}, v_0)$, we get that $W^- = \Omega^-(\underline{d}, v_0)$. It follows that $\Omega^-(\underline{e}, v_0) \supseteq \Omega^-(\underline{d}, v_0)$. Using again formulas (3.30) and (3.6), together with the fact that $\Omega^-(\underline{d}, v_0) \in S^-_{\underline{d}, v_0}$, we compute

$$\begin{aligned} \eta(\underline{e}, \Omega^-(\underline{d}, v_0)) &= \eta(\underline{d}, \Omega^-(\underline{d}, v_0)) - \mathrm{val}_{\Gamma\setminus S}(\Omega^-(\underline{d}, v_0), \Omega^-(\underline{d}, v_0)^c) \\ &= -\mathrm{val}_{\Gamma\setminus S}(\Omega^-(\underline{d}, v_0), \Omega^-(\underline{d}, v_0)^c) < 0, \end{aligned}$$

because $\Gamma \setminus S$ is connected by hypothesis and $\Omega^-(\underline{d}, v_0)$ is a proper subset of $V(\Gamma \setminus S) = V(\Gamma)$ by our initial assumption $\underline{d} \in B_{\Gamma\setminus S}(\underline{q}) \setminus B^{v_0}_{\Gamma\setminus S}(\underline{q})$ (see (3.27)) together with the fact that $v_0 \in \Omega^-(\underline{d}, v_0)$. Therefore $\Omega^-(\underline{d}, v_0) \notin S^-_{\underline{e}, v_0}$ and hence $\Omega^-(\underline{e}, v_0) \supsetneq \Omega^-(\underline{d}, v_0)$, i.e., we get part (ii) of the Claim. ☐

Remark 3.6 The previous result was obtained for $S = \emptyset$ in [6, Lemma 3.1.5], building upon ideas from [9, Proposition 4.1]. Marco Pacini [36] has communicated to us a different proof of the above result.

By putting together Lemma 3.4, Proposition 3.5 and Equation (3.5), we deduce the following

Corollary 3.7 *The cardinality of set $B^{v_0}_{\Gamma\setminus S}(\underline{q})$ is equal to the complexity $c(\Gamma \setminus S)$ of $\Gamma \setminus S$. In particular, $B^{v_0}_{\Gamma\setminus S}(\underline{q}) \neq \emptyset$ if and only if $\Gamma \setminus S$ is connected.*

4 Fine compactified Jacobians and Néron models

Let $f : \mathcal{X} \to B = \mathrm{Spec}(R)$ be a one-parameter regular local smoothing of $X = \mathcal{X}_k$ (see Section 2.3). Fix a section $\sigma : B \to \mathcal{X}$ and a polarization $\underline{q}$ on X (see Section 2.4) such that $d := |\underline{q}|$. Consider the B-scheme $J_f^\sigma(\underline{q})$ of Section 2.6 and denote by $J_f^\sigma(\underline{q})_{\mathrm{sm}}$ its smooth locus over B.

Theorem 4.1 *Let $f : \mathcal{X} \to B$ be a one-parameter regular local smoothing of $X = \mathcal{X}_k$. Let σ be a section of f and $\underline{q}$ a polarization on X such that $d := |\underline{q}|$. Then $J_f^\sigma(\underline{q})_{sm}$ is isomorphic to the Néron model $\mathrm{N}(\mathrm{Pic}^d \mathcal{X}_K)$ of the degree-d Jacobian of the generic fiber $\mathcal{X}_K$ of f.*

Proof. According to Fact (2.13), $J_f^\sigma(\underline{q})_{\mathrm{sm}}$ parametrizes line bundles on $\mathcal{X}$ of relative degree d and whose special fiber is $\underline{q}$-P-quasistable, where $P := \sigma(\mathrm{Spec}\, k) \in X_{\mathrm{sm}}$. If we denote by v_0 the vertex of the dual graph Γ_X of X corresponding to the irreducible component to which P belongs, then the $\underline{q}$-P-quasistable multidegrees on X correspond to the 0-cochains belonging to $B^{v_0}_{\Gamma_X}(\underline{q})$ in the notation of Definition 3.2. Therefore, we get a canonical B-isomorphism

$$J_f^\sigma(\underline{q})_{\mathrm{sm}} \cong \frac{\coprod_{\underline{d}\in B^{v_0}_{\Gamma_X}(\underline{q})} \mathrm{Pic}^{\underline{d}}_f}{\sim_K}, \tag{4.1}$$

where $\sim_K$ denotes the gluing along the general fibers of $\mathrm{Pic}^{\underline{d}}_f$ which are isomorphic to $\mathrm{Pic}^d(\mathcal{X}_K)$. Since the general fiber of $J_f^\sigma(\underline{q})_{\mathrm{sm}}$ is isomorphic to $\mathrm{Pic}^d(\mathcal{X}_K)$, the Néron mapping property gives a map (see Fact 2.3):

$$r : J_f^\sigma(\underline{q})_{\mathrm{sm}} \cong \frac{\coprod_{\underline{d}\in B^{v_0}_{\Gamma_X}(\underline{q})} \mathrm{Pic}^{\underline{d}}_f}{\sim_K} \longrightarrow \mathrm{N}(\mathrm{Pic}^d \mathcal{X}_K) \cong \frac{\coprod_{\delta\in\Delta^d_X} \mathrm{Pic}^{\delta}_f}{\sim_K}.$$

Since we have a natural inclusion $i : J_f^\sigma(\underline{q})_{\mathrm{sm}} \hookrightarrow \mathrm{Pic}^d_f$ which is the identity on the general fibers, the map r factors through the map q of (2.3). Therefore the map r sends each $\mathrm{Pic}^{\underline{d}}_f$ into $\mathrm{Pic}^{[\underline{d}]}_f$. Since the natural map $B^{v_0}_{\Gamma_X}(\underline{q}) \to \Delta^d_X$ is a bijection according to Proposition 3.5, we conclude that the map r is an isomorphism. □

Remark 4.2

(i) In the terminology of [11, Definition 2.3.5] and [13, Definition 1.4 and Proposition 1.6], the above Theorem 4.1 says that the fine compactified Jacobians $J^P_X(\underline{q})$ are always of Néron-type (or N-type).

(ii) Using Theorem 7.1 and Remark 7.8, the above Theorem 4.1 recovers [13, Theorem 2.9], which is a generalization of [10, Theorem 6.1]: $\overline{P^d_X}$ is of Néron-type if X is weakly d-general.

5 A stratification of the fine compactified Jacobians

In the present section we shall exhibit a stratification of $J^P_X(\underline{q})$ in terms of fine compactified Jacobians of partial normalizations of X.

For each subset $S \subseteq X_{\mathrm{sing}}$, denote by $J^P_{X,S}(\underline{q})$ the subset of $J^P_X(\underline{q})$ corresponding to torsion-free sheaves which are not free exactly at S. Each $J^P_{X,S}(\underline{q})$ is a locally closed subset of $J^P_X(\underline{q})$ that we endow with the reduced schematic structure. Similarly, we endow the closure $\overline{J^P_{X,S}(\underline{q})}$ of each stratum $J^P_{X,S}(\underline{q})$ with the reduced schematic structure. We have the following stratification

$$J^P_X(\underline{q}) = \coprod_{S\subseteq X_{\mathrm{sing}}} J^P_{X,S}(\underline{q}). \tag{5.1}$$

Theorem 5.1 *The stratification of $J^P_X(\underline{q})$ given in* (5.1) *satisfies the following properties*:

(i) *Each stratum $J^P_{X,S}(\underline{q})$ is a disjoint union of $c(\Gamma_{X_S})$ torsors for the generalized Jacobian $J(X_S)$ of the partial normalization of X at S. In particular, $J^P_{X,S}(\underline{q})$ is non-empty if and only if X_S is connected.*

(ii) *The closure of each stratum is given by*

$$\overline{J^P_{X,S}(\underline{q})} = \coprod_{S\subset S'} J^P_{X,S'}(\underline{q}).$$

(iii) *The pushforward $(\nu_S)_*$ along the partial normalization map $\nu_S : X_S \to X$ gives isomorphisms*:

$$\begin{cases} J^P_{X_S}(\underline{q^S})_{sm} \cong J^P_{X,S}(\underline{q}), \\ J^P_{X_S}(\underline{q^S}) \cong \overline{J^P_{X,S}(\underline{q})}, \end{cases}$$

where $\underline{q^S}$ is the polarization on X_S defined in Lemma-Definition 2.5 *and P is seen as a smooth point of X_S using the isomorphism $(X_S)_{sm} \cong X_{sm}$.*

Remark 5.2 It is easy to see that if $\underline{q}$ is the canonical polarization of degree d (see Remark 2.11(ii)) then $\underline{q^S}$ is again a canonical polarization for every $S \subseteq X_{\text{sing}}$ if and only if $d = g-1$. This explains why the stratification found by Caporaso for $\overline{P^{g-1}_X}$ in [12, Section 4.1] can work only in degree $d = g-1$. In the general case, even if one is interested only in coarse or fine compactified Jacobians with respect to canonical polarizations, non-canonical polarizations naturally show-up in the above stratification.

