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From Lie Algebras to Quantum Groups

Helena Albuquerque
Samuel Lopes
Joana Teles

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Morada: Almas de Freire, Coimbra
Código Postal: 3040 COIMBRA
Correio Electrónico: cim@mat.uc.pt
Telefone: 239 802 370
Fax: 239 445 380

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Workshop *From Lie Algebras to Quantum Groups*

Helena Albuquerque* Samuel Lopes† Joana Teles‡

Foreword

This workshop brought together leading specialists in the topics of Lie algebras, quantum groups and related areas. It aimed to present the latest developments in these areas as well as to stimulate the interaction between young researchers and established specialists.

We remark on the significant role that, for the last decades, the theory of algebras has played in the development of some areas of physics, and conversely, the importance that the growth of physics has had in the implementation of new algebraic structures. In fact, with the development of physics, more complex algebraic structures have arisen and the mathematical structures that were used to explain certain physical phenomena became insufficient. For example, there are two structures of considerable importance in contemporary mathematical research that are deeply connected to the theory of Lie algebras: Lie superalgebras and quantum groups.

The new supersymmetry theories that appeared in the 80's presented, within the same structure, particles that satisfied commutation relations and others that satisfied anti-commutation relations. The algebras that existed up until then did not exhibit such a structure, and that led to the emergence of a new mathematical structure: Lie superalgebras. The genesis of quantum groups is quite similar. Introduced independently by Drinfel'd and Jimbo in 1985, quantum groups appeared in connection with a quantum mechanical problem in statistical mechanics: the quantum Yang-Baxter equation. It was realized then that quantum groups had far-reaching applications in theoretical physics (e.g. conformal and quantum field theories), knot theory, and virtually all other areas of mathematics. The notion of a quantum group is also closely related to the study of integrable dynamical systems, from which the concept of a Poisson-Lie group emerged, and to the classical Weyl-Moyal quantization.

*Departamento de Matemática, Universidade de Coimbra, Portugal.
†Departamento de Matemática Pura, Universidade do Porto, Portugal.
‡Departamento de Matemática, Universidade de Coimbra, Portugal.

The Organizing Committee members were:

- Helena Albuquerque, *Departamento de Matemática, Universidade de Coimbra, Portugal*
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From Clifford algebras to Cayley algebras

Helena Albuquerque*

1 Introduction

In this paper I present some of the main results on the theory of quasiassociative algebras that were published untill June 2006, in a study developed in collaboration with Shahn Majid, Alberto Elduque, José Pérez-Izquierdo and A.P. Santana. It is a kind of structure that has begin to be studied by the author and S. Majid in 1999 in the context of the theory of cathegories of quasi-Hopf algebras and comodules [6] that generalizes the known theory of associative algebras. Octonions appears in [6] as an algebra in the monoidal cathegory of G-graded vector spaces. At the algebraic point of view this theory presents new examples of a kind of quasilinear algebra that deals with a set of matrices endowed with a non associative multiplication. In [6] the notion of a dual quasi-Hopf algebra was obtained dualizing Drinfeld's axioms.

A *dual quasibialgebra* is a $(H, \Delta, \epsilon, \phi)$ where the coproduct $\Delta : H \to H \otimes H$ and counit $\epsilon : H \to k$ form a coalgebra and are multiplicative with respect to a 'product' $H \otimes H \to H$. Besides, H is associative up to'conjugation' by ϕ in the sense

$$\sum a_{(1)} \cdot (b_{(1)} \cdot c_{(1)}) \phi(a_{(2)}, b_{(2)}, c_{(2)}) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) (a_{(2)} \cdot b_{(2)}) \cdot c_{(2)} \tag{1}$$

for all $a, b, c \in H$ where $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ is a notation and ϕ is a unital 3-cocycle in the sense

$$\begin{aligned} &\sum \phi(b_{(1)}, c_{(1)}, d_{(1)}) \phi(a_{(1)}, b_{(2)} c_{(2)}, d_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) = \\ &\sum \phi(a_{(1)}, b_{(1)}, c_{(1)} d_{(1)}) \phi(a_{(2)} b_{(2)}, c_{(2)}, d_{(2)}), \end{aligned} \tag{2}$$

for all $a, b, c, d \in H$, and $\phi(a, 0, b) = \epsilon(a)\epsilon(b)$ for all $a, b \in H$. Also ϕ is convolution-invertible in the algebra of maps $H^{\otimes 3} \to k$, i.e. that there exists $\phi^{-1} : H^{\otimes 3} \to k$ such that

$$\begin{aligned} &\sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)}) = \epsilon(a)\epsilon(b)\epsilon(c) \\ &= \sum \phi^{-1}(a_{(1)}, b_{(1)}, c_{(1)}) \phi(a_{(2)}, b_{(2)}, c_{(2)}) \end{aligned} \tag{3}$$

*Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal. E-mail:lena@mat.uc.pt. Supported by CMUC-FCT.

for all $a, b, c \in H$. A dual quasibialgebra is a *dual quasi-Hopf algebra* if there is a linear map $S : H \to H$ and linear functionals $\alpha, \beta : H \to k$ such that

$$\begin{array}{l}\sum(Sa_{(1)})a_{(3)}\alpha(a_{(2)}) = 1\alpha(a), \sum a_{(1)}Sa_{(3)}\beta(a_{(2)}) = 1\beta(a),\\ \sum\phi(a_{(1)}, Sa_{(3)}, a_{(5)})\beta(a_{(2)})\alpha(a_{(4)}) = \epsilon(a),\\ \sum\phi^{-1}(Sa_{(1)}, a_{(3)}, Sa_{(5)})\alpha(a_{(2)}\beta(a_{(4)}) = \epsilon(a)\end{array} \tag{4}$$

for all $a \in H$.

H is called *dual quasitriangular* if there is a convolution-invertible map $R : H \otimes H \to k$ such that

$$R(a\cdot b, c) = \sum\phi(c_{(1)}, a_{(1)}, b_{(1)})R(a_{(2)}, c_{(2)})\phi^{-1}(a_{(3)}, c_{(3)}, b_{(2)})R(b_{(3)}, c_{(4)})\phi(a_{(4)}, b_{(4)}, c_{(5)}), \tag{5}$$

$$\begin{array}{l}R(a, b\cdot c) =\\ \sum\phi^{-1}(b_{(1)}, c_{(1)}, a_{(1)})R(a_{(2)}, c_{(2)})\phi(b_{(2)}, a_{(3)}, c_{(3)})R(a_{(4)}, b_{(3)})\phi^{-1}(a_{(5)}, b_{(4)}, c_{(4)})\end{array} \tag{6}$$

$$\sum b_{(1)}\cdot a_{(1)}R(a_{(2)}, b_{(2)}) = \sum R(a_{(1)}, b_{(1)})a_{(2)}\cdot b_{(2)} \tag{7}$$

for all $a, b, c \in H$.

A *corepresentation* or comodule under a coalgebra means vector space V and a map $\beta : V \to V \otimes H$ obeying $(id \otimes \Delta)\circ\beta = (\beta\otimes id)\circ\Delta$ and $(id\otimes\epsilon)\circ\beta = id$.

If F is any convolution-invertible map, $F : H \otimes H \to k$ obeying $F(a, 0) = F(0, a) = \epsilon(a)$ for all $a \in H$ (a 2-cochain) and H is a dual quasi-Hopf algebra then so is H_F with the new product $\cdot_F$, $\phi_F, R_F, \alpha_F, \beta_F$ given by

$$\begin{array}{rcl}a\cdot_F b &=& \sum F^{-1}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}F(a_{(3)}, b_{(3)})\\ \phi_F(a, b, c) &=& \sum F^{-1}(b_{(1)}, c_{(1)})F^{-1}(a_{(1)}, b_{(2)}c_{(2)})\phi(a_{(2)}, b_{(3)}, c_{(3)})F(a_{(3)}b_{(4)}, c_{(4)})F(a_{(4)}, b_{(5)})\\ \alpha_F(a) &=& \sum F(Sa_{(1)}, a_{(3)})\alpha(a_{(2)})\\ \beta_F(a) &=& \sum F^{-1}(a_{(1)}, Sa_{(3)})\beta(a_{(2)})\\ R_F(a, b) &=& \sum F^{-1}(b_{(1)}, a_{(1)})R(a_{(2)}, b_{(2)})F(a_{(3)}, b_{(3)})\end{array} \tag{8}$$

for all $a, b, c \in H$. This is the dual version of the twisting operation or 'gauge equivalence' of Drinfeld, so called because it does not change the category of comodules up to monoidal equivalence.

So the notion of comodule quasialgebra with the approach described in [6] is,

Definition 1.1. *A G-graded algebra A is quasiassociative if is a G-graded vector space $A = \oplus_{g\in G}A_g$ that satisfies*

$$(ab)c = \phi(\bar{a}, \bar{b}, \bar{c})a(bc), \forall_{a\in A_{\bar{a}}, b\in A_{\bar{b}}, c\in A_{\bar{c}}}, \tag{9}$$

for any invertible group cocycle $\phi : G\times G\times G \to K^*$ with

$$\phi(x, y, z)\phi(y, z, t) = \frac{\phi(xy, z, t)\phi(x, y, zt)}{\phi(x, yz, t)}, \quad \phi(x, 0, y) = 1, \forall x, y, z, t \in G. \tag{10}$$

($\Rightarrow A_0$ is an associative algebra and A_g is an A_0- bimodule, $\forall g \in G$)

Examples of quasiassociative algebras are K_FG algebras: K_FG is the same vector space as the group algebra KG but with a different product $a.b = F(a,b)ab, \forall_{a,b \in G}$, where F is a 2-cochain on G. For this class of algebras the cocycle ϕ depends on F,

$$\phi(x,y,z) = \frac{F(x,y)F(xy,z)}{F(y,z)F(x,yz)}, x,y,z \in G \tag{11}$$

and measures the associativity of the algebra. Also the map $R(x,y) = \frac{F(x,y)}{F(y,x)}, x,y \in G$ has an important role in the study of these algebras because it measures the commutativity.

2 Examples of K_FG algebras

In this section we exemplify some results that we have proved in [4,7]. Some known classes of K_FG algebras were studied and were characterized by the properties of the cochain F. For example, we have studied some conexions between alternative K_FG algebras and composition algebras,

Theorem 2.1. *k_FG is an alternative algebra if and only if for all $x, y \in G$ we have,*

$$\begin{array}{rcl} \phi^{-1}(y,x,z) + R(x,y)\phi^{-1}(x,y,z) & = & 1 + R(x,y) \\ \phi(x,y,z) + R(z,y)\phi(x,z,y) & = & 1 + R(z,y). \end{array} \tag{12}$$

In this case, $\phi(x,x,y) = \phi(x,y,y) = \phi(x,y,x) = 1$

Theorem 2.2. *If $G \simeq (Z_2)^n$ then the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ makes k_FG a composition algebra if and only if $F^2(x,y) = 1$ for all $x, y \in G$ and $F(x,xz)F(y,yz) + F(x,yz)F(y,xz) = 0$ for all $x,y,z \in G$ with $x \neq y$.*

Theorem 2.3. *If $\sigma(x) = F(x,x)x$ for all $x \in G$ is a strong involution, and $F^2 = 1$, then the following are equivalent,*

i) k_FG is an alternative algebra,

ii) k_FG is a composition algebra.