Before proving the theorem, we need to analyze the multidegrees of the sheaves $\mathcal{I}$ belonging to the strata $J^P_{X,S}(\underline{q})$.

5.1 Multidegrees of sheaves $\mathcal{I} \in J^P_X(\underline{q})$

For a torsion-free, rank 1 sheaf $\mathcal{I}$ on X, the subset $NF(\mathcal{I}) \subset X_{\text{sing}}$ where $\mathcal{I}$ is not free (see Section 2.5) admits a partition

$$NF(\mathcal{I}) = NF_e(\mathcal{I}) \coprod NF_i(\mathcal{I}),$$

where $NF_e(\mathcal{I}) := NF(\mathcal{I}) \cap X_{\text{ext}}$ and $NF_i(\mathcal{I}) := NF(\mathcal{I}) \cap X_{\text{int}}$.

Given a sheaf $\mathcal{I}$ on X, we define its multidegree $\underline{\deg(\mathcal{I})}$ as the 0-cochain in $C^0(\Gamma_X, \mathbb{Z})$ such that $\underline{\deg(\mathcal{I})}_v := \deg_{X[v]}(\mathcal{I})$ for every $v \in V(\Gamma_X)$. Given a subset $W \subset V(\Gamma_X)$, we define

$$\underline{\deg(\mathcal{I})}_W := \sum_{v\in V(\Gamma_{X[W]})} \underline{\deg(\mathcal{I})}_v = \sum_{v\in V(\Gamma_{X[W]})} \deg_{X[v]}(\mathcal{I}).$$

In what follows we analyze the difference between $\deg_{X[W]}(\mathcal{I})$ and $\underline{\deg(\mathcal{I})}_W$ where $\mathcal{I}$ is a torsion-free, rank 1 sheaf on X.

Lemma 5.3 *Let Y be a subcurve of X and let $Y_1, \ldots, Y_m$ be the irreducible components of Y. Then*

$$\deg_Y(\mathcal{I}) = \sum_{i=1}^m \deg_{Y_i}(\mathcal{I}) + |NF_e(\mathcal{I}) \cap X \setminus Y^c|.$$

Proof. We will first prove that if Y and Z are two subcurves of X without common irreducible components then

$$\deg_{Y\cup Z}(\mathcal{I}) = \deg_Y(\mathcal{I}) + \deg_Z(\mathcal{I}) + |NF(\mathcal{I}) \cap Y \cap Z|. \tag{5.2}$$

Using Proposition 2.9(i), there exists a line bundle L on X_S where $S = NF(\mathcal{I})$ such that $\mathcal{I} = (\nu_S)_*(L)$. By Proposition 2.9(iii), we have the equalities

$$\begin{cases} \deg_{Y\cup Z}\mathcal{I} = \deg_{Y_S\cup Z_S} L + \left|S_i^{Y\cup Z}\right|, \\ \deg_Y \mathcal{I} = \deg_{Y_S} L + \left|S_i^Y\right|, \\ \deg_Z \mathcal{I} = \deg_{Z_S} L + \left|S_i^Z\right|. \end{cases} \tag{a}$$

Since L is a line bundle, we have that

$$\deg_{Y_S \cup Z_S} L = \deg L_{|Y_S \cup Z_S} = \deg L_{|Y_S} + \deg L_{|Z_S} = \deg_{Y_S} L + \deg_{Z_S} L. \tag{b}$$

We have already observed in (2.6) that

$$\left|S_i^{Y \cup Z}\right| = \left|S_i^Y\right| + \left|S_i^Z\right| + |S \cap Y \cap Z|. \tag{c}$$

Equation (5.2) is easily proved by putting together Equations (a), (b) and (c).

The proof of the lemma is now by induction on the number m of irreducible components of Y. If $m = 1$ then the formula follows from the fact that $X \setminus Y_1^c$ contains only internal nodes. As for the induction step, using (5.2), we can write

$$\deg_Y(\mathcal{I}) = \deg_{Y_1 \cup \cdots \cup Y_{m-1}}(\mathcal{I}) + \deg_{Y_m}(\mathcal{I}) + |NF_e(\mathcal{I}) \cap (Y_1 \cup \cdots \cup Y_{m-1}) \cap Y_m|. \tag{*}$$

By the induction hypothesis, we have that

$$\deg_{Y_1 \cup \cdots \cup Y_{m-1}}(\mathcal{I}) = \sum_{i=1}^{m-1} \deg_{Y_i}(\mathcal{I}) + |NF_e(\mathcal{I}) \cap X \backslash (Y_1 \cup \cdots \cup Y_{m-1})^c|. \tag{**}$$

Since an external node in $X \backslash Y^c$ either is an external node of $Y_1 \cup \cdots \cup Y_{m-1}$ or is node at which Y_m intersects $Y_1 \cup \cdots \cup Y_{m-1}$, we have that

$$\begin{aligned} |NF_e(\mathcal{I}) \cap X \setminus Y^c| \; = \; & |NF_e(\mathcal{I}) \cap X \setminus (Y_1 \cup \cdots \cup Y_{m-1})^c| \\ & + |NF_e(\mathcal{I}) \cap (Y_1 \cup \cdots \cup Y_{m-1}) \cap Y_m)|. \end{aligned} \tag{***}$$

We conclude by putting together (*), (**), (***). □

For every subset $S \subseteq X_{\text{sing}}$, denote by $B_{X,S}^P\left(\underline{q}\right)$ the set of possible multidegrees of sheaves $\mathcal{I} \in J_{X,S}^P(\underline{q})$. Write $S = S_e \coprod S_i$, where $S_e := S \cap X_{\text{ext}}$ and $S_i = S \cap X_{\text{int}}$. We need the following version of the dual graph of X: the *loop-less* dual graph of X, denoted by $\widetilde{\Gamma_X}$, is the graph obtained from Γ_X by removing all the loops. In particular, $V\left(\widetilde{\Gamma_X}\right) = V(\Gamma_X)$ while $E\left(\widetilde{\Gamma_X}\right)$ can be identified with X_{int}.

Proposition 5.4 *For any $S \subseteq X_{sing}$ we have that*

$$B_{X,S}^P\left(\underline{q}\right) = B_{\widetilde{\Gamma_X} \setminus S_e}^{v_P}(\underline{q}).$$

In particular, the cardinality of $B_{X,S}^P\left(\underline{q}\right)$ is equal to $c\left(\widetilde{\Gamma_X} \setminus S_e\right) = c(\Gamma_X \backslash S) = c(\Gamma_{X_S})$.

Proof. Consider the loop-less dual graph $\widetilde{\Gamma_X}$ of X and a sheaf $\mathcal{I} \in J_X^P(\underline{q})$. Then, Lemma 5.3 translated in terms of $\widetilde{\Gamma_X}$ says that, for every $W \subset V(\Gamma_X) = V\left(\widetilde{\Gamma_X}\right)$, the multidegree $\underline{\deg(\mathcal{I})}$ of $\mathcal{I}$ satisfies:

$$\deg_{X[W]}(\mathcal{I}) = \underline{\deg(\mathcal{I})}_W + \left|NF_e(\mathcal{I}) \cap \widetilde{\Gamma_X}[W]\right|.$$

In particular, $\deg(\mathcal{I}) = \left|\underline{\deg(\mathcal{I})}\right| + |NF_e(\mathcal{I})|$. Using this formula together with the fact that, for every $W \subset V(\Gamma_X) = V\left(\widetilde{\Gamma_X}\right)$, $\delta_{X[W]} = \text{val}_{\widetilde{\Gamma_X}}(W)$ we deduce that a torsion-free, rank 1 sheaf $\mathcal{I}$ is P-quasistable with respect to $\underline{q}$ (in the sense of Definition 2.10(ii)) if and only if its multidegree $\underline{\deg(\mathcal{I})} \in C^0\left(\widetilde{\Gamma_X}, \mathbb{Z}\right)$ is v_P-quasistable with respect to $\underline{q}$ (in the sense of Definition 3.2(ii)). The last assertion follows from Corollary 3.7 together with the easy facts that the operation of removing loops from a graph does not change its complexity and that $\Gamma \backslash S = \Gamma_{X_S}$. □

Proof of Theorem 5.1. Part (i): By Proposition 2.9(i), the subvariety of $J_{X,S}^P(\underline{q})$ consisting of sheaves with a fixed multidegree $\underline{d}$ is isomorphic to $\text{Pic}^{\underline{d}'}(X_S)$, where $\underline{d}'$ is related to $\underline{d}$ according to the formula of Proposition 2.9(iii). Each $\text{Pic}^{\underline{d}'}(X_S)$ is clearly a torsor for $J(X_S)$. We conclude by the fact that the set $B_{X,S}^P\left(\underline{q}\right)$ of multidegrees of sheaves belonging to $J_{X,S}^P(\underline{q})$ has cardinality $c(\Gamma_{X_S})$ by Proposition 5.4.