Cayley Dickson Process was studied also in [6], and it was proved that after applying this process to a K_FG algebra we obtain another $K_{\bar{F}}\bar{G}$ algebra related to the first one which properties are predictable. In fact,

Theorem 2.4. *Let G be a finite abelian group, F a cochain on it (k_FG is a G-graded quasialgebra). For any $s : G \to k^*$ with $s(e) = 1$ we define $\bar{G} = G \times Z_2$ and on it the cochain $\bar{F}$ and function $\bar{s}$,*

$\bar{F}(x,y) = F(x,y), \bar{F}(x,vy) = s(x)F(x,y),$

$\bar{F}(vx,y) = F(y,x), \bar{F}(vx,vy) = \alpha s(x)F(y,x),$

$\bar{s}(x) = s(x), \bar{s}(vx) = -1$ *for all* $x, y \in G$.

Here $x \equiv (x,e)$ *and* $vx \equiv (x,\nu)$ *denote elements of* $\bar{G}$*, where* $Z_2 = \{e,\nu\}$ *with product* $\nu^2 = e$.

If $\sigma(x) = s(x)x$ *is a strong involution, then* $k_{\bar{F}}\bar{G}$ *is the algebra obtained from Cayley-Dickson process applied to* $k_F G$.

We conclude that if $G = (Z_2)^n$ and $F = (-1)^f$ then the standard Cayley-Dickson process has $\bar{G} = (Z_2)^{n+1}$ and $\bar{F} = (-1)^{\bar{f}}$. Using the notation $\vec{x} = (x_1, \cdots, x_n) \in (Z_2)^n$ where $x_i \in \{0,1\}$ we have

$\bar{f}((\vec{x}, x_{n+1}), (\vec{y}, y_{n+1})) = f(\vec{x},\vec{y})(1 - x_{n+1}) + f(\vec{y},\vec{x})x_{n+1} + y_{n+1}f(\vec{x},\vec{x}) + x_{n+1}y_{n+1}$.

Theorem 2.5.

(i) The 'complex number' algebra has this form with $G = Z_2, f(x,y) = xy, x, y \in Z_2$ *where we identify* G *as the additive group* Z_2 *but also make use of its product.*

(ii) The quaternion algebra is of this form with

$$\bar{G} = Z_2 \times Z_2, \bar{f}(\vec{x},\vec{y}) = x_1y_1 + (x_1 + x_2)y_2$$

where $\vec{x} = (x_1, x_2) \in \bar{G}$ *is a vector notation.*

(iii) The octonion algebra is of this form with

$$\bar{\bar{G}} = Z_2 \times Z_2 \times Z_2, \bar{\bar{f}}(\vec{x},\vec{y}) = \sum_{i \le j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3.$$

(iv) The 16-onion algebra is of this form with

$$\bar{\bar{\bar{G}}} = Z_2 \times Z_2 \times Z_2 \times Z_2$$

and

$$\bar{\bar{\bar{f}}}(\vec{x},\vec{y}) = \sum_{i \le j} x_i y_j + \sum_{i \ne j \ne k \ne i} x_i x_j y_k + \sum_{distinct\ i,j,k,l} x_i x_j y_k y_l + \sum_{i \ne j \ne k \ne i} x_i y_j y_k x_4.$$

Now consider a n dimensional vector space V over a field K with characteristic not 2. Define in V a nondegenerate quadratic form $\mathbf{q}$. We know that there is an orthogonal basis $\{e_1, \cdots, e_n\}$ of V with $\mathbf{q}(e_i) = q_i$ for some $q_i \ne 0$. The Clifford algebra $C(V, \mathbf{q})$, is the associative algebra generated by 1 and $\{e_i\}$ with the relations $e_i^2 = q_i.1, e_ie_j + e_je_i = 0, \forall i \ne j$. The dimension of $C(V, \mathbf{q})$ is 2^n and it has a canonical basis $\{e_{i_1} \cdots e_{i_p} | 1 \le i_1 < i_2 \cdots < i_p \le n\}$.

In [9] we have studied Clifford Algebras as quasialgebras $K_F G$ and some of their representations,

Theorem 2.6. *The algebra $k_F Z_2^n$ can be identified with $C(V,\mathbf{q})$, where $F \in Z^2(G,k)$ is defined by $F(x,y) = (-1)^{\sum_{j<i} x_i y_j} \prod_{i=1}^{n} q_i^{x_i y_i}$ where $x = (x_1, \cdots x_n) \in Z_2^n$ and twists kG into a cotriangular Hopf algebra with $R(x,y) = (-1)^{\rho(x)\rho(y)+x\cdot y}$ where $\rho(x) = \sum_i x_i$ and $x \cdot y$ is the dot product of Z_2-valued vectors.*

$C(V,\mathbf{q})$ is a superalgebra considering the Z_2 graduation induced by the map $\sigma(e_x) = (-1)^{\rho(x)} e_x$ extended linearly.

Theorem 2.7. $C(V \oplus W, \mathbf{q} \oplus \mathbf{p}) \simeq C(V,\mathbf{q}) \otimes C(W,\mathbf{p})$ *as super algebras.*

(This result is known in the classical theory but with our approach is easier to prove. It is clear from the form of F that $F((x,x'),(y,y')) = F(x,y)F(x',y')(-1)^{\rho(x')\rho(y)}$ where $\{e_x\}$ is a basis of V and $\{e_{x'}\}$ of W with the multiplication defined in the algebra product by $(a \otimes c)(b \otimes d) = a \cdot b \otimes c \cdot d(-1)^{\rho(c)\rho(b)}$.)

We now use the above convenient description of Clifford algebras to express a 'doubling process' similar to the Cayley Dickson process. Let A be a finite-dimensional algebra with identity 1 and σ an involutive automorphism of A. For any fixed element $q \in k^*$ there is a new algebra of twice the dimension, $\bar{A} = A \oplus Av$, which multiplication is given by

$$(a + bv) \cdot (c + dv) = a \cdot c + qb \cdot \sigma(d) + (a \cdot d + b \cdot \sigma(c))v,$$

and with a new involutive automorphism,

$$\bar{\sigma}(a + vb) = \sigma(a) - \sigma(b)v.$$

We will say that $\bar{A}$ is obtained from A by Clifford process.

Theorem 2.8. *Let G be a finite Abelian group and F a cochain as above. So $k_F G$ is a G-graded quasialgebra and for any $s : G \to k^*$ with $s(e) = 1$ and any $q \in k^*$, define $\bar{G} = G \times Z_2$ and*

$$\bar{F}(x, yv) = F(x,y) = \bar{F}(x,y),$$

$$\bar{F}(xv, y) = s(y)F(x,y), \bar{F}(xv, yv) = qs(y)F(x,y),$$

$$\bar{s}(x) = s(x), \bar{s}(xv) = -s(x) \text{ for all } x, y \in G.$$

Here $x \equiv (x,e)$ and $xv \equiv (x,\eta)$ where η with $\eta^2 = e$ is the generator of the Z_2. If $\sigma(e_x) = s(x)e_x$ is an involutive automorphism then $k_{\bar{F}}\bar{G}$ is the Clifford process applied to $k_F G$.

Theorem 2.9. *For any $s : G \to k^*$ and $q \in k^*$ as above the $k_{\bar{F}}\bar{G}$ given by the generalised Clifford process has associator and braiding*

$$\bar{\phi}(x, yv, z) = \bar{\phi}(x, y, zv) = \bar{\phi}(x, yv, zv) = \phi(x, y, z),$$

$$\bar{\phi}(xv, y, z) = \bar{\phi}(xv, yv, z) = \phi(xv, y, zv) = \phi(xv, yv, zv) = \phi(x, y, z)\frac{s(yz)}{s(y)s(z)}$$

$$\bar{R}(x, y) = R(x, y), \bar{R}(xv, y) = s(y)R(x, y)$$

$$\bar{R}(x, yv) = \frac{R(x, y)}{s(x)},$$

$$\bar{R}(xv, yv) = R(x, y)\frac{s(y)}{s(x)}.$$

Theorem 2.10. *If s defines an involutive automorphism σ then*

1. *$k_{\bar{F}}\bar{G}$ is alternative iff $k_F G$ is alternative and for all $x, y, z \in G$, either $\phi(x, y, z) = 1$ or $s(x) = s(y) = s(z) = 1$.*
2. *$k_{\bar{F}}\bar{G}$ is associative iff $k_F G$ is associative.*

If F and s have the form $F(x, y) = (-1)^{f(x,y)}, s(x) = (-1)^{\xi(x)}, q = (-1)^{\zeta}$ for some Z_2-valued functions f, ξ and $\zeta \in Z_2$ then the generalised Clifford process yields the same form with $G = Z_2^{n+1}$ and $\bar{f}((x, x_{n+1}), (y, y_{n+1})) = f(x, y)+(y_{n+1}\zeta+\xi(y))x_{n+1}, \bar{\xi}(x, x_{n+1}) = \xi(x)+x_{n+1}$.

Theorem 2.11. *Starting with $C(r, s)$ the Clifford process with $q = 1$ yields $C(r+1, s)$. With $q = -1$ it gives $C(r, s + 1)$. Hence any $C(m, n)$ with $m \geq r, n \geq s$ can be obtained from successive applications of the Clifford process from $C(r, s)$.*

So as an immediate consequence of the last theorem we can say that starting with $C(0, 0) = k$ and iterating the Clifford process with a choice of $q_i = (-1)^{\zeta_i}$ at each step, we arrive at the standard $C(V, \mathbf{q})$ and the standard automorphism $\sigma(e_x) = (-1)^{\rho(x)}e_x$.

Besides we must note that the Clifford Process is a twisting tensor product $\bar{A} = A \otimes_\sigma C(k, q)$ where $C(k, q) = k[v]$ with the relation $v^2 = q$, and $va = \sigma(a)v$ for all $a \in A$.

3 Quasirepresentations and quasilinear algebra

In [9] we have extended Clifford process also to representations proving that,

Theorem 3.1. *If W is an irreducible representation of A not isomorphic to W_σ defined by the action of $\sigma(a)$ then $\bar{W} = W \oplus W_\sigma$ is an irreducible representation π of $\bar{A}$ obtained via the Clifford process with q. Here*

$$\pi(v) = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix},$$

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$$

are the action on $W \oplus W$ in block form (here $\pi(a)$ is the explicit action of a in the direct sum representation $W \oplus W_\sigma$). If W, W_σ are isomorphic then W itself is an irreducible representation of $\bar{A}$ for a suitable value of q.

More generaly we have defined an action of a quasialgebra in [6],

Theorem 3.2. *A representation or 'action' of a G-graded quasialgebra A is a G-graded vector space V and a degree-preserving map $\circ : A \otimes V \to V$ such that $(ab) \circ v = \phi(|a|, |b|, |v|) a \circ (b \circ v), 1 \circ v = v$ on elements of homogeneous degree. Here $|a \circ v| = |a||v|$.*

Theorem 3.3. *Let $|i| \in G$ for $i = 1, \cdots, n$ be a choice of grading function. Then the usual $n \times n$ matrices M_n with the new product*

$$(\alpha \cdot \beta)^i{}_j = \sum_k \alpha^i{}_k \beta^k{}_j \frac{\phi(|i|, |k|^{-1}, |k||j|^{-1})}{\phi(|k|^{-1}, |k|, |j|^{-1})}, \forall \alpha, \beta \in M_n$$

form a G-graded quasialgebra $M_{n,\phi}$, where $|E_i{}^j| = |i||j|^{-1} \in G$ is the degree of the usual basis element of M_n. An action of a G-graded quasialgebra A in the n-dimensional vector space with grading $|i|$ is equivalent to an algebra map $\rho : A \to M_{n,\phi}$.

Several properties of quasilinear algebra can be proved using this class of matrices with a "quasi-associative" multiplication. For example we can study a quasi-LU decomposition for some matrix $X \in M_{n,\phi}$:(Note that this study generalizes the known LU decomposition of a matrix in the usual associative algebra of square matrices $n \times n$.) We began studying elementary "quasi-matrices" and "quasi-elementary" operations and we can conclude that,

Theorem 3.4. *Let $X \in M_{n,\phi}$. There are quasi-permutation matrices $P_{i_1,j_1}, P_{i_2,j_2}, ..., P_{i_k,j_k}$, an upper triangular matrix U and a lower triangular matrix L with $l_{ii} = \phi(i^{-1}, i, i^{-1})$ such that $P_{i_1,j_1}(P_{i_2,j_2}(...(P_{i_k,j_k}X))...) = L.U$*

4 Division quasialgebras

In [7] we have studied cocycles and some properties of quasialgebras graded over the ciclic group Z_n. Using the definition of a 3-cocycle in a group G we proved that a Z_2-graded quasialgebra is either an associative superalgebra or an antiassociative superalgebra (aaq-algebra) with cocycle defined by,$\phi(x, y, z) = (-1)^{xyz}, \forall x, y, z, \in Z_2$.

In Z_3 every cocycle has the form

$$\phi_{111} = \alpha, \phi_{112} = \beta, \phi_{121} = \frac{1}{\omega\alpha}, \phi_{122} = \frac{\omega}{\beta}, \phi_{211} = \frac{\alpha}{\beta\omega},$$

$$\phi_{212} = \alpha\omega, \phi_{221} = \frac{\beta}{\omega\alpha}, \phi_{222} = \frac{\omega}{\alpha}$$

for some $\alpha, \beta \in K^*$ with ω a cubic root of the unity. Here $\phi(1, 1, 1) = \phi_{111}$, etc. is a shorthand.

Definition 4.1. *The quasiassociative algebra $A = \oplus_{g\in G} A_g$ is said to be a quasiassociative division algebra if it is unital ($1 \in A_0$) and any nonzero homogeneous element has a right and a left inverse. Given such an algebra we denote the right inverse of $0 \neq u \in A_g$ by u^{-1} and the left inverse by u_L^{-1}. Notice that, since A_0 is a division associative algebra, the left and right inverses of any nonzero element of A_0 coincide.*

For any nonzero $u \in A_g$, the elements u^{-1} and u_L^{-1} are in A_{-g}, and we have $u^{-1} = \phi(-g, g, -g)u_L^{-1}$. For $0 \neq u \in A_g$ and $0 \neq w \in A_h$, we have $(uw)^{-1} = \frac{\phi(h,-h,-g)}{\phi(g,h,-g-h)} w^{-1}u^{-1}$.

It is easy to prove that for division quasiassociative algebra, the null part A_0 is a division associative algebra and A_g is an A_0-bimodule satisfying $A_g = A_0 u_g = u_g A_0$, for any nonzero $u_g \in A_g$, any $g \in G$.

In [5] we have studied division antiassociative algebras and characterized antiassociative algebras with semisimple (artinian) even part and odd part that is a unital bimodule for the even part.

Theorem 4.1. *Given a division (associative) algebra D, σ an automorphism of D such that there is an element $a \in D^*$ with $\sigma^2 = \tau_a : d \mapsto ada^{-1}$ and with $\sigma(a) = -a$, on the direct sum of two copies of D: $\Delta = D \oplus Du$ (here u is just a marking device), define the multiplication*

$$(d_0 + d_1u)(e_0 + e_1u) = (d_0e_0 + d_1\sigma(e_1)a) + (d_0e_1 + d_1\sigma(e_0))u. \tag{13}$$

Then with $\Delta_0 = D$ and $\Delta_1 = Du$, this is easily seen to be a division aaq-algebra. We will denote by $< D, \sigma, a >$ the division aaq-algebra Δ described before. The aaq-algebras $< D, \sigma, a >$ exhaust, up to isomorphism, the division aaq-algebras with nonzero odd part.

In general, it was proved in [2] that,

Theorem 4.2. *Let D be a division associative algebra over K, G a finite abelian group and $\phi : G \times G \times G \to K^*$ a cocycle. Suppose that for each $g, h, l \in G$, there are automorphisms ψ_g of D and non zero elements $c_{g,h}$ of D satisfying*

$$\psi_g \psi_h = c_{g,h} \psi_{g+h} c_{g,h}^{-1} \tag{14}$$

and

$$c_{g,h} c_{g+h,l} = \phi(g,h,l) \psi_g(c_{h,l}) c_{g,h+l} \tag{15}$$

In the direct sum of $|G|$ copies of D, $\Delta = \oplus_{g \in G} D u_g$ (u_g is a marking device with $u_0 = 1$), consider the multiplication defined by

$$\begin{array}{rcl} d_1(d_2 u_g) & = & (d_1 d_2) u_g \\ (d_1 u_g) d_2 & = & (d_1 \psi_g(d_2)) u_g \\ (d_1 u_g)(d_2 u_h) & = & (d_1 \psi_g(d_2) c_{g,h}) u_{g+h}, \end{array} \tag{16}$$

for $d_1, d_2 \in D$ and $g, h \in G$. Then with $\Delta_0 = D$ and $\Delta_g = D u_g$,Δ is a quasiassociative G-graded division algebra. Conversely, every quasiassociative G graded division algebra can be obtained this way.

As an interesting particular case of division quasialgebras we have studied the alternative ones in [4]. We called strictly alternative algebras the alternative algebras that are not associative.

Theorem 4.3. *There are no strictly alternative division quasialgebras over fields of characteristic* 2*.*

Let $A = \oplus_{g \in G} A_g$ be a strictly alternative division quasialgebra. Then $G/N \cong Z_2 \times Z_2 \times Z_2$ where $N = \{x \in A : (x, A, A) = 0\}$.

In [4] we have proved that any strictly alternative division quasialgebra over a field of characteristic different from 2 can be obtained by a sort of "graded Cayley-Dickson doubling process" built from the associative division algebra $A_N = \oplus_{g \in N} A_g$.

Definition 4.2. *Let K be a field, G an abelian group and $N \leq S \leq T \leq G$ a chain of subgroups with $[T : S] = 2$ and $T = S \cup Sg$ with $g^2 \in N$. Let $\beta : G \mapsto K$ be the map given by $\beta(g) = 1$ if $g \in N$ and $\beta(g) = -1$ if $g \notin N$. Let $F : S \times S \to K^\times$ be a 2-cochain such that*

$$\frac{F(s_1, s_2)}{F(s_2, s_1)} = \beta(s_1)\beta(s_2)\beta(s_1 s_2)\big(= (\partial\beta)(s_1, s_2)\big)$$

for any $s_1, s_2 \in S$ and let $0 \neq \alpha \in K$. Then the 2-cochain $\bar{F} = F_{(T,g,\alpha)} : T \times T \to K^\times$ defined by

$$\begin{array}{rcl} \bar{F}(x,y) & = & F(x,y) \\ \bar{F}(x,yg) & = & F(y,x) \\ \bar{F}(xg,y) & = & \beta(y)F(x,y) \\ \bar{F}(xg,yg) & = & \beta(y)F(y,x)\alpha \end{array} \tag{17}$$

for any $x, y \in S$, is said to be the 2-cochain extending F by means of (T, g, α).

Theorem 4.4. *Let k be a field of characteristic $\neq 2$ and K/k a field extension. Let G be an abelian group, N a subgroup of G such that $G/N \cong Z_2 \times Z_2 \times Z_2$ and N_1 and N_2 subgroups of G such that $N < N_1 < N_2 < G$. Let $\beta : G \mapsto K^\times$ be the map given by $\beta(x) = 1$ for $x \in N$ and $\beta(x) = -1$ for $x \notin N$ and $F_0 : N \times N \mapsto K^\times$ a symmetric 2-cocycle. Let $g_1, g_2, g_3 \in G$ with $N_1 = \langle N, g_1 \rangle$, $N_2 = \langle N_1, g_2 \rangle$ and $G = \langle N_2, g_3 \rangle$, and let $\alpha_1, \alpha_2, \alpha_3$ be nonzero elements in K. Consider the extended 2-cochains $F_1 = (F_0)_{(N_1,g_1,\alpha_1)}$, $F_2 = (F_1)_{(N_2,g_2,\alpha_2)}$ and $F_3 = (F_2)_{(G,g_3,\alpha_3)}$. Then $K_{F_3}G$ is a strictly alternative division quasialgebra over k.*

Conversely, if k is a field and A is a strictly alternative division quasialgebra over k then the characteristic of k is $\neq 2$ and there are $K, G, N, F_0, \alpha_1, \alpha_2, \alpha_3$ satisfying the preceding conditions such that $A \cong K_{F_3}G$.

5 Quasialgebras with simple null part

In [5] we have classified antiassociative quasialgebras which even part is semisimple (artinian) and the odd part is a unital bimodule for the even part. We have concluded that,

Theorem 5.1. *Any unital aaq-algebra with semisimple even part is a direct sum of ideals which are of one of the following types:*

(i) $A = A_0 \oplus A_1$ *with* A_0 *simple artinian and* $(A_1)^2 \neq 0$.

(ii) $A = A_0 \oplus A_1$ *with* $A_0 = B_1 \oplus B_2$, *where* B_1 *and* B_2 *are simple artinian ideals of* A_0, *and* $A_1 = V_{12} \oplus V_{21}$, *where* $V_{12}^2 = 0 = V_{21}^2$, $V_{12}V_{21} = B_1$ *and* $V_{21}V_{12} = B_2$.

(iii) A trivial unital aaq-algebra with a semisimple even part.

To better understand the last theorem consider two types of aaq-algebras with semisimple artinian even part:

- Let Δ be a division aaq-algebra and consider a natural number n. The set $Mat_n(\Delta)$ is an aaq-algebra whose even part is $Mat_n(\Delta_0)$;

- Let Δ be a division aaq-algebra and consider natural numbers n, m. The set $\widetilde{Mat}_{n,m}(D)$ of $(n+m)\times(n+m)$ matrices over D, with the chess-board Z_2-grading:

$$\begin{aligned}
\widetilde{Mat}_{n,m}(D)_0 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in Mat_n(D),\, b \in Mat_m(D) \right\} \\
\widetilde{Mat}_{n,m}(D)_1 &= \left\{ \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} : v \in Mat_{n\times m}(D),\, w \in Mat_{m\times n}(D) \right\}
\end{aligned} \tag{18}$$

with multiplication given by,

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + v_1w_2 & a_1v_2 + v_1b_2 \\ w_1a_2 + b_1w_2 & -w_1v_2 + b_1b_2 \end{pmatrix} .$$

$\widetilde{Mat}_{n,m}(D)$ is an aaq-algebra, whose even part is isomorphic to $Mat_n(D) \times Mat_m(D)$.

Then,

Theorem 5.2. *Any unital aaq-algebras with semisimple even part $A = A_0 \oplus A_1$ is a finite direct sum of ideals $A = A^1 \oplus \cdots \oplus A^r \oplus \hat{A}^1 \oplus \cdots \oplus \hat{A}^s \oplus \tilde{A}$ where:*

For $i = 1, \cdots, r$, A^i is isomorphic to $Mat_{n_i}(\Delta^i)$ for some n_i and some division aaq-algebra Δ^i.

For $j = 1, \cdots, s$, $\hat{A}^j$ is isomorphic to $\widetilde{Mat}_{n_j,m_j}(D^j)$ for some division algebra D^j and natural numbers n_j and m_j.