Part (ii): The inclusion

$$\overline{J^P_{X,S}(\underline{q})} \subset \coprod_{S \subset S'} J^P_{X,S'}\big(\underline{q}\big)$$

is clear since under specialization the set $\mathrm{NF}(\mathcal{I})$ can only increase. In order to prove the reverse inclusion, it is enough to show that if $\mathcal{I} \in J^P_X\big(\underline{q}\big)$ is such that $n \in \mathrm{NF}(\mathcal{I})$ then there exists a sheaf $\mathcal{I}' \in J^P_X\big(\underline{q}\big)$ specializing to $\mathcal{I}$ and such that $\mathrm{NF}(\mathcal{I}') = \mathrm{NF}(\mathcal{I}) \setminus \{\mathrm{n}\}$.

Suppose first that n is an external node and, up to reordering the components of X, assume that $n \in C_1 \cap C_2$. By looking at the miniversal deformation ring of $\mathcal{I}$ (see e.g., [16, Lemma 3.14]), we can find a torsion free, rank 1 sheaf $\mathcal{I}'$ specializing to $\mathcal{I}$ with $\mathrm{NF}(\mathcal{I}') = \mathrm{NF}(\mathcal{I}) \setminus \{\mathrm{n}\}$ and such that the multidegree of $\mathcal{I}'$ is related to the one of $\mathcal{I}$ by means of the following

$$\deg_{C_i} \mathcal{I}' = \begin{cases} \deg_{C_1} \mathcal{I} + 1 & \text{if} \quad i = 1, \\ \deg_{C_i} \mathcal{I} & \text{if} \quad i \neq 1. \end{cases} \tag{5.3}$$

Since the condition of being $\underline{q}$-P-quasistable is an open condition, we get that $\mathcal{I}'$ is $\underline{q}$-P-quasistable and we are done.

Suppose now that n is an internal node. By looking at the miniversal deformation ring of $\mathcal{I}$, we can find a torsion-free rank 1 sheaf $\mathcal{I}'$ specializing to $\mathcal{I}$ with $\mathrm{NF}(\mathcal{I}') = \mathrm{NF}(\mathcal{I}) \setminus \{\mathrm{n}\}$ and such that the multidegree of $\mathcal{I}'$ is equal to the one of $\mathcal{I}$. Clearly $\mathcal{I}'$ is $\underline{q}$-P-quasistable and we are done.

Part (iii): First of all, observe that the pushforward map $(\nu_S)_*$ is a closed embedding since it is induced by a functor between the categories of torsion-free rank one sheaves on X_S and on X which is fully faithful, as it follows from [21, Lemma 3.4] (note that the result in loc. cit. extends easily from the case of integral curves to the case of reduced curves).[1] Therefore, in order to conclude the proof of part (iii), it is enough to show that the map $(\nu_S)_*$ induces a bijection on geometric points.

Consider first the bijection of Proposition 2.9(i). We claim that a line bundle $L \in \mathrm{Pic}(X_S)$ is $\underline{q^S}$-P-quasistable on X_S if and only if $(\nu_S)_*L$ is $\underline{q}$-P- quasistable on X. This amounts to prove that for any subcurve $Y \subset X$ we have

$$\deg_{Y_S} L \geq \underline{q^S}_{Y_S} - \frac{\delta_{Y_S}}{2} \iff \deg_Y (\nu_S)_* L \geq \underline{q}_Y - \frac{\delta_Y}{2},$$

and similarly with the strict inequality $>$ (since $P \in Y$ if and only if $P \in Y_S$). This equivalence follows from the equalities

$$\begin{cases} \deg_{Y_S} L = \deg_Y (\nu_S)_* L - \left|S^Y_i\right|, \\ \underline{q^S}_{Y_S} = \underline{q}_Y - \dfrac{\left|S^Y_e\right|}{2} - \left|S^Y_i\right|, \\ \delta_{Y_S} = \delta_Y - \left|S^Y_e\right|, \end{cases}$$

where the first equality follows from Proposition 2.9(iii), the second follows from the definition of $\underline{q^S}$ (see Lemma-Definition 2.5) and the third is easily checked. Therefore, using Fact 2.13(i), the push-forward via the normalization map ν_S induces a morphism

$$(\nu_S)_* : J^P_{X_S}\big(\underline{q^S}\big)_{\mathrm{sm}} \longrightarrow J^P_{X,S}\big(\underline{q}\big), \tag{5.4}$$

which is bijective on geometric points. This proves the first isomorphism in Part (iii).

Let us now prove the second isomorphism of Part (iii). To that aim, consider two subsets $\emptyset \subseteq S \subseteq S' \subseteq X_{\mathrm{sing}}$. We have a commutative diagram

$$\begin{tikzcd}
X_{S'} \arrow[rr, "\nu_{S'\setminus S}"] \arrow[dr, "\nu_{S'}"'] & & X_S \arrow[dl, "\nu_S"] \\
& X &
\end{tikzcd}$$

---

[1] We are grateful to Eduardo Esteves for pointing out to us this argument.

where $\nu_{S'\setminus S}$ is the partial normalization of X_S at the nodes corresponding to $S'\setminus S$. By abuse of notation, we denote by P the inverse image of $P \in X$ in X_S and in $X_{S'}$. We claim that the above diagram induces, via push-forwards, a commutative diagram

$$\begin{tikzcd}
J^P_{X_{S'}}\left(\underline{q^{S'}}\right)_{\text{sm}} \arrow[rr, "(\nu_{S'\setminus S})_*", "\cong"'] \arrow[dr, "\cong", "(\nu_{S'})_*"'] & & J^P_{X_S, S'\setminus S}\left(\underline{q^S}\right) \arrow[dl, "(\nu_S)_*", "\cong"'] \\
& J^P_{X,S'}\left(\underline{q}\right) &
\end{tikzcd} \tag{5.5}$$

where all the maps are isomorphisms. Indeed, from (5.4) with S replaced by S', it follows that the map $(\nu_{S'})_*$ is an isomorphism. Similarly, if we apply (5.4) with X replaced by X_S, S replaced by $S'\setminus S$ and $\underline{q}$ replaced by $\underline{q^S}$, we obtain that $(\nu_{S'\setminus S})_*$ is an isomorphism since it is easily checked that $(X_S)_{S'\setminus S} \cong X_{S'}$ and $\underline{\left(\underline{q^S}\right)^{S'\setminus S}} = \underline{q^{S'}}$. Since the diagram (5.5) is clearly commutative, we get that $(\nu_S)_*$ is well-defined and that it is an isomorphism.

From the fact that the map $(\nu_S)_*$ in diagram (5.5) is an isomorphism, using the stratification (5.1) and the one in part (ii), we deduce that the natural map

$$(\nu_S)_* : J^P_{X_S}\left(\underline{q^S}\right) = \coprod_{S\subseteq S'\subseteq X\text{sing}} J^P_{X_S, S'\setminus S}\left(\underline{q^S}\right) \to \coprod_{S\subseteq S'\subseteq X\text{sing}} J^P_{X,S'}\left(\underline{q}\right) = \overline{J^P_{X,S}\left(\underline{q}\right)} \tag{5.6}$$

is bijective on geometric points, which concludes the proof. ☐

Corollary 5.5 *For the stratification in* (5.1)*, it holds*:

(i) $J^P_{X,S}(\underline{q})$ *has pure codimension equal to* $|S|$.

(ii) $\overline{J^P_{X,S}(\underline{q})} \supset J^P_{X,S'}\left(\underline{q}\right)$ *if and only if* $S \subseteq S'$.

(iii) *The smooth locus of* $\overline{J^P_{X,S}(\underline{q})}$ *is equal to* $J^P_{X,S}(\underline{q})$.

Proof. Part (i) follows from Theorem 5.1(i) together with the equality

$$\dim J(X) - \dim J(X_S) = g(X) - g(X_S) = |S|,$$

where we used that X_S is connected.

Part (ii) follows from Theorem 5.1(ii).