$\tilde{A}$ is a trivial unital aaq-algebras with semisimple even part. Moreover, r, s, the n_i's and the pairs $\{n_j, m_j\}$ are uniquely determined by A; and so are (up to isomorphism) $\tilde{A}$, the division aaq-algebras Δ^i's and the division algebras D^j's.

In general if $A = \oplus_{g\in G} A_g$ is a quasiassociative algebra over the field K , with simple artinian null part $B = A_0$, then it is isomorphic to $V \otimes_D \Delta \otimes_D V^*$, where Δ is a quasiassociative division algebra, V is a simple A_0-module and $D = End_{A_0}(V)$. This is equivalent to the following theorem [1],

Theorem 5.3. *Any quasiassociative algebra A with simple artinian null part is isomorphic to an algebra of matrices $Mat_n(\Delta)$, for some integer n and quasiassociative division algebra Δ. The integer n is uniquely determined by A and so is, up to isomorphism, the division algebra Δ.*

6 Wedderburn quasialgebras

A *Wedderburn quasialgebra* is a unital quasialgebra that satisfies the descending chain condition on graded left ideals and with no nonzero nilpotent graded ideals [3]. In this paper it was proved an analogous of the Wedderburn-Artin Theorem for quasialgebras.

The first result obtained in [3] it was that if A is a Wedderburn quasialgebra, then the null part A^0 is a Wedderburn algebra.

Consider natural numbers $n_1, \ldots, n_r$, define the matrix algebra $M_{n_1+\cdots+n_r}(k)$ having a basis consisting of the elements

$$E_{ij}^{kl} = E_{n_1+\cdots+n_{i-1}+k, n_1+\cdots+n_{j-1}+l} \tag{19}$$

where, as usual, E_{ij} denotes the matrix with 1 in the (i,j) entry and 0's elsewhere. The multiplication is,

$$E_{ij}^{kl} E_{mn}^{pq} = \delta_{jm}\delta_{lp} E_{in}^{kq}. \tag{20}$$

Let R be a quasialgebra with cocycle ϕ. Consider in G a grading function $|1|, \ldots, |r| \in G$ and take natural numbers $n_1, \cdots, n_r$. Define the set of matrices,

$$\tilde{M}^{\phi}_{n_1,\cdots,n_r}(R) =< x_{ij}^{kl} >=< E_{ij}^{kl} x >, : x \in R, 1 \leq i,j \leq r, 1 \leq k \leq n_i, 1 \leq l \leq n_j \tag{21}$$

that is a G-graded algebra for the gradation,

$$|x_{ij}^{kl}| = |i||x||j|^{-1} \tag{22}$$

for homogeneous x, and multiplication given by

$$x_{ij}^{kl} y_{mn}^{pq} = \delta_{jm}\delta_{lp} \frac{\phi(|i|, |x||j|^{-1}, |j||y'||n|^{-1})\phi(|x||j|^{-1}, |j|, |y||n|^{-1})}{\phi(|x|, |j|^{-1}, |j|)\phi(|x|, |y|, |n|^{-1})} (xy)_{in}^{kq}. \tag{23}$$

In fact, $\tilde{M}^{\phi}_{n_1,\ldots,n_r}(R)$ is a quasialgebra with cocycle ϕ.

The main result in [3] is,

Theorem 6.1. *Let A be a Wedderburn quasialgebra. Then A is isomorphic to a finite direct sum of quasialgebras of the form $\tilde{M}^{\phi}_{n_1,\ldots,n_r}(\hat{D})$, where $\hat{D}$ is a division quasi-algebra with cocycle ϕ.*

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Freudenthal's magic supersquare in characteristic 3

Isabel Cunha* Alberto Elduque†

Abstract

A family of simple Lie superalgebras over fields of characteristic 3, with no counterpart in Kac's classification in characteristic 0 have been recently obtained related to an extension of the classical Freudenthal's Magic Square. This article gives a survey of these new simple Lie superalgebras and the way they are obtained.

1 Introduction

Over the years, many different constructions have been given of the excepcional simple Lie algebras in Killing-Cartan's classification, involving some nonassociative algebras or triple systems. Thus the Lie algebra G_2 appears as the derivation algebra of the octonions (Cartan 1914), while F_4 appears as the derivation algebra of the Jordan algebra of 3×3 hermitian matrices over the octonions and E_6 as an ideal of the Lie multiplication algebra of this Jordan algebra (Chevalley-Schafer 1950).

In 1966 Tits [Tit66] gave a unified construction of the exceptional simple Lie algebras, valid over arbitrary fields of characteristic $\neq 2, 3$ which uses a couple of ingredients: a unital composition algebra (or Hurwitz algebra) C, and a central simple Jordan algebra J of degree 3:

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J,$$

where C_0 and J_0 denote the sets of trace zero elements in C and J. By defining a suitable Lie bracket on $\mathcal{T}(C, J)$, Tits obtained Freudenthal's Magic Square ([Sch95, Fre64]):

*Departamento de Matemática, Universidade da Beira Interior, 6200 Covilhã, Portugal. Email: icunha@mat.ubi.pt. Supported by Supported by Centro de Matemática da Universidade de Coimbra – FCT.

†Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain. Email: elduque@unizar.es. Supported by the Spanish Ministerio de Educación y Ciencia and FEDER (MTM 2004-081159-C04-02) and by the Diputación General de Aragón (Grupo de Investigación de Álgebra)

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k\times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k\times k$	A_2	$A_2\oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

At least in the split cases, this is a construction which depends on two unital composition algebras, since the Jordan algebra involved consists of the 3×3 hermitian matrices over a unital composition algebra. Even though the construction is not symmetric on the two Hurwitz algebras involved, the result (the Magic Square) is symmetric.

Over the years, several symmetric constructions of Freudenthal's Magic Square based on two Hurwitz algebras have been proposed: Vinberg [Vin05], Allison and Faulkner [AF93] and more recently, Barton and Sudbery [BS, BS03], and Landsberg and Manivel [LM02, LM04] provided a different construction based on two Hurwitz algebras C, C', their Lie algebras of triality $\mathfrak{tri}(C)$, $\mathfrak{tri}(C')$, and three copies of their tensor product: $\iota_i(C\otimes C'), i=0,1,2$. The Jordan algebra $J=H_3(C')$, its subspace of trace zero elements and its derivation algebra can be split naturally as:

$$\begin{aligned}
&J=H_3(C')\cong k^3\oplus\bigl(\oplus_{i=0}^2\iota_i(C')\bigr),\\
&J_0\cong k^2\oplus\bigl(\oplus_{i=0}^2\iota_i(C')\bigr),\\
&\mathfrak{Der}\,J\cong\mathfrak{tri}(C')\oplus\bigl(\oplus_{i=0}^2\iota_i(C')\bigr),
\end{aligned}$$

and the above mentioned symmetric constructions are obtained by rearranging Tits construction as follows:

$$\begin{aligned}
\mathcal{T}(C,J)&=\mathfrak{Der}\,C\oplus(C_0\otimes J_0)\oplus\mathfrak{Der}\,J\\
&\cong\mathfrak{Der}\,C\oplus(C_0\otimes k^2)\oplus\bigl(\oplus_{i=0}^2C_0\otimes\iota_i(C')\bigr)\oplus\bigl(\mathfrak{tri}(C')\oplus(\oplus_{i=0}^2\iota_i(C'))\bigr)\\
&\cong\bigl(\mathfrak{tri}(C)\oplus\mathfrak{tri}(C')\bigr)\oplus\bigl(\oplus_{i=0}^2\iota_i(C\otimes C')\bigr).
\end{aligned}$$

This construction, besides its symmetry, has the advantage of being valid too in characteristic 3. Simpler formulas are obtained if symmetric composition algebras are used, instead of the more classical Hurwitz algebras. This led the second author to reinterpret the above construction in terms of two symmetric composition algebras [Eld04].

An algebra endowed with a nondegenerate quadratic form $(S,*,q)$ is said to be a *symmetric composition algebra* if it satisfies

$$\begin{cases} q(x*y)=q(x)q(y),\\ q(x*y,z)=q(x,y*z).\end{cases}$$

for any $x, y, z \in S$, where $q(x, y) = q(x + y) - q(x) - q(y)$ is the polar of q.

Any Hurwitz algebra C with norm q, standard involution $x \mapsto \bar{x} = q(x, 1)1 - x$, but with new multiplication

$$x * y = \bar{x}\bar{y},$$

is a symmetric composition algebra, called the associated *para-Hurwitz algebra*.

The classification of symmetric composition algebras was given by Elduque, Okubo, Osborn, Myung and Pérez-Izquierdo (see [EM93, EP96, KMRT98]). In dimension 1,2 or 4, any symmetric composition algebra is a para-Hurwitz algebra, with a few exceptions in dimension 2 which are, nevertheless, forms of para-Hurwitz algebras; while in dimension 8, apart from the para-Hurwitz algebras, there is a new family of symmetric composition algebras termed *Okubo algebras*.

If $(S, *, q)$ is a symmetric composition algebra, the subalgebra of $\mathfrak{so}(S, q)^3$ defined by

$$\mathfrak{tri}(S) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : d_0(x * y) = d_1(x) * y + x * d_2(y) \forall x, y \in S\}$$

is the *triality Lie algebra* of S, which satisfies:

$$\mathfrak{tri}(S) = \begin{cases} 0 & \text{if } \dim S = 1, \\ \text{2-dim'l abelian} & \text{if } \dim S = 2, \\ \mathfrak{so}(S_0, q)^3 & \text{if } \dim S = 4, \\ \mathfrak{so}(S, q) & \text{if } \dim S = 8. \end{cases}$$

The construction given by Elduque in [Eld04] starts with two symmetric composition algebras S, S' and considers the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra

$$\mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(S \otimes S')\big),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$, with anticommutative multiplication given by:

- $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$ is a Lie subalgebra of $\mathfrak{g}(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' * y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

for any $x, y \in S$, $x', y' \in S'$, $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$ and $(d'_0, d'_1, d'_2) \in \mathfrak{tri}(S')$. The triple $t_{x,y} = \big(q(x,.)y - q(y,.)x, \frac{1}{2}q(x,y)1 - r_x l_y, \frac{1}{2}q(x,y)1 - l_x r_y\big)$ is in $\mathfrak{tri}(S)$ and $\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$ is the *triality automorphism* in $\mathfrak{tri}(S)$; and similarly for t' and θ' in $\mathfrak{tri}(S')$.

With this multiplication, $\mathfrak{g}(S, S')$ is a Lie algebra and, if the characteristic of the ground field is $\neq 2, 3$, Freudenthal's Magic Square is recovered.

$\dim S'$

$\dim S$	$\mathfrak{g}(S,S')$	1	2	4	8
	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

In characteristic 3, some attention has to be paid to the second row (or column), where the Lie algebras obtained are not simple but contain a simple codimension 1 ideal.

$\dim S'$

$\dim S$	$\mathfrak{g}(S,S')$	1	2	4	8
	1	A_1	$\tilde{A}_2$	C_3	F_4
	2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
	4	C_3	$\tilde{A}_5$	D_6	E_7
	8	F_4	$\tilde{E}_6$	E_7	E_8

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

The characteristic 3 presents also another exceptional feature. Only over fields of this characteristic there are nontrivial composition superalgebras, which appear in dimensions 3 and 6. This fact allows to extend Freudenthal's Magic Square with the addition of two further rows and columns, filled with (mostly simple) Lie superalgebras.

A precise description of those superalgebras can be done as contragredient Lie superalgebras (see [CEa] and [BGL]).

Most of the Lie superalgebras in the extended Freudenthal's Magic Square in characteristic 3 are related to some known simple Lie superalgebras, specific to this characteristic, constructed in terms of orthogonal and symplectic triple systems, which are defined in terms of central simple degree three Jordan algebras.

2 The Supersquare and Jordan algebras

It turns out that the Lie superalgebras $\mathfrak{g}(S_r, S_{1,2})$ and $\mathfrak{g}(S_r, S_{4,2})$, for $r = 1$, 4 and 8, and their derived subalgebras for $r = 2$, are precisely the simple Lie superalgebras defined in [Eld06b] in terms of orthogonal and symplectic triple systems and strongly related to simple Jordan algebras of degree 3.

Let S be a para-Hurwitz algebra. Then,

$$\begin{aligned}\mathfrak{g}(S_{1,2}, S) &= \big(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)\big) \\ &= \big((\mathfrak{sp}(V) \oplus V) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(1 \otimes S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(V \otimes S)\big).\end{aligned}$$

Consider the Jordan algebra of 3×3 hermitian matrices over the associated Hurwitz algebra:

$$\begin{aligned}J = H_3(\bar{S}) &= \left\{ \begin{pmatrix} \alpha_0 & \bar{a}_2 & a_1 \\ a_2 & \alpha_1 & \bar{a}_0 \\ \bar{a}_1 & a_0 & \alpha_2 \end{pmatrix} : \alpha_0, \alpha_1, \alpha_2 \in k,\ a_0, a_1, a_2 \in S \right\} \\ &\cong k^3 \oplus \big(\oplus_{i=0}^{2} \iota_i(S)\big).\end{aligned}$$

In [CEb, Theorem 4.9.] it is proved that:

$$\begin{cases} \mathfrak{g}(S_{1,2}, S)_{\bar{0}} \cong \mathfrak{sp}(V) \oplus \mathfrak{Der}\, J \quad \text{(as Lie algebras)}, \\ \mathfrak{g}(S_{1,2}, S)_{\bar{1}} \cong V \otimes \hat{J} \quad \text{(as modules for the even part)}. \end{cases}$$

Therefore, some results concerning $\mathbb{Z}_2$-graded Lie superalgebras and orthogonal triple systems ([Eld06b]) allow us to conclude that $\hat{J} = J_0/k1$ is an *orthogonal triple system* with product given by

$$[\hat{x}\hat{y}\hat{z}] = \big(x \circ (y \circ z) - y \circ (x \circ z)\big)\hat{}$$

($\hat{x} = x + k1$).

Orthogonal triple systems were first considered by Okubo [Oku93].

Theorem 1 (Elduque-Cunha [CEb])**.** *The Lie superalgebra $\mathfrak{g}(S_{1,2}, S)$ is the Lie superalgebra associated to the orthogonal triple system $\hat{J} = J_0/k1$, for $J = H_3(\bar{S})$.*

In order to analyze the Lie superalgebras $\mathfrak{g}(S_{4,2}, S)$, let V be, as before, a two dimensional vector space endowed with a nonzero alternating bilinear form. Assume that the ground field is algebraically closed. Then (see [CEb]) it can be checked that $\mathfrak{g}(S_8, S)$ can be identified to a $\mathbb{Z}_2$-graded Lie algebra

$$\mathfrak{g}(S_8, S) \simeq \big(\mathfrak{sp}(V) \oplus \mathfrak{g}(S_4, S)\big) \oplus \big(V \otimes T(S)\big),$$

for a suitable $\mathfrak{g}(S_4, S)$-module $T(S)$ of dimension $8 + 6 \dim S$.

In a similar vein, one gets that $\mathfrak{g}(S_{4,2}, S)$ can be identified with

$$\mathfrak{g}(S_{4,2}, S) \simeq \mathfrak{g}(S_4, S) \oplus T(S).$$

Hence, according to [CEb, Theorem 5.3, Theorem 5.6]:

Corollary 1 (Elduque-Cunha)**.** *Let S be a para-Hurwitz algebra, then:*

1. *$T(S)$ above is a symplectic triple system.*
2. *$\mathfrak{g}(S_{4,2}, S) \cong \mathfrak{inder}\, T \oplus T$ is the Lie superalgebra attached to this triple system.*

Remark 1. *Symplectic triple systems are closely related to the so called Freudenthal triple systems (see [YA75]). The classification of the simple finite dimensional symplectic triple systems in characteristic* 3 *appears in [Eld06b, Theorem 2.32] and it follows from this classification that the symplectic system triple $T(S)$ above is isomorphic to a well-known triple system defined on the set of* 2×2*-matrices* $\begin{pmatrix} k & J \\ J & k \end{pmatrix}$, *with $J = H_3(\bar{S})$.*

The next table summarizes the previous arguments:

$\mathfrak{g}$	S_1	S_2	S_4	S_8
$S_{1,2}$	Lie superalgebras attached to orthogonal triple systems $\hat{J} = J_0/k1$			
$S_{4,2}$	Lie superalgebras attached to symplectic triple systems $\begin{pmatrix} k & J \\ J & k \end{pmatrix}$			

(J a degree 3 central simple Jordan algebra)

3 Final remarks

In the previous section, the Lie superalgebras $\mathfrak{g}(S_r, S_{1,2})$ and $\mathfrak{g}(S_r, S_{4,2})$ $(r = 1, 2, 4, 8)$ in the extended Freudenthal's Magic Square have been shown to be related to Lie superalgebras which had been constructed in terms of orthogonal and symplectic triple systems in [Eld06b]. Let us have a look here at the remaining Lie superalgebras in the supersquare:

The result in [CEa, Corollary 5.20] shows that the even part of $\mathfrak{g}(S_{1,2}, S_{1,2})$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_7(k)$, while its odd part is the direct sum of two copies of the spin module for $\mathfrak{so}_7(k)$, and therefore, over any algebraically closed field of characteristic 3, $\mathfrak{g}(S_{1,2}, S_{1,2})$ is isomorphic to the simple Lie superalgebra in [Eld06b, Theorem 4.23(ii)], attached to a simple *null orthogonal triple system*.

For $\mathfrak{g}(S_{4,2}, S_{4,2})$, as shown in [CEa, Proposition 5.10 and Corollary 5.11], its even part is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_{13}(k)$, while its odd part is the spin module for the even part, and hence that $\mathfrak{g}(S_{4,2}, S_{4,2})$ is the simple Lie superalgebra in [Elda, Theorem 3.1(ii)] for $l = 6$.

Only the simple Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{4,2})$ has not previously appeared in the literature. Its even part is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_8(k)$, while its odd part is the irreducible module of dimension 40 which appears as a subquotient of the third exterior power of the natural module for $\mathfrak{sp}_8(k)$ (see [CEa, §5.5]).

Also $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ are related to some *orthosymplectic triple systems* [CEb], which are triple systems of a mixed nature.

In conclusion, notice that

the main feature of Freudenthal's Magic Supersquare is that among all the Lie superalgebras involved, only $\mathfrak{g}(S_{1,2}, S_1) \cong \mathfrak{psl}_{2,2}$ *has a counterpart in Kac's classification in characteristic* 0*. The other Lie superalgebras in Freudenthal's Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic* 3*.*

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On graded Lie algebras and intersection cohomology

G. Lusztig*

This is the text of a lecture given at the University of Coimbra in June 2006; a more complete version will appear elsewhere.

1

Let G be a reductive connected group over $\mathbf{C}$. Let LG be the Lie algebra of G. Let $\iota : \mathbf{C}^* \longrightarrow G$ be a homomorphism of algebraic groups. Let

$$G^\iota = \{g \in G; g\iota(t) = \iota(t)g \text{ for all } t \in \mathbf{C}^*\}.$$

We have $LG = \oplus_{n \in \mathbf{Z}} L_n G$ where

$$L_n G = \{x \in LG; \mathrm{Ad}(\iota(t))x = t^n x \quad \forall t \in \mathbf{k}^*\}.$$

Now G^ι acts on $L_k G$ by the adjoint action. If $k \neq 0$ this action has finitely many orbits. Fix

$$\Delta = \{n, -n\}$$

where $n \in \mathbf{Z} - \{0\}$. For $n \in \Delta$ let $\mathfrak{I}_{L_n G}$ be the set of all isomorphism classes of irreducible G^ι-equivariant local systems on various G^ι-orbits in $L_n G$. For $\mathcal{L}, \mathcal{L}'$ in $\mathfrak{I}_{L_n G}$ (on the orbits $\mathcal{O}, \mathcal{O}'$) and $i \in \mathbf{Z}$ let $m_{i;\mathcal{L},\mathcal{L}'}$ be the multiplicity of $\mathcal{L}'$ in the local system obtained by restricting to $\mathcal{O}'$ the i-th cohomology sheaf of the intersection cohomology complex $IC(\bar{\mathcal{O}}, \mathcal{L})$.

Let $m_{\mathcal{L},\mathcal{L}'} = \sum_i m_{i,\mathcal{L},\mathcal{L}'} v^{\dim \mathcal{O} - \dim \mathcal{O}' - \rangle}$. These form the entries of the *multiplicity matrix*. The problem is to compute it.

Example 1. Let V be a vector space of dimension 4 with a nonsingular symplectic form. Let $G = Sp(V)$. Let V_1, V_{-1} be Lagangian subspaces of V such that $V = V_1 \oplus V_{-1}$. Let $\Delta = \{2, -2\}$. Define $\iota : \mathbf{C}^* \longrightarrow G$ by $\iota(t)x = tx$ if $x \in V_1$, $\iota(t)x = t^{-1}x$ if $x \in V_{-1}$.

We have $L_2 G = S^2 V_1, L_{-2} V = S^2(V_1^*)$ and $\mathfrak{I}_{L_2 G}$ consists of 4 local systems

$\mathcal{L}_{\prime}$, $\mathbf{C}$ on the orbit of dimension 0;

$\mathcal{L}_{\in}$, $\mathbf{C}$ on the orbit of dimension 2;

$\mathcal{L}_{\ni}$, $\mathbf{C}$ on the orbit of dimension 3;

$\mathcal{L}_{\triangle}$, non-trivial on the orbit of dimension 3.

*Department of Mathematics, M.I.T., Cambridge, MA 02139

Example 2. Let V be a vector space of dimension d. Let $G = GL(V)$. Assume that V is $\mathbf{Z}$-graded: $V = \oplus_{k\in\mathbf{Z}} V_k$. Define $\iota : \mathbf{C}^* \longrightarrow G$ by $\iota(t)x = t^k x$ for $x \in V_k$. Take $\Delta = \{1, -1\}$. Let $L_1G = \{T \in \mathrm{End}(V); TV_k \subset V_{k+1} \forall k\}$. The G^ι-orbits on L_1G are in bijection with the isomorphism classes of representations of prescribed dimension of a quiver of type A. In this case our problem is equivalent to the problem of describing the transition matrix from a PBW basis to the canonical basis of a quantized enveloping algebra U^+ of type A.

This suggests that to solve our problem we must immitate and generalize the construction of canonical bases of U_v^+.

2

Let B be the canonical basis of U_v^+ the plus part of a quantized enveloping algebra of finite simply laced type.

B was introduced in the author's paper in J.Amer.Math.Soc. (1990) by two methods:

- topological: study of perverse sheaves on the moduli space of representations of a quiver.
- algebraic: define a $\mathbf{Z}[v]$-lattice $\mathcal{L}$ in U_v^+ and a $\mathbf{Z}$- basis B_0 of $\mathcal{L}/v\mathcal{L}$ (basis at $v = 0$) in terms of PBW-bases then lift B_0 to B.