Part (iii) follows from Theorem 5.1(iii). ☐

Remark 5.6 A result similar to Corollary 5.5 was proved by Caporaso in [10, Theorem 6.7] for the compactified Jacobian $\overline{P^d_X}$ (see Remark 2.12(v)) of a d-general curve X in the sense of Remark 7.5. Indeed, by using Theorem 7.1, our Corollary 5.5 recovers [10, Theorem 6.7] and extends it to the case of X weakly d-general in the sense of Remark 7.8.

6 Fine compactified Jacobians as quotients

Recall from 2.1 that we denote by $\widehat{X_S}$ $\big($resp. $\widehat{X}\big)$ the partial blowup of X at $S \subseteq X$sing (resp. the total blowup of X) and the natural blow-down morphisms by $\pi_S : \widehat{X_S} \to X$ $\big($resp. $\pi : \widehat{X} \to X\big)$. Moreover, for each $S \subseteq X$sing, we have a commutative diagram

$$\begin{tikzcd}
\widehat{X} \arrow[rr, "\pi^S"] \arrow[dr, "\pi"'] & & \widehat{X_S} \arrow[dl, "\pi_S"] \\
& X &
\end{tikzcd} \tag{6.1}$$

where π^S is the blow-down of all the exceptional subcurves of $\widehat{X}$ lying over the nodes of Xsing$\setminus S$.

Given a polarization $\underline{q}$ on X, consider the polarizations $\widehat{\underline{q}^S}$ $\big($resp. $\widehat{\underline{q}}\big)$ on $\widehat{X_S}$ $\big($resp. $\widehat{X}\big)$ introduced in Lemma-Definition 2.6. Given $P \in X_{\mathrm{sm}}$, we denote also with P the inverse image of P in $\widehat{X_S}$ and in $\widehat{X}$, in a slight abuse of notation.

Given $S \subseteq X_{\mathrm{sing}}$, denote by $J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}}$ the open and closed subset of $J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{sm}}$ consisting of all line bundles that have degree -1 on all the exceptional components of $\widehat{X}_S$. Note that $J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}}$ may be empty for some $S \subseteq X_{\mathrm{sing}}$.

Theorem 6.1

(i) *For any* $S \subseteq X_{sing}$, $J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}}$ *is a disjoint union of* $c(\Gamma_{X_S})$ *torsors for the generalized Jacobian* $J\big(\widehat{X}_S\big) \cong J\big(\widehat{X}\big) \cong J(X)$. *In particular* $J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}}$ *is non-empty if and only if* X_S *is connected.*

(ii) *The pull-back via the map* π^S *induces an open and closed embedding*

$$\big(\pi^S\big)^* : J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}} \hookrightarrow J^P_{\widehat{X}}(\widehat{\underline{q}})_{sm}. \tag{6.2}$$

Via the above identification, $J^P_{\widehat{X}}(\widehat{\underline{q}})_{sm}$ *decomposes into a disjoint union of open and closed strata*

$$J^P_{\widehat{X}}(\widehat{\underline{q}})_{sm} = \coprod_{\emptyset \subseteq S \subseteq Xsing} J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}}. \tag{6.3}$$

(iii) *The push-forward along the map* π *induces a surjective morphism*

$$\pi_* : J^P_{\widehat{X}}(\widehat{\underline{q}})_{sm} \twoheadrightarrow J^P_X\big(\underline{q}\big),$$

which is compatible with the stratifications (5.1) *and* (6.3) *in the sense that it induces a cartesian diagram*

$$\begin{tikzcd}
J^P_{\widehat{X}_S}\big(\widehat{\underline{q}^S}\big)_{\mathrm{prim}} \arrow[r, hook, "\left(\pi^S\right)^*"] \arrow[d, "(\pi_S)_*"'] & J^P_{\widehat{X}}(\widehat{\underline{q}})_{sm} \arrow[d, "\pi_*"] \\
J^P_{X,S}\big(\underline{q}\big) \arrow[r, hook] & J^P_X\big(\underline{q}\big)
\end{tikzcd}$$

Moreover, the map $(\pi_S)_*$ *on the left-hand side of the above diagram is given by taking a quotient by the algebraic torus* $\mathbb{G}_m^{|S|}$ *of dimension* $|S|$.

Proof. Let us start by proving Part (ii). First of all, observe that the pull-backs via the maps of diagram (6.1) induce canonical isomorphisms between the generalized Jacobians

$$\pi^* : J(X) \xrightarrow{\cong} J\big(\widehat{X_S}\big) \xrightarrow{\cong} J\big(\widehat{X}\big),$$

so that we will freely identify them during this proof.

Let us prove that the map (6.2) is well-defined, that is, given a P-$\widehat{\underline{q}^S}$-quasistable line bundle L on $\widehat{X_S}$, then $\big(\pi^S\big)^* L$ is a P-$\widehat{\underline{q}}$-quasistable line bundle on $\widehat{X}$. Clearly we have that $\deg\big(\pi^S\big)^* L = \deg L = \big|\widehat{\underline{q}^S}\big| = |\,\widehat{\underline{q}}\,|$. Moreover, if Z is a subcurve of $\widehat{X}$ and we denote by $\pi^S(Z)$ its image in $\widehat{X_S}$, then it is easily checked that $\delta_Z \geq \delta_{\pi^S(Z)}$, which implies that

$$\deg_Z\big(\pi^S\big)^* L = \deg_{\pi^S(Z)} L \geq \widehat{\underline{q}^S}_{\pi^S(Z)} - \frac{\delta_{\pi^S(Z)}}{2} \geq \widehat{q}_Z - \frac{\delta_Z}{2},$$

where the first inequality is strict if $P \in \pi^S(Z)$ which happens if and only if $P \in Z$. Hence, $\big(\pi^S\big)^* L$ is a P-$\widehat{\underline{q}}$-quasistable.

The map (6.2) is equivariant with respect to the action of the generalized Jacobians $J\big(\widehat{X_S}\big) \cong J\big(\widehat{X}\big)$ and both the sides are disjoint union of torsors for these generalized Jacobians. Therefore, $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}}$ is mapped via (6.2) isomorphically onto a disjoint union of connected components of $J^P_{\widehat{X}}(\widehat{\underline{q}})_{\mathrm{sm}}$. The image of $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}}$ inside $J^P_{\widehat{X}}(\widehat{\underline{q}})_{\mathrm{sm}}$ consists of all P-$\widehat{\underline{q}}$-quasistable line bundles on $\widehat{X}$ that have degree -1 on the exceptional components lying over the nodes belonging to S and degree 0 on the other exceptional components.

In order to prove that the decomposition description (6.3) holds, it remains to show that any line bundle L on $\widehat{X}$ which is P-$\widehat{\underline{q}}$-quasistable must have degree -1 or 0 on each exceptional component E of $\widehat{X}$. Indeed, by applying (2.10) to E and to $E^c = \overline{\widehat{X} \setminus E}$ and using that $\delta_E = 2$, we get that $\deg_E L$ must be equal to -1, 0 or 1. However, since $P \in E^c$, strict inequality must hold when applying (2.10) to E^c, so $\deg_E L$ cannot be equal to 1. Part (ii)is now complete.

Claim: The commutative diagram (2.1) induces a commutative diagram

$$\begin{tikzcd}
J^P_{X_S}\big(\underline{q^S}\big)_{\mathrm{sm}} \arrow[dr, "\cong", "(\nu_S)_*"'] & & J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}} \arrow[ll, two heads, "i_S^*"'] \arrow[dl, "(\pi_S)_*"] \\
& J^P_{X,S}\big(\underline{q}\big) &
\end{tikzcd} \tag{6.4}$$

where $(\nu_S)_*$ is an isomorphism and the maps i_S^* and $(\pi_S)_*$ are surjective. The fact that the map $(\nu_S)_*$ is well-defined and is an isomorphism is proved in Theorem 5.1(iii). Therefore, the commutativity of the diagram, together with the fact that it is well-defined, will follow from Proposition 2.9(i) if we show that i_S^* is well-defined, i.e., if L is a P-$\widehat{\underline{q^S}}$-quasistable line bundle on $\widehat{X_S}$ having degree -1 on each exceptional component of $\widehat{X_S}$ then $i_S^*(L)$ is a P-$\underline{q^S}$-quasistable line bundle on X_S. Indeed, we have that

$$\deg i_S^*(L) = \deg L - |S| = \big|\widehat{\underline{q^S}}\big| - |S| = \big|\underline{q}\big| - |S| = \big|\underline{q^S}\big|.$$

Moreover, for any subcurve $Y_S \subseteq X_S$, it is easily checked that (in the notations of Lemma-Definition 2.5)

$$\begin{cases} \deg_{Y_S} i_S^*(L) = \deg_{i_S(Y_S)} L, \\ \underline{q^S}_{Y_S} = \underline{q}_Y - \dfrac{\big|S_e^Y\big|}{2} - \big|S_i^Y\big| = \widehat{\underline{q^S}}_{i_S(Y_S)} - \dfrac{\big|S_e^Y\big|}{2} - \big|S_i^Y\big|, \\ \delta_{Y_S} = \delta_Y - \big|S_e^Y\big| = \delta_{i_S(Y_S)} - 2\big|S_i^Y\big| - \big|S_e^Y\big|. \end{cases}$$

Using the above relations, it turns out that the inequality (2.10) for the subcurve $Y_S \subseteq X_S$ and the line bundle $i_S^* L$ follows form the same inequality (2.10) applied to the subcurve $i_S(Y_S) \subseteq \widehat{X_S}$ and the line bundle L. Hence i_S^* is well-defined.

In order to conclude the proof of the claim, it remains to prove that the map i_S^* is surjective. Clearly $J^P_{X_S}\big(\underline{q^S}\big)_{\mathrm{sm}}$ is a disjoint union of torsors for $J(X_S)$ of the form $\mathrm{Pic}^{\underline{d}'}(X_S)$ for some suitable multidegrees $\underline{d}'$; the number of such components is $c(\Gamma_{X_S})$ by Theorem 5.1. Similarly, $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}}$ is a disjoint union of torsors for $J\big(\widehat{X_S}\big)$ of the form $\mathrm{Pic}^{\underline{d}}\big(\widehat{X_S}\big)$ for some suitable multidegrees $\underline{d}$ on $\widehat{X_S}$; call n_S the number of such components. It is clear that the map i_S^* is equivariant with respect to the actions of $J(X_S)$ and $J\big(\widehat{X_S}\big)$ and of the natural surjective map

$$J\big(\widehat{X_S}\big) \twoheadrightarrow J(X_S). \tag{6.5}$$

This implies that each connected component $\mathrm{Pic}^{\underline{d}}\big(\widehat{X_S}\big)$ of $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}}$ is sent surjectively onto the connected component $\mathrm{Pic}^{\underline{d}_{X_S}}(X_S)$ of $J^P_{X_S}\big(\underline{q^S}\big)_{\mathrm{sm}}$, where $\underline{d}_{X_S}$ is the restriction of the multidegree $\underline{d}$ to X_S. Since $\underline{d}$ has degree -1 on each exceptional component of $\widehat{X_S}$, the multidegree $\underline{d}$ is completely determined by its restriction $\underline{d}_{X_S}$. This means that different components of $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\mathrm{prim}}$ are sent to different components of $J^P_{X_S}\big(\underline{q^S}\big)$. In particular, we get that

$$n_S \leq c(\Gamma_{X_S}). \tag{*}$$

Let us now show that $n_S = c(\Gamma_{X_S})$, which will conclude the proof of the claim and also the proof of Part (i). By Theorem 4.1 and Fact 2.3, it follows that the number of connected components of $J^P_{\widehat{X}}(\widehat{\underline{q}})_{\rm sm}$ is equal to $c(\Gamma_{\widehat{X}})$. Using the decomposition (6.3) and the inequality (*), we get that

$$c(\Gamma_{\widehat{X}}) = \sum_{\emptyset \subseteq S \subseteq X_{\rm sing}} n_S \leq \sum_{\emptyset \subseteq S \subseteq X_{\rm sing}} c(\Gamma_{X_S}). \tag{**}$$

Fact 2.2 applied to the graph $\Gamma = \Gamma_{\widehat{X}}$ and $S = E(\Gamma_X)$ gives that equality must hold in (**) and hence, a fortiori, also in (*) for every $S \subset X_{\rm sing}$. Part (i) follows.

Finally, let us prove Part (iii). The image of the stratum $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\rm prim} \subset J^P_{\widehat{X}}(\widehat{\underline{q}})_{\rm sm}$ via π_* coincides with its image via the map $(\pi_S)_*$, which by the above Claim, is equal to $J^P_{X,S}\big(\underline{q}\big)$. Therefore π_* is surjective and compatible with the filtrations (5.1) and (6.3). For all the subsets $S \subseteq X_{\rm sing}$ such that $J^P_{\widehat{X}_S}\big(\widehat{\underline{q^S}}\big)_{\rm prim} \neq \emptyset$, the map $(\pi_S)_*$ is given by taking the quotient by the kernel of the surjection (6.5), which is equal to $\mathbb{G}_m^{|S|}$ since X_S is connected by Part (i). The proof is now complete. ☐

6.1 Relating one-parameter regular local smoothings of X and of $\widehat{X}$

Let $f : \mathcal{X} \to \operatorname{Spec} R = B$ be a one-parameter regular local smoothing of X (see Section 2.3) and assume that f admits a section σ.

Then, as shown in [10, Section 8.4], there exists a one-parameter regular local smoothing $\widehat{f} : \widehat{\mathcal{X}} \to B_1$ of $\widehat{X}$ endowed with a section $\widehat{\sigma}$ in such a way that there is a commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{X}} \arrow[r] \arrow[d, "\widehat{f}"] & \mathcal{X} \arrow[d, "f"'] \\
B_1 \arrow[r] \arrow[u, bend left, "\widehat{\sigma}"] & B \arrow[u, bend right, "\sigma"']
\end{tikzcd} \tag{6.6}$$

which, moreover, is a cartesian diagram on the general fibers of f and $\widehat{f}$.

For the reader's convenience, we review Caporaso's construction. Let t be a uniformizing parameter of R (i.e., a generator of the maximal ideal of R) and consider the degree-2 extension $K \hookrightarrow K_1 := K(u)$ where $u^2 = t$. Denote by R_1 the integral closure of R inside K_1 so that $B_1 := \operatorname{Spec}(R_1) \to B = \operatorname{Spec}(R)$ is a degree-2 ramified cover. Note that R_1 is a DVR having quotient field K_1 and residue field $k = \overline{k}$. Consider the base change

$$f_1 : \mathcal{X}_1 := \mathcal{X} \times_B B_1 \longrightarrow B_1,$$

and let $\sigma_1 : B_1 \to \mathcal{X}_1$ be the section of f_1 obtained by pulling back the section σ of f. The special fiber of $\mathcal{X}_1$ is isomorphic to X and the total space $\mathcal{X}_1$ has a singularity formally equivalent to $xy = u^2$ at each of the nodes of the special fiber. It is well-known that the relatively minimal regular model of $f_1 : \mathcal{X}_1 \to B_1$, call it $\widehat{f} : \widehat{\mathcal{X}} \to B_1$, is obtained by blowing-up $\mathcal{X}_1$ once at each one of these singularities. Moreover, the section σ_1 of f_1 admits a lifting to a section $\widehat{\sigma}$ of $\widehat{f}$ since the image of σ_1 is contained in the smooth locus of $\mathcal{X}_1$. It is easy to check that the general fiber of $\widehat{f}$ is equal to $\widehat{\mathcal{X}}_{K_1} = \mathcal{X}_K \times_K K_1$ while its special fiber is equal to $\widehat{\mathcal{X}}_k = \widehat{X}$. In other words, $\widehat{f} : \widehat{\mathcal{X}} \to B_1$ is a one-parameter regular local smoothing of $\widehat{X}$. By construction, it follows that we have a commutative diagram as in (6.6) which, moreover, is cartesian on the general fibers of f and $\widehat{f}$.

Theorem 6.2 *In the set up of* 6.1, *let* $\underline{q}$ *be a polarization on* X *of total degree* $d = |\underline{q}|$ *and let* $\widehat{\underline{q}}$ *be the associated polarization on* $\widehat{X}$ (*see Section* 2.4). *Then there is a surjective* B_1*-morphism*

$$\tau_{\widehat{f}} : J^{\hat{\sigma}}_{\hat{f}}\left(\underline{\hat{q}}\right)_{sm} \cong N\big(\operatorname{Pic}^d \widehat{\mathcal{X}}_{K_1}\big) \longrightarrow J^{\sigma}_{f}\big(\underline{q}\big) \times_B B_1,$$

which is an isomorphism over the general point of B_1.