Another proof of the existence of B was later given by Kashiwara (Duke Math.J., 1991).

3

Let $n \in \Delta$. Let $\mathcal{L} \in \mathfrak{I}_{L_nG}$ on an orbit $\mathcal{O}$. We say that $\mathcal{L}$ is ordinary if there exists a G-equivariant local system $\mathcal{F}$ on C (the unique nilpotent G-orbit in LG containig $\mathcal{O}$) such that

$\mathcal{L}$ appears in $\mathcal{F}|_{\mathcal{O}}$ and $(C, \mathcal{F})$ appears in the Springer correspondence for G.

It is known that for $\mathcal{L}, \mathcal{L}' \in \mathfrak{I}_{\mathcal{L}\setminus\mathcal{G}}$ we have

$\mathcal{L}$ ordinary, $m_{\mathcal{L},\mathcal{L}'} \neq 0$ implies $\mathcal{L}'$ ordinary;

$\mathcal{L}'$ ordinary, $m_{\mathcal{L},\mathcal{L}'} \neq 0$ implies $\mathcal{L}$ ordinary.

Let $\mathfrak{I}^{ord}_{L_nG} = \{\mathcal{L} \in \mathfrak{I}_{L_nG}; \mathcal{L} \text{ ordinary}\}$.

We say that (G, ι) is *rigid* if there exists a homomorphism $\gamma : SL_2(\mathbf{C}) \longrightarrow G$ which maps the diagonal matrix with diagonal entries t^n, t^{-n} to $\iota(t^2)\times$(centre of G) for any $t \in \mathbf{C}^*$.

Definition. Let $\mathcal{K}(L_nG)$ be the $\mathbf{Q}(v)$-vector space with two bases $(\mathcal{L})$, $(\underline{\mathcal{L}})$ indexed by $\mathfrak{I}^{ord}_{L_nG}$ where $\underline{\mathcal{L}} = \sum_{\mathcal{L}'} m_{\mathcal{L},\mathcal{L}'}\mathcal{L}'$.

4

Let U_v^+ be as in §2. Let $(e_i)_{i\in I}$ be the standard generators of U_v^+. Then U_v^+ is the quotient of the free associative $\mathbf{Q}(v)$-algebra with generators e_i by the radical of an explicit symmetric bilinear form on it. We have

$$U_v^+ = \oplus_\nu U_{v,\nu}^+$$

where $U_{v,\nu}^+$ is spanned by the monomials

$$e_{i_1}^{c_1} e_{i_2}^{c_2} \dots e_{i_r}^{c_r}$$

with $\sum_{s;i_s=i} c_s = \nu(i)$ for any $i \in I$. Thus U_v^+ is the quotient of the $\mathbf{Q}(v)$-vector space spanned by $e_{i_1}^{c_1} e_{i_2}^{c_2} \dots e_{i_r}^{c_r}$ as above by the radical of an explicit symmetric bilinear form on it.

5 Combinatorial parametrization of set of G^ι-orbits in L_nG

Let $\mathcal{P}$ be the set of parabolic subgroups of G. Let

$$\mathcal{P}^\iota = \{\mathcal{P} \in \mathcal{P}; \iota(\mathbf{C}) \subset \mathcal{P}\}.$$

If $P \in \mathcal{P}$ we set $\bar{P} = P/U_P$; we have $\iota : \mathbf{C}^* \longrightarrow \bar{P}$ naturally. Let $n \in \Delta$. Let $P \in \mathcal{P}^\iota$ be such that $(\bar{P}, \iota)$ is rigid. Choose a Levi subgroup M of P such that $\iota(\mathbf{C}^*) \subset M$. Let $s \in [LM, LM]$ be such that $[s, x] = kx$ for any $k \in \mathbf{Z}, x \in L_kM$. (Note that s is unique.) Let

$$L^rG = \{x \in LG; [s, x] = rx\},$$

$$L_t^rG = L^rG \cap L_tG.$$

Then

$$LG = \oplus_{r\in(n/2)\mathbf{Z};s\in\mathbf{Z}} L_t^rG.$$

We say that P is n-*good* if

$$LU_P = \oplus_{r\in(n/2)\mathbf{Z};s\in\mathbf{Z};2t/n<2r/n} L_t^rG.$$

Then automatically

$$LM = \oplus_{r\in(n/2)\mathbf{Z};s\in\mathbf{Z};2t/n=2r/n} L_t^rG.$$

Let

$$\mathbf{P}_n = \{P \in \mathcal{P}^\iota; (\bar{\mathcal{P}}, \iota) \text{ is rigid, } P \text{ is } n\text{-good}\},$$

$$\underline{\mathbf{P}}_n = G^\iota \backslash \mathbf{P}_n.$$

We claim that

$$\underline{\mathbf{P}}_n \xrightarrow{\sim} \{G^\iota - \text{orbits in } L_nG\}.$$

Let P, L_t^rG be as above. Then $L_0^0G = LG'$ where G' acts by Ad on L_n^nG. Let $\mathcal{O}_\prime$ be the unique G'-orbit on L_n^nG. Now $G' \subset G^\iota$ hence there is a unique G^ι-orbit $\mathcal{O}$ in L_nG containing $\mathcal{O}_\prime$. The bijection above is given by $P \mapsto \mathcal{O}$. Other parametrizations of $\{G^\iota - \text{orbits in } L_nG\}$ were given by Vinberg (1979), Kawanaka (1987).

6

Let $\mathcal{B}$ be the variety of Borel subgroups of G. Let $\mathcal{B}^\iota = \mathcal{B} \cap \mathcal{P}^\iota$. Let $\mathcal{I}_\mathcal{G} = \{\mathcal{B}; \mathcal{B} \in \mathcal{B}^\iota\}$. Let $\underline{\mathcal{J}}_G = G^\iota \backslash \mathcal{J}_\mathcal{G}$. Let $\tilde{K}_G$ be the $\mathbf{Q}(v)$-vector space with basis $(I_\mathcal{S})_{\mathcal{S} \in \underline{\mathcal{J}}_G}$. For $Q \in \mathcal{P}^\iota$ we define a linear map (induction)

$$ind_Q^G \longrightarrow \tilde{K}_{\bar{Q}} \longrightarrow \tilde{K}_G$$

by

$$I_{\bar{Q}^\iota-\text{orbit of } B'} \mapsto I_{G^\iota-\text{orbit of } B}$$

where B is the inverse image of B' under $Q \longrightarrow \bar{Q}$. Let$^-$: $\mathbf{Q}(v) \longrightarrow \mathbf{Q}(v)$ be the $\mathbf{Q}$-linear map such that $\overline{v^m} = v^{-m}$ for $m \in \mathbf{Z}$. Define a "bar operation" $\beta : \tilde{K}_G \longrightarrow \tilde{K}_G$ by $\beta(fI_\mathcal{S}) = \bar{f} I_\mathcal{S}$. Inspired by the results in §4 we define a bilinear pairing

$$(:) : \tilde{K}_G \times \tilde{K}_G \longrightarrow \mathbf{Q}(v)$$

by

$$(I_\mathcal{S} : I_{\mathcal{S}'}) = \sum_\Omega (-v)^{\tau(\Omega)}$$

where Ω runs over the G^ι-orbits on $\mathcal{B}^\iota \times \mathcal{B}^\iota$ such that $pr_1\Omega = \mathcal{S}, pr_2\Omega = \mathcal{S}'$ and if $(B, B') \in \Omega$ we set

$$\tau(\Omega) = -\dim \frac{L_0U' + L_0U}{L_0U' \cap L_0U} + \dim \frac{L_nU' + L_nU}{L_nU' \cap L_nU};$$

here U, U' are the unipotent radicals of B, B'. Let $\mathcal{R}$ be the radical of (:). Let $K_G = \tilde{K}_G/\mathcal{R}$. Then (:) induces a nondegenerate bilinear pairing (:) on K_G. Note that β induces a map $K_G \longrightarrow K_G$ denoted again by β. Moreover ind_Q^G induces a map $K_{\bar{Q}} \longrightarrow K_G$ denoted again by ind_Q^G.

For $n \in \Delta$ and $\eta \in \underline{\mathbf{P}}_n$ we define subsets Z_n^η of K_G by induction on $\dim G$.

Assume that $\eta \in \underline{\mathbf{P}}_n$, $\eta \neq \{G\}$. We set $Z_n^\eta = ind_P^G(Z_n^{\bar{P}})$ where $P \in \eta$.

We set $Z'_n = \cup_{\eta \in \underline{\mathbf{P}}_n; \eta \neq \{G\}} Z_n^\eta$. Then:

the last union is disjoint and $ind_P^G : Z^{\{\bar{P}\}} \longrightarrow Z_n^\eta$ *is a bijection.*

For $\eta \in \underline{\mathbf{P}}_n$ and $P \in \eta$ we set $d_\eta = \dim L_0G - \dim L_0P + \dim L_nP$. Define a partial order $\leq$ on $\mathbf{P}_n - \{G\}$ by

$\eta' < \eta$ if and ony if $d_{\eta'} < d_\eta$.

$\eta' \leq \eta$ if and only if $\eta = \eta'$ or $\eta' < \eta$.

If $\xi \in Z'_n$ we have $\beta(\xi) = \sum_{\xi' \in Z'_n} a_{\xi,\xi_1}\xi_1$ where $a_{\xi,\xi_1} \in \mathbf{Z}[v, v^{-1}]$ are unique and

$a_{\xi,\xi_1} \neq 0$ implies $\eta_1 < \eta$ or $\xi = \xi_1$ (here $\xi \in Z_n^\eta, \xi_1 \in Z_n^{\eta_1}$)

$a_{\xi,\xi_1} = 1$ if $\xi_1 = \xi$.

Also, $\sum_{\xi_2 \in Z'_n} \overline{a_{\xi,\xi_2}} a_{\xi_2,\xi_1} = \delta_{\xi,\xi_1}$ for $\xi, \xi_1 \in Z'_n$. By a standard argument there is a unique family of elements $c_{\xi,\xi_1} \in \mathbf{Z}[v]$ defined for ξ, ξ_1 in Z'_n such that

$c_{\xi,\xi_1} = \sum_{\xi_2 \in Z'_n} \overline{c_{\xi,\xi_2}} a_{\xi_2,\xi_1}$;

$c_{\xi,\xi_1} \neq 0$ implies $\eta_1 < \eta$ or $\xi = \xi_1$ (where $\xi \in Z_n^\eta, \xi_1 \in Z_n^{\eta_1}$)

$c_{\xi,\xi_1} \neq 0$, $\xi \neq \xi_1$ implies $c_{\xi,\xi_1} \in v\mathbf{Z}[v]$;

$c_{\xi,\xi_1} = 1$ if $\xi = \xi_1$.

For $\xi \in Z'_n$ we set $W_n^\xi = \sum_{\xi_1 \in Z'_n} c_{\xi,\xi_1}\xi_1$. Then $\beta(W_n^\xi) = W_n^\xi$.

Define $Y_n : K_G \longrightarrow K_G$ by $(Y_n(x) : Z'_n) = 0$ and $x = Y_n(x) + \sum_{\xi \in Z'_n} Y\gamma_\xi \xi$ where $\gamma_\xi \in \mathbf{Q}(v)$. This is well defined since the matrix $((\xi,\xi'))_{\xi,\xi' \in Z'_n}$ is invertible. Let

$$J_{-n} = \{\xi_0 \in Z'_{-n}; Y_n(W_{-n}^{\xi_0}) \neq 0\}.$$

Let $Z_n = Z'_n$ if (G,ι) is not rigid. If (G,ι) is rigid let

$$Z_n = Z'_n \cup \{\xi; \xi = Y_n(W_{-n}^{\xi_0}) \text{ for some } \xi_0 \in J_{-n}\}.$$

We say that Z_n a PBW-basis of K_G. It depends on n. For $\xi \in Z_n$ we define W_n^ξ as follows. W_n^ξ is as above if $\xi \in Z'_n$. $W_n^\xi = W_{-n}^\xi$ where $\xi = Y_n(W_{-n}^{\xi_0})$, $\xi_0 \in J_{-n}$. We say that (W_n^ξ) is the canonical basis of K_G. It does not depend on n.

Theorem. The transition matrix expressing the canonical basis in terms of the PBW basis Z_n coincides with the multiplicity matrix $(m_{\mathcal{L},\mathcal{L}'})$ in §1 (with $\mathcal{L},\mathcal{L}'$ ordinary). There is an appropriate generalization for not necessarily ordinary $\mathcal{L},\mathcal{L}'$.

On the quantization of a class of non-degenerate triangular Lie bialgebras over $\mathbb{K}[[\hbar]]$

Carlos Moreno* Joana Teles†

Abstract

Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathbb{K}[[\hbar]]$ be the principal ideal domain of formal series in $\hbar$ over $\mathbb{K}$. Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a finite dimensional non-degenerate triangular Lie-bialgebra over $\mathbb{K}$. Let $(a_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r_1(\hbar))$ be a deformation non-degenerate triangular Lie-bialgebra over $\mathbb{K}[[\hbar]]$ corresponding to $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$. The aim of this talk is

a) To quantize $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}} = d_c(\hbar) r_1(\hbar))$ in the framework by Etingof-Kazhdan for the quantization of Lie-bialgebras. Let $A_{\mathfrak{a}_\hbar, \tilde{J}^{-1}_{r_1(\hbar)}}$ be the Hopf Q.U.E. algebra over $\mathbb{K}[[\hbar]]$ so obtained.

b) Let $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r'_1(\hbar))$ be another deformation non-degenerate triangular Lie-bialgebra corresponding to $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}_\hbar} = d_c r'_1)$. Let $A_{\mathfrak{a}_\hbar, \tilde{J}^{-1}_{r'_1(\hbar)}}$ be its quantization. Let $\beta_1(\hbar), \beta'_1(\hbar) \in \mathfrak{a}^*_\hbar \hat{\otimes}_{\mathbb{K}[[\hbar]]} \mathfrak{a}^*_\hbar$ be the corresponding 2-forms associated respectively to $r_1(\hbar)$ and $r'_1(\hbar)$. We prove that $\tilde{J}^{-1}_{r_1(\hbar)}$ and $\tilde{J}^{-1}_{r'_1(\hbar)}$ are equivalent in the Hochschild cohomology of the universal enveloping algebra $\mathcal{U}\mathfrak{a}_\hbar$ if and only if $\beta_1(\hbar)$ and $\beta'_1(\hbar)$ are in the same class in the Chevalley cohomology of the Lie algebra $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$ over $\mathbb{K}[[\hbar]]$.

Keywords: Quantum Groups, Quasi-Hopf algebras, Lie bialgebras.

1 Some definitions

1.1 Lie bialgebras and Manin triples

1.1.1 Lie bialgebras over $\mathbb{K}$

Let $\mathbb{K}$ be a field of characteristic 0. Let $\mathbb{K}[[\hbar]] = \mathbb{K}_\hbar$ be the ring of formal power series in $\hbar$ with coefficients in $\mathbb{K}$. It is a principal ideal domain (PID) and its unique maximal ideal is $\hbar\mathbb{K}_\hbar$.

Let $(\mathfrak{a}, [,]_{\mathfrak{a}})$ be a finite dimensional Lie algebra over $\mathbb{K}$.

*Departamento de Física Teórica, Universidad Complutense, E-28040 Madrid, Spain.

†CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal. E-mail: jteles@mat.uc.pt

- A finite dimensional Lie bialgebra over $\mathbb{K}$ is a set $(\mathfrak{a}, [,]_{\mathfrak{a}}, \mathbb{K}, \varepsilon_{\mathfrak{a}})$ where $\varepsilon_{\mathfrak{a}} : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ is a 1-cocycle of $\mathfrak{a}$, with values in $\mathfrak{a} \otimes \mathfrak{a}$, with respect to the adjoint action of $\mathfrak{a}$ such that $\varepsilon_{\mathfrak{a}}^{\mathfrak{t}} : \mathfrak{a}^* \otimes \mathfrak{a}^* \to \mathfrak{a}^*$ is a Lie bracket on $\mathfrak{a}^*$.

- It is called quasi-triangular if $\varepsilon_{\mathfrak{a}} = d_c r_1$, where d_c is the Chevalley-Eilenberg coboundary, $r_1 \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution to CYBE ($[r_1, r_1] = 0$) and $(r_1)_{12} + (r_1)_{21}$ is $\mathrm{ad}_{\mathfrak{a}}$-invariant.

- In case r_1 is skew-symmetric, it is said to be a triangular Lie bialgebra. Moreover if $det(r_1) \neq 0$ it is called a non-degenerate triangular Lie bialgebra.

1.1.2 Deformation Lie bialgebras

Consider now the $\mathbb{K}[[\hbar]]$-module obtained from extension of the scalars $\mathbb{K}[[\hbar]] \otimes_{\mathbf{K}} \mathfrak{a}$.

Let $(\mathfrak{a}_\hbar = \mathbb{K}[[\hbar]] \otimes_{\mathbf{K}} \mathfrak{a} = \mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}_\hbar}, \mathbb{K}_\hbar = \mathbb{K}[[\hbar]], \varepsilon_{\mathfrak{a}_\hbar})$, where $\varepsilon_{\mathfrak{a}_\hbar} : \mathfrak{a}_\hbar \to \mathfrak{a}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{a}_\hbar$ is a 1-cocycle of $\mathfrak{a}_\hbar$ with values in tha adjoint representation, be a Lie bialgebra over $\mathbb{K}[[\hbar]]$.

It is a deformation Lie bialgebra of the Lie bialgebra over $\mathbb{K}$, $\mathfrak{a}$.

1.1.3 Manin triples

A Manin triple over $\mathbb{K}_\hbar$ is a set $(\overline{\mathfrak{g}}_\hbar = (\mathfrak{g}_\hbar)_+ \oplus (\mathfrak{g}_\hbar)_-, [,]_{\overline{\mathfrak{g}}_\hbar}, <;>_{\overline{\mathfrak{g}}_\hbar})$ where

- $(\overline{\mathfrak{g}}_\hbar, [,]_{\overline{\mathfrak{g}}_\hbar})$ is a (finite dimensional) Lie algebra over $\mathbb{K}_\hbar$;

- $\big((\mathfrak{g}_\hbar)_\pm, [,]_{(\mathfrak{g}_\hbar)_\pm}, \mathbb{K}_\hbar\big)$ is a (finite dimensional) Lie algebra over $\mathbb{K}_\hbar$;

- $<;>_{\overline{\mathfrak{g}}_\hbar}$ is a non-degenerate, symmetric, bilinear, $ad_{\overline{\mathfrak{g}}_\hbar}$-invariant form on $\overline{\mathfrak{g}}_\hbar$;

- $[,]_{\overline{\mathfrak{g}}_\hbar}|_{(\mathfrak{g}_\hbar)_\pm} = [,]_{(\mathfrak{g}_\hbar)_\pm}$.

1.1.4 From Manin triples to Lie bialgebras

If $(\overline{\mathfrak{g}}_\hbar = (\mathfrak{g}_\hbar)_+ \oplus (\mathfrak{g}_\hbar)_-, [,]_{\overline{\mathfrak{g}}_\hbar}, <;>_{\overline{\mathfrak{g}}_\hbar})$ is a Manin triple then

- There is a $\mathbb{K}_\hbar$-linear isomorphism $\chi^{-1} : (\mathfrak{g}_\hbar)_- \longrightarrow (\mathfrak{g}_\hbar)_+^*$ because $<;>_{\overline{\mathfrak{g}}_\hbar}$ is non-degenerate.

- We may define in $\mathfrak{g}_\hbar = (\mathfrak{g}_\hbar)_+ \oplus (\mathfrak{g}_\hbar)_+^*$ a structure of a Manin triple.

- There is a structure of Lie bialgebra on $(\mathfrak{g}_\hbar)_+$.

1.1.5 From Lie bialgebras to Manin triples

Let $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \mathbb{K}_\hbar, \varepsilon_{\mathfrak{a}_\hbar})$ be a Lie bialgebra over $\mathbb{K}_\hbar$. Then $(\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*,\ [,]_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*}, \langle;\rangle_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*})$, where

$$[(x;\xi),(y;\eta)]_{\mathfrak{g}_\hbar} = ([x,y]_{\mathfrak{a}_\hbar} + ad^*_\xi y - ad^*_\eta x; [\xi,\eta]_{\mathfrak{a}_\hbar^*} + ad^*_x \eta - ad^*_y \xi),$$

is a Manin triple.

The set $(\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}^*_\hbar, [,]_{\mathfrak{a}_\hbar \oplus \mathfrak{a}^*_\hbar}, \varepsilon_{\mathfrak{g}_\hbar} = d_c(\hbar)r)$, where $r \in \mathfrak{g}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{g}_\hbar$ (canonical), is a quasi-triangular Lie bialgebra with $[,]_{\mathfrak{g}^*_\hbar} = \Big([,]_{\mathfrak{a}^*_\hbar}; -[,]_{\mathfrak{a}_\hbar}\Big)$.

It is called the (quasi-triangular Lie bialgebra) classical double of the Lie bialgebra $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \mathbb{K}_\hbar, \varepsilon_{\mathfrak{a}_\hbar})$.

1.1.6 Drinfeld Theorem

Drinfeld quasi-Hopf quasi-triangular QUE algebra corresponding to the deformation Lie algebra $(\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}^*_\hbar, [,]_{\mathfrak{g}_\hbar}, r = a + t \in \mathfrak{g}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{g}_\hbar$ invariant) is

$$(\widehat{\mathcal{U}\mathfrak{g}}_\hbar, \cdot, 1, \Delta_{\mathfrak{g}_\hbar}, \epsilon_{\mathfrak{g}_\hbar}, \Phi_{\mathfrak{g}_\hbar} = e^{P(\hbar t^{12};\hbar t^{23})}, S_{\mathfrak{g}_\hbar}, \alpha = c^{-1}, \beta = 1, R_{\mathfrak{g}_\hbar} = e^{\frac{\hbar}{2}t})$$

where $c = \sum_i X_i S_{\mathfrak{g}_\hbar}(Y_i) Z_i$, $\Phi_{\mathfrak{g}_\hbar} = \sum_i X_i \otimes Y_i \otimes Z_i$ and P is a formal Lie series with coefficients in $\mathbb{K}$ (or just in $\mathbb{Q}$.)

2 Etingof-Kazhdan quantization

2.1 The classical double

2.1.1 Quantization of the double

Given the bialgebra $(\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}^*_\hbar, [,]_{\mathfrak{a}_\hbar \oplus \mathfrak{a}^*_\hbar}, \varepsilon_{\mathfrak{g}_\hbar} = d_c(\hbar)r)$ double of the Lie bialgebra $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \mathbb{K}_\hbar, \varepsilon_{\mathfrak{a}_\hbar})$, we will construct a twist of Drinfeld quasi-Hopf QUE algebra

$$J \in \widehat{\mathcal{U}\mathfrak{g}}_\hbar \hat{\otimes}_{\mathbf{K}_\hbar} \widehat{\mathcal{U}\mathfrak{g}}_\hbar$$

obtaining a Hopf QUE algebra. To do it, we need the following elements.