Proof. Let $P := \widehat{\sigma}(k_1) \in \widehat{X}_{\rm sm}$ and denote by v_0 the vertex of the dual graph $\Gamma_{\widehat{X}}$ of $\widehat{X}$ corresponding to the irreducible component of $\widehat{X}$ containing P. The fact that $J^{\hat{\sigma}}_{\hat{f}}\,(\underline{\hat{q}})_{\rm sm} \cong N\big(\operatorname{Pic}^d \widehat{\mathcal{X}}_{K_1}\big)$ is an immediate consequence of Theorem 4.1. By (4.1), we have

$$J^{\widehat{\sigma}}_{\widehat{f}}\,(\underline{\widehat{q}})_{\rm sm} \cong \frac{\coprod_{\underline{d} \in B^{v_0}_{\Gamma_{\widehat{X}}}(\underline{\widehat{q}})} \operatorname{Pic}^{\underline{d}}_{\widehat{f}}}{\sim_{K_1}},$$

where $\sim_{K_1}$ denotes the gluing along the general fibers of $\operatorname{Pic}^{\underline{d}}_{\widehat{f}}$ which are isomorphic to $\operatorname{Pic}^d\big(\widehat{X}_{K_1}\big)$, where $d = \big|\underline{\widehat{q}}\big| = \big|\underline{q}\big|$. We will start by showing the existence of a B_1-morphism

$$\tau^{\underline{d}}_{\widehat{f}} : \operatorname{Pic}^{\underline{d}}_{\widehat{f}} \longrightarrow J^{\sigma}_{f}\big(\,\underline{q}\,\big) \times_B B_1$$

for every $\underline{d} \in B^{v_0}_{\Gamma_{\widehat{X}}}(\underline{\widehat{q}})$. By the universal property of fiber products, the existence of $\tau^{\underline{d}}_{\widehat{f}}$ is equivalent the existence of a morphism $\mu^{\underline{d}}_{\widehat{f}} : \operatorname{Pic}^{\underline{d}}_{\widehat{f}} \to J^{\sigma}_{f}\big(\,\underline{q}\,\big)$ making the following diagram commute

$$\begin{tikzcd}
\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \arrow[rrd, bend left, "\mu^{\underline{d}}_{\widehat{f}}"] \arrow[rdd, bend right] & & \\
& J^{\sigma}_{f}\big(\,\underline{q}\,\big) \times_B B_1 \arrow[r] \arrow[d] & J^{\sigma}_{f}\big(\,\underline{q}\,\big) \arrow[d] \\
& B_1 \arrow[r] & B
\end{tikzcd}$$

Now, since $J^{\sigma}_{f}\big(\,\underline{q}\,\big)$ is a fine moduli space, such a morphism $\mu^{\underline{d}}_{\widehat{f}}$ is uniquely determined by a family of $(1,\sigma)$-quasistable torsion-free sheaves on $\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_B \mathcal{X}$ with respect to $\underline{q}$ $\big($since all the singular fibers of $\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_B \mathcal{X} \to \operatorname{Pic}^{\underline{d}}_{\widehat{f}}$ are isomorphic to X, we are slightly abusing the notation here: $\operatorname{Pic}^{\underline{d}}_{\widehat{f}}$ may very well not be a DVR$\big)$: we fix our notation according to the following commutative diagram, where both the outward and the left inward diagrams are cartesian and the morphism $\widehat{\pi}$ is the morphism induced by the inner commutativity of the diagram on the fiber product $\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_B \mathcal{X}$.

$$\begin{tikzcd}
\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_B \mathcal{X} \arrow[rrrd] \arrow[rdd, "\tilde{f}"] & & & \\
& \operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_{B_1} \widehat{\mathcal{X}}_1 \arrow[lu, "\widehat{\pi}"'] \arrow[r] \arrow[d, shift right, "\bar{f}"'] & \widehat{\mathcal{X}}_1 \arrow[r] \arrow[d, shift right, "\widehat{f}"'] & \mathcal{X} \arrow[d, shift right, "f"'] \\
& \operatorname{Pic}^{\underline{d}}_{\widehat{f}} \arrow[r] \arrow[u, shift right, bend right, "{(1,\widehat{\sigma})}"'] \arrow[luu, bend left, "{(1,\sigma)}"] & B_1 \arrow[r] \arrow[u, shift right, bend right, "\widehat{\sigma}"'] & B \arrow[u, shift right, bend right, "\sigma"']
\end{tikzcd}$$

The morphism $\widehat{\pi}$ is then a B-morphism that is an isomorphism over the general point of B while over the closed point of B consists of blowing down all the exceptional components of the morphism $\pi : \widehat{X} \to X$. Since $\widehat{f}$ is a family of projective curves with reduced and connected fibers having geometrically integral irreducible components and admitting a section $\widehat{\sigma}$, it follows from the work of Mumford in [33] that the relative Picard functor of $\widehat{f}$ is representable (see [22], Theorems 9.2.5 and 9.4.18.1). Therefore, there exists a Poincaré sheaf $\mathcal{P}$ on $\operatorname{Pic}^{\underline{d}}_{\widehat{f}} \times_{B_1} \widehat{\mathcal{X}}_1$ (see [22], Exercise 9.4.3), i.e., a sheaf whose restriction to a fiber of $\bar{f}$ at a point $[C, L]$ of $\operatorname{Pic}^{\underline{d}}_{\widehat{f}}$ is isomorphic to L. The above description of $\widehat{\pi}$ together with Theorem 6.1(iii) implies that $\mathcal{I} := \widehat{\pi}_*(\mathcal{P})$ is a family of $(1, \sigma)$-quasistable torsion-free sheaves with respect to $\underline{q}$ over the family $\tilde{f}$. This yields uniquely a morphism $\mu^{\underline{d}}_{\tilde{f}}$ as already observed.

By construction, over the general point $\operatorname{Spec} K_1$ of B_1, the morphism $\tau^{\underline{d}}_{\widehat{f}}$ restricts to the natural isomorphism

$$\operatorname{Pic}^{d}\left(\widehat{\mathcal{X}}_{K_1}\right) = \operatorname{Pic}^{d}(\mathcal{X}_K \times_K K_1) \xrightarrow{\cong} \operatorname{Pic}^{d}(\mathcal{X}_K) \times_K K_1.$$

Therefore, as $\underline{d}$ varies on $B^{v_0}_{\Gamma_{\widehat{X}}}\left(\widehat{\underline{q}}\right)$, we can glue the morphisms $\tau^{\underline{d}}_{\widehat{f}}$ along the general fiber to obtain the desired B_1-morphism $\tau_{\widehat{f}}$. By construction the B_1-morphism $\tau_{\widehat{f}}$ is an isomorphism over the general point of B_1 and, by Theorem 6.1(iii), it is surjective over the closed point of B_1. This concludes the proof of the statement. □

7 Comparing fine and coarse compactified Jacobians

In this section, we investigate when a fine compactified Jacobian is isomorphic to its coarse compactified Jacobian. Indeed, it turns out that the sufficient condition given by Esteves in [20, Theorem 4.4] is also necessary (for nodal curves).

Throughout the whole section we will use the terminology introduced in Section 2.4 above.

Theorem 7.1 *Let X be a nodal curve and $\underline{q}$ a polarization on X. The following conditions are equivalent*

(i) *The polarization $\underline{q}$ is non-degenerate*;

(ii) *For every $P \in X_{sm}$ the map $\Phi : J^P_X\left(\underline{q}\right) \to U_X\left(\underline{q}\right)$ is an isomorphism*;

(iii) *There exists a point $P \in X_{sm}$ such that the map $\Phi : J^P_X\left(\underline{q}\right) \to U_X\left(\underline{q}\right)$ is an isomorphism*;

(iv) *The number of irreducible components of $U_X\left(\underline{q}\right)$ is equal to the complexity $c(\Gamma_X)$ of the dual graph Γ_X of X.*

Proof. The implication (i) ⇒ (ii) follows from [20, Theorem 4.4]. In fact, note that, although the theorem of loc. cit. is stated in a weaker form, namely assuming the stronger hypothesis that $\underline{q}_Y - \frac{\delta_Y}{2} \notin \mathbb{Z}$ for all subcurves $Y \subsetneq X$ which are not spines, a closer look at its proof reveals that the theorem holds under the weaker hypothesis that $\underline{q}$ is not integral at all the subcurves $Y \subsetneq X$ which are not spines.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (iv) follows from the fact that the number of irreducible components of $J^P_X\left(\underline{q}\right)$ is equal to $c(\Gamma_X)$. Indeed, according to Theorem 5.1, the number of irreducible components of $J^P_X\left(\underline{q}\right)$ is equal to the number of irreducible components of $J^P_X(\underline{q})_{\mathrm{sm}}$, which, according to Proposition 5.4 applied to the case $S = \emptyset$, is equal to $c(\Gamma_X)$.

(iv) ⇒ (i): Fix a one-parameter regular local smoothing $f : \mathcal{X} \to B = \operatorname{Spec}(R)$ of X (see Section 2.3). Such a one-parameter smoothing determines a commutative diagram:

$$\begin{tikzcd}
 & N^d_X \arrow[d, "s"] & \\
J^{ss}_X\left(\underline{q}\right)_{\mathrm{sm}} \arrow[ur, two heads, "t"] \arrow[r, "p"] & U_X\left(\underline{q}\right) & \\
 & J^{ss}_X\left(\underline{q}\right)^0_{\mathrm{sm}} \arrow[ul, hook, "j'"'] \arrow[u, phantom, "\square"] \arrow[r, "p'"'] & U_X(\underline{q})_{\mathrm{sm}} \arrow[ul, hook, "j"'] \arrow[uul, "u"']
\end{tikzcd} \tag{7.1}$$

that we now explain. $N^d_X := N\left(\operatorname{Pic}^d \mathcal{X}_K\right)_k$ is the special fiber of the Néron model of $\operatorname{Pic}^d(\mathcal{X}_K)$ relative to f, where $d := \left|\underline{q}\right|$. $U_X\left(\underline{q}\right)_{\mathrm{sm}}$ denotes the smooth locus of $U_X\left(\underline{q}\right)$ and j is its open immersion into $U_X\left(\underline{q}\right)$. $J^{ss}_X\left(\underline{q}\right)_{\mathrm{sm}}$ denotes the variety parametrizing line bundles on X that are $\underline{q}$-semistable and p is the natural map sending a $\underline{q}$-semistable line bundle into its class in $U_X\left(\underline{q}\right)$, or in other words p is induced by the universal family of $\underline{q}$-semistable line bundles over $J_X^{ss}\left(\underline{q}\right)_{\text{sm}} \times X$. $J_X^{ss}\left(\underline{q}\right)_{\text{sm}}^0$ is, by definition, equal to

$$J_X^{ss}\left(\underline{q}\right)_{\text{sm}}^0 := U_X\left(\underline{q}\right)_{\text{sm}} \times_{U_X\left(\underline{q}\right)} J_X^{ss}\left(\underline{q}\right)_{\text{sm}},$$

and j', p' are the induced maps. The maps t and u are the special fibers of two maps over B induced by the Néron mapping property: indeed $J_X^{ss}\left(\underline{q}\right)_{\text{sm}}$ $\left(\text{resp. } U_X\left(\underline{q}\right)_{\text{sm}}\right)$ is the special fiber of a B-scheme $\operatorname{Pic}_f^{ss}$ $\left(\text{resp. } U_f\left(\underline{q}\right)_{\text{sm}}\right)$ smooth over B whose generic fiber is $\operatorname{Pic}^d(\mathcal{X}_K)$. Note also that the map t is the restriction to $J_X^{ss}\left(\underline{q}\right)_{\text{sm}} \subset \operatorname{Pic}^d(X)$ of the special fiber of the map $q : \operatorname{Pic}_f^d \to N\left(\operatorname{Pic}^d(\mathcal{X}_K)\right)$ (see (2.3)). From the explicit description of the map q given in Section 2.3 and the fact that every element in the degree class group Δ_X^d of X can be represented by a $\underline{q}$-semistable line bundle on X (as it follows from Proposition 3.5), we deduce that t is surjective. Finally, the map s is induced by the fact that $U_f\left(\underline{q}\right)$ is separated over B and $N\left(\operatorname{Pic}^d(\mathcal{X}_K)\right)$ is the biggest separated quotient of the non-separated B-scheme $\operatorname{Pic}_f^{ss}$ (see Section 2.3).

Claim 1: p' is surjective.

Consider a polystable sheaf $\mathcal{I} \in U_X\left(\underline{q}\right)_{\text{sm}}$. According to Fact 2.13(ii), the set of nodes $\operatorname{NF}(\mathcal{I})$ at which $\mathcal{I}$ is not free is contained in X_{sep}. The surjectivity of p' is equivalent to showing that there exists a $\underline{q}$-semistable line bundle L in the same S-equivalence class of $\mathcal{I}$. By decreasing induction on the cardinality of $\operatorname{NF}(\mathcal{I})$, it is enough to show that given $n \in \operatorname{NF}(\mathcal{I})$ there exists $\mathcal{I}' \in U_X\left(\underline{q}\right)_{\text{sm}}$ such that $\mathcal{I}'$ is S-equivalent to $\mathcal{I}$ and $\operatorname{NF}(\mathcal{I}') = \operatorname{NF}(\mathcal{I}) \setminus \{\mathrm{n}\}$. Let T_1 and T_2 be the tails attached to n, and set $I_i := I_{T_i}$. Since n is a separating node, it follows from [19, Example 38] that $\mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2$. To conclude, it is enough to take a non-trivial extension

$$0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}' \longrightarrow \mathcal{I}_2 \longrightarrow 0,$$

whose existence follows from [19, Lemma 4].

Claim 2: If u is surjective then $\operatorname{Im} p \subseteq U_X\left(\underline{q}\right)_{\text{sm}}$.

If u is surjective then, using that p' is surjective by Claim 1, we get that $t \circ j' = u \circ p'$ is surjective. From the diagram (7.1) we easily get that $\operatorname{Im}(s \circ t \circ j') \subseteq U_X\left(\underline{q}\right)_{\text{sm}}$. This, together with the surjectivity of $t \circ j'$ implies that $\operatorname{Im} s \subseteq U_X\left(\underline{q}\right)_{\text{sm}}$. Since $\operatorname{Im} p \subseteq \operatorname{Im} s$ because t is surjective, we get the conclusion.

Let us now conclude the proof of the implication (iv) $\Rightarrow$ (i). Assume that the number of irreducible components of $U_X\left(\underline{q}\right)$ is equal to $c(\Gamma_X)$. This means that u is surjective (and hence an isomorphism). By Claim 2, we deduce that $\operatorname{Im} p \subseteq U_X\left(\underline{q}\right)_{\text{sm}}$. We claim that this implies that $\underline{q}$ is non-degenerate. Indeed, if this were not the case then, by Lemma 7.2 below, there would exist a $\underline{q}$-semistable line bundle L such that $\deg_Z L = \underline{q}_Z - \frac{\delta_Z}{2}$ for some proper subcurve $Z \subsetneq X$ which is not a spine. But then clearly $Z \cap Z^c \subset \operatorname{NF}(\operatorname{Gr}(\mathrm{L})) \not\subset X_{\text{sep}}$ which would imply that $p(L) = [\operatorname{Gr}(L)] \notin U_X\left(\underline{q}\right)_{\text{sm}}$ by Fact (2.13). □

Lemma 7.2 *If a polarization $\underline{q}$ on X is not general then there exists a subcurve $Z \subsetneq X$ with both Z and Z^c connected and a $\underline{q}$-semistable line bundle L on X such that* $\deg_Z L = \underline{q}_Z - \frac{\delta_Z}{2}$. *Moreover, if $\underline{q}$ is not non-degenerate, then we can choose Z not to be a spine.*

Proof. By assumption, $\underline{q}$ is integral at a proper subcurve $Y \subsetneq X$. Chose a connected component of Y and call it Z'. Set Z to be one of the connected components of Z'^c. Clearly Z and Z^c are connected.

If moreover $\underline{q}$ is not non-degenerate then there exists a subcurve $Y \subsetneq X$ as before which, moreover, is not a spine. Then we can chose a subcurve Z' as before in such a way that is it not a spine. This easily implies that Z is not a spine as well.

From the assumption that $\underline{q}$ is integral at Y and from the construction of Z, we deduce that $\underline{q}_Z - \frac{\delta_Z}{2} \in \mathbb{Z}$ and that $\underline{q}_{Z^c} - \frac{\delta_{Z^c}}{2} = \left|\underline{q}\right| - \underline{q}_Z - \frac{\delta_Z}{2} \in \mathbb{Z}$.

Consider the restriction $\underline{q}_{|Z}$ of the polarization $\underline{q}$ at Z (see Section 2.4). Since Z is connected, the complexity of its dual graph Γ_Z is at least one and therefore Proposition 5.4 implies that, for any chosen smooth point $P \in X_{\text{sm}}$, there exists a line bundle L_1 on Z that is $\underline{q}_{|Z}$-P-quasistable, and in particular $\underline{q}_{|Z}$-semistable. This means that for any subcurve $W_1 \subset Z$ it holds:

$$\begin{cases} \deg_Z L_1 = \left|\underline{q}_{|Z}\right| = \underline{q}_Z - \dfrac{\delta_Z}{2}, \\ \deg_{W_1} L_1 \geq \left(\underline{q}_{|Z}\right)_{W_1} - \dfrac{\left|W_1 \cap \overline{Z \setminus W_1}\right|}{2} = \underline{q}_{W_1} - \dfrac{|W_1 \cap Z^c|}{2} - \dfrac{\left|W_1 \cap \overline{Z \setminus W_1}\right|}{2} \\ \qquad\qquad = \underline{q}_{W_1} - \dfrac{\delta_{W_1}}{2}. \end{cases} \tag{7.2}$$

Analogously, consider the polarization $\widetilde{\underline{q}}$ on Z^c given by

$$\widetilde{\underline{q}}_R := \underline{q}_R + \frac{|R \cap Z|}{2} \quad \text{for any subcurve } R \subset Z^c.$$

Since Z^c is connected, there exists a line bundle L_2 on Z^c that is $\widetilde{\underline{q}}$-semistable, i.e., such that for any subcurve $W_2 \subset Z^c$ it holds:

$$\begin{cases} \deg_{Z^c} L_2 = |\widetilde{\underline{q}}| = \underline{q}_{Z^c} + \dfrac{\delta_{Z^c}}{2}, \\ \deg_{W_2} L_2 \geq \widetilde{\underline{q}}_{W_2} - \dfrac{\left|W_2 \cap \overline{Z^c \setminus W_2}\right|}{2} = \underline{q}_{W_2} + \dfrac{|W_2 \cap Z|}{2} - \dfrac{\left|W_2 \cap \overline{Z^c \setminus W_2}\right|}{2} \\ \qquad\qquad = \underline{q}_{W_2} - \dfrac{\delta_{W_2}}{2} + |W_2 \cap Z|. \end{cases} \tag{7.3}$$

Now let L be a line bundle on X such that $L_Z = L_{|Z} = L_1$ and $L_{Z^c} = L_{|Z^c} = L_2$ (obviously such an L exists). Using Equations (7.2) and (7.3), we have that

$$\deg L = \deg_Z L_1 + \deg_{Z^c} L_2 = \underline{q}_Z - \frac{\delta_Z}{2} + \underline{q}_{Z^c} + \frac{\delta_{Z^c}}{2} = \left|\underline{q}\right|. \tag{7.4}$$

For any subcurve $W \subset X$, let $W = W_1 \cup W_2$ where $W_1 := W \cap Z$ and $W_2 := W \cap Z^c$. Using Equations (7.2) and (7.3), we compute

$$\begin{aligned} \deg_W L \ &= \ \deg_{W_1} L_1 + \deg_{W_2} L_2 \geq \underline{q}_{W_1} - \frac{\delta_{W_1}}{2} + \underline{q}_{W_2} - \frac{\delta_{W_2}}{2} + |W_2 \cap Z| \\ &\geq \ \underline{q}_W - \frac{\delta_{W_1}}{2} - \frac{\delta_{W_2}}{2} + |W_1 \cap W_2| = \underline{q}_W - \frac{\delta_W}{2}. \end{aligned} \tag{7.5}$$

The above Equations (7.4) and (7.5) says that L is $\underline{q}$-semistable. On the other hand, from Equation (7.2) we get $\deg_Z L = \underline{q}_Z - \frac{\delta_Z}{2}$. □

7.1 Relation between non-degenerate and general polarizations

The aim of this subsection is to discuss the relation between a polarization $\underline{q}$ being non-degenerate and the stronger condition of being general (see Definition 2.8). We begin by describing the geometric meaning of being general.

Proposition 7.3 *The following conditions are equivalent*:

(i) $\underline{q}$ *is general* (*see Definition* 2.8(i));

(ii) *Every* $\underline{q}$*-semistable sheaf is* $\underline{q}$*-stable, i.e.,* $U_X^s\left(\underline{q}\right) = U_X\left(\underline{q}\right)$;

(iii) *Every* $\underline{q}$*-semistable simple sheaf is* $\underline{q}$*-stable, i.e.,* $J_X^s\left(\underline{q}\right) = J_X^{ss}\left(\underline{q}\right)$;

(iv) *Every* $\underline{q}$*-semistable line bundle is* $\underline{q}$*-stable.*

Proof. (i) ⇒ (ii): If $\underline{q}$ is general then the right-hand side of the inequality (2.10) is never an integer. Hence the inequality in (2.10), if satisfied, is always strict, from which the conclusion follows.

The implications (ii) ⇒ (iii) ⇒ (iv) are clear.

(iv) ⇒ (i): If $\underline{q}$ is not general, then Lemma 7.2 implies that there exists a $\underline{q}$-semistable line bundle L on X that is not $\underline{q}$-stable. □

Remark 7.4 The implication (i) $\Rightarrow$ (iii) was proved in [20, Proposition 3.5].

Remark 7.5 The canonical polarization of degree d on X of Remark 2.11(ii) is general if and only X is d-general in the sense of [10, Corolary-Definition 4.13] (see also [13, Definition 1.13]), as it follows easily by comparing the definition of loc. cit. with the above Proposition 7.3.

In the remaining of this subsection, we want to give an answer to the following

Question 7.6 *How far is a non-degenerate polarization from being general?*

Denote by X^2 any smoothing of X at the set of separating nodes X_{sep} of X. Given a subcurve $Z \subset X^2$, denote by $\overline{Z}$ the subcurve of X to which Z specializes. Observe that $g_{\overline{Z}} = g_Z$ and $\delta_{\overline{Z}} = \delta_Z$. A subcurve $Y \subset X$ is of the form $Y = \overline{Z}$ for some subcurve $Z \subset X^2$ if and only if

$$Y \cap Y^c \cap X_{\text{sep}} = \emptyset. \tag{7.6}$$

Given a polarization $\underline{q}$ on X, we define a polarization $\underline{q}^2$ on any smoothing X^2 by $\underline{q}^2_Z := \underline{q}_{\overline{Z}}$ for any subcurve $Z \subset X^2$. Observe that, although the smoothing X^2 is not unique, its combinatorial type (i.e., its weighted dual graph) and the polarization $\underline{q}^2$ are uniquely determined.

Proposition 7.7 *A polarization $\underline{q}$ on X is non-degenerate if and only if, for every* (*or equivalently, for some*) *smoothing X^2 of X at its set of separating nodes, the induced polarization $\underline{q}^2$ on X^2 is general.*

Proof. Assume that $\underline{q}$ is non-degenerate on X. Let Z be a proper subcurve of any fixed smoothing X^2 and W a connected component of Z or Z^c. We want to show that $\underline{q}^2_W - \frac{\delta_W}{2} \notin \mathbb{Z}$. Consider the subcurve $\overline{Z} \subset X$. Clearly $\overline{Z}$ is a proper subcurve and is not a spine because of (7.6). Moreover $\overline{W}$ is a connected component of $\overline{Z}$ or $\overline{Z}^c$. Therefore, because of the assumption and the definition of $\underline{q}^2$, we get $\underline{q}^2_W - \frac{\delta_W}{2} = \underline{q}_{\overline{W}} - \frac{\delta_{\overline{W}}}{2} \notin \mathbb{Z}$.

Conversely, assume that $\underline{q}^2$ is general for some fixed smoothing X^2 and, by contradiction, assume also that $\underline{q}$ is not non-degenerate on X. Then there exists some subcurve Y of X such that

$$\begin{cases} Y \text{ is connected,} \\ Y \cap Y^c \not\subset X_{\text{sep}} \quad \text{(i.e., } Y \text{ is not a spine)}, \\ \underline{q} \text{ is integral at } Y. \end{cases} \tag{7.7}$$

If we chose Y maximal among the subcurves satisfying the properties (7.7), then we claim that $Y \cap Y^c \cap X_{\text{sep}} = \emptyset$. Indeed, if this is not the case, then there exists a separating node $n \in Y \cap Y^c$. Since Y is connected, one of the two tails attached to n, call it T, is a connected component of Y^c. Consider the subcurve $Y' := Y \cup T$. It is easily checked that Y' is connected, $Y' \cap Y'^c = (Y \cap Y^c) \setminus \{n\} \not\subset X_{\text{sep}}$ and that $\underline{q}$ is integral at Y'. Therefore Y' satisfies the properties (7.7) and, since $Y \subsetneq Y'$, this contradicts the maximality of Y.

Since the chosen maximal subcurve Y satisfies property (7.6), we known that there exists a subcurve $Z \subsetneq X^2$ such that $\overline{Z} = Y$. But then the same argument as before gives that $\underline{q}^2$ is integral at Z, which contradicts the initial assumption on $\underline{q}^2$. ☐

Remark 7.8 The canonical polarization of degree d on X of Remark 2.11(ii) is non-degenerate if and only X is weakly d-general in the sense of [13, Definition 1.13], as it follows easily by comparing the definition of loc. cit. with the above Proposition 7.7. Using this, the equivalence (i) $\Leftrightarrow$ (iv) of our Theorem 7.1 recovers [13, Theorem 2.9] in the case of the canonical polarization of degree d.

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