2.1.2 Drinfeld category

The category $\mathcal{M}_{\mathfrak{g}_\hbar}$ is defined as

- $Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}} = \{$topologically free $\mathfrak{g}_\hbar$-modules$\}$, that is $X \in Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}}$ iff
  1. $X = V[[\hbar]]$, where V is a vector space over $\mathbb{K}$
  2. $V[[\hbar]]$ is a $\mathfrak{g}_\hbar$-module
- Of course, $Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}} = \{$topologically free $\mathcal{U}\mathfrak{g}_\hbar$-modules$\}$
- $Hom_{\mathcal{M}_{\mathfrak{g}_\hbar}}(U[[\hbar]], V[[\hbar]])$ is the set of $\mathfrak{g}_\hbar$-module morphisms, $U[[\hbar]] \xrightarrow{f} V[[\hbar]]$
  1. $f(a(\hbar).x(\hbar)) = a(\hbar).f(x(\hbar)),\ a(\hbar) \in \mathfrak{g}_\hbar$
  2. $f((a(\hbar) \circ b(\hbar) - b(\hbar) \circ a(\hbar))x(\hbar)) = [a(\hbar), b(\hbar)]_{\mathfrak{g}_\hbar} f(x(\hbar))$

and f extends in a unique way to a $\mathcal{U}(\mathfrak{g}_\hbar)$-module morphism.

Theorem 2.1.1. *$Hom_{\mathfrak{g}_\hbar}(U[[\hbar]], V[[\hbar]])$ is a torsion free $\mathbb{K}_\hbar$-module.*
In the topology defined by its natural filtration

$$\{\hbar^p Hom_{\mathfrak{g}_\hbar}(U[[\hbar]], V[[\hbar]])\}$$

its completion $\widehat{Hom_{\mathfrak{g}_\hbar}}(U[[\hbar]], V[[\hbar]])$ is separated, complete and a torsion free $\mathbb{K}_\hbar$-module.
It is then (see Kassel) a topologically free $\mathbb{K}_\hbar$-module, so

$$\widehat{Hom_{\mathfrak{g}_\hbar}}(U[[\hbar]], V[[\hbar]]) = Hom_{\mathfrak{a}\oplus\mathfrak{a}^*}(U, V)[[\hbar]].$$

The canonical element $r \in \mathfrak{g}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{g}_\hbar$ defines $\Omega_{V_i(\hbar)V_j(\hbar)} \in End_{\mathbf{K}_\hbar}(V_1[[\hbar]] \otimes V_2[[\hbar]] \otimes V_3[[\hbar]])$, $i, j = 1, 2, 3; i \neq j$ by

$$\Omega_{V_i(\hbar)V_j(\hbar)}(v_1(\hbar) \otimes v_2(\hbar) \otimes v_3(\hbar)) = (\cdots \overset{i}{\widehat{f_k}} \otimes \cdots \otimes \overset{j}{\widehat{f^k}} \otimes \cdots +$$
$$+ \cdots \overset{i}{\widehat{f^k}} \otimes \cdots \otimes \overset{j}{\widehat{f_k}} \otimes \cdots).(v_1(\hbar) \otimes v_2(\hbar) \otimes v_3(\hbar)).$$

Lemma 2.1.2. $\Omega_{V_i(\hbar)V_j(\hbar)} \in Hom_{\mathfrak{g}_\hbar}(V_1[[\hbar]] \otimes V_2[[\hbar]] \otimes V_3[[\hbar]], V_1[[\hbar]] \otimes V_2[[\hbar]] \otimes V_3[[\hbar]])$.

2.1.3 Tensor structure on $\mathcal{M}_{\mathfrak{g}_\hbar}$

The element r is $ad_{\mathfrak{g}_\hbar}$-invariant. The Lie associator $\Phi_{\mathfrak{g}_\hbar} \in \widehat{\mathcal{U}\mathfrak{g}_\hbar} \hat{\otimes} \widehat{\mathcal{U}\mathfrak{g}_\hbar} \hat{\otimes} \widehat{\mathcal{U}\mathfrak{g}_\hbar}$ is also invariant; that is

$$\Phi_{\mathfrak{g}_\hbar} \cdot (\Delta_{\mathfrak{g}_\hbar} \otimes 1)\Delta_{\mathfrak{g}_\hbar}(x(\hbar)) = (1 \otimes \Delta_{\mathfrak{g}_\hbar})\Delta_{\mathfrak{g}_\hbar}(x(\hbar)) \cdot \Phi_{\mathfrak{g}_\hbar},\ x(\hbar) \in \mathfrak{g}_\hbar.$$

¿From this equality and the fact that $\Phi_{\mathfrak{g}_\hbar} = expP(\hbar t^{12}, \hbar t^{23})$ we may define a $\mathfrak{g}_\hbar$-morphism

$$\Phi_{V_1(\hbar),V_2(\hbar),V_3(\hbar)} \in Hom_{\widehat{\mathcal{U}\mathfrak{g}_\hbar}}\big((V_1(\hbar) \hat{\otimes} V_2(\hbar)) \hat{\otimes} V_3(\hbar)\ ,\ V_1(\hbar) \hat{\otimes} (V_2(\hbar) \hat{\otimes} V_3(\hbar))\big).$$

Theorem 2.1.3. *$\Phi_{V_1(\hbar),V_2(\hbar),V_3(\hbar)}$ is an isomorphism in category $\mathcal{M}_{\mathfrak{g}_\hbar}$ and the set of these isomorphisms defines a natural isomorphism between functors:*

$$\otimes(\otimes \times Id) \longrightarrow \otimes(Id \times \otimes).$$

2.1.4 Braided tensor structure on $\mathcal{M}_{\mathfrak{g}_\hbar}$

For any couple

$$V_1[[\hbar]], V_2[[\hbar]] \in Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}}$$

consider the isomorphism

$$\beta_{V_1(\hbar)V_2(\hbar)} : V_1(\hbar) \hat{\otimes} V_2(\hbar) \longrightarrow V_2(\hbar) \hat{\otimes} V_1(\hbar)$$
$$u(\hbar) \hat{\otimes} v(\hbar) \longmapsto \sigma\left(e^{\frac{\hbar}{2}\Omega_{12}(\hbar)} u(\hbar) \hat{\otimes} v(\hbar)\right),$$

where σ is the usual permutation.

Then $\beta_{V_1(\hbar)V_2(\hbar)} \in Hom_{\widehat{\mathcal{U}\mathfrak{g}_\hbar}} \left(V_1(\hbar)\hat{\otimes}V_2(\hbar), V_2(\hbar)\hat{\otimes}V_1(\hbar)\right).$

Theorem 2.1.4. *The set of isomorphisms $\beta_{V_1(\hbar)V_2(\hbar)}$ of $\mathcal{M}_{\mathfrak{g}_\hbar}$ defines a natural isomorphism between functors $\otimes \longrightarrow \otimes \circ \sigma$. Then the category $\mathcal{M}_{\mathfrak{g}_\hbar}$ has a braided tensor structure.*

2.1.5 The category $\mathcal{A}$

- $\mathrm{Ob}_{\mathcal{A}} = \{$ topologically free $\mathbb{K}_\hbar$-modules $\}$
- $\mathrm{Hom}_{\mathcal{A}}(V_1[[\hbar]], V_2[[\hbar]]) = \{f : V_1[[\hbar]] \longrightarrow V_2[[\hbar]]$, $\mathbb{K}_\hbar$-linear maps preserving filtrations ($\Longleftrightarrow$ continuous) $\}$

Lemma 2.1.5. *The category $\mathcal{A}$ is a strict monoidal symmetric category:*

$$\Phi = Id : \left(V[[\hbar]]\hat{\otimes}U[[\hbar]]\right)\hat{\otimes}W[[\hbar]] \simeq V[[\hbar]]\hat{\otimes}\left(U[[\hbar]]\hat{\otimes}W[[\hbar]]\right),$$

and

$$\begin{aligned} \sigma : V[[\hbar]]\hat{\otimes}U[[\hbar]] &\longrightarrow U[[\hbar]]\hat{\otimes}V[[\hbar]] \\ u(\hbar)\otimes v(\hbar) &\longmapsto v(\hbar)\otimes u(\hbar). \end{aligned}$$

2.1.6 The functor $\mathcal{F}$

Let $\mathcal{F}$ be the following map

$$\mathcal{F} : \mathcal{M}_{\mathfrak{g}_\hbar} \longrightarrow \mathcal{A}$$

where

- $\begin{aligned} \mathcal{F}(V[[\hbar]]) &= Hom_{\widehat{\mathcal{U}\mathfrak{g}_\hbar}}(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, V[[\hbar]]) \\ &= Hom_{\mathfrak{a}\oplus\mathfrak{a}^*=\mathfrak{g}_0}(\mathcal{U}\mathfrak{g}_0, V)[[\hbar]] \end{aligned}$
- for $f \in Hom_{\mathfrak{g}_\hbar}(V[[\hbar]], U[[\hbar]])$, then

  $$\mathcal{F}(f) \in Hom_{\mathcal{A}}\left(Hom_{\mathfrak{g}_\hbar}(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, V[[\hbar]]), Hom_{\mathfrak{g}_\hbar}(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, U[[\hbar]])\right)$$

  is defined as $(\mathcal{F}(f))(g) = f \circ g \in \mathcal{F}(U[[\hbar]]) \in Ob_{\mathcal{A}}$, and $g \in \mathcal{F}(V[[\hbar]])$.

Then $\mathcal{F}$ is a functor.

2.1.7 Tensor structure on $\mathcal{F}$

We should now equip this functor $\mathcal{F}$ with a tensor structure. We will use the decomposition

$$\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^* = (\mathfrak{g}_\hbar)_+ \oplus (\mathfrak{g}_\hbar)_-$$

to produce such a structure.

2.1.8 $\mathfrak{g}_\hbar$-modules $M(\hbar)_+$ and $M(\hbar)_-$

- By Poincaré-Birkhoff-Witt theorem we have $\mathbb{K}_\hbar$-module isomorphisms
  - $\mathcal{U}(\mathfrak{g}_\hbar)_+ \otimes_{\mathbf{K}_\hbar} \mathcal{U}(\mathfrak{g}_\hbar)_- \longrightarrow \mathcal{U}\mathfrak{g}_\hbar \qquad$ (product in $\mathcal{U}\mathfrak{g}_\hbar$)
  - $\mathcal{U}(\mathfrak{g}_\hbar)_- \otimes_{\mathbf{K}_\hbar} \mathcal{U}(\mathfrak{g}_\hbar)_+ \longrightarrow \mathcal{U}\mathfrak{g}_\hbar \qquad$ (product in $\mathcal{U}\mathfrak{g}_\hbar$)
- Let the $\mathbb{K}_\hbar$-modules of rang 1

$$W_\pm(\hbar) = \{a(\hbar).e_\pm : a(\hbar) \in \mathbb{K}_\hbar,\ e_\pm \text{ basis}\}$$

  We endow $W_\pm$ with a trivial $\mathcal{U}(\mathfrak{g}_\hbar)_\pm$-module structure.
  - $x(\hbar)_\pm(a(\hbar)e_\pm) = a(\hbar)(x(\hbar)_\pm e_\pm) = 0_\pm,\ x(\hbar)_\pm \in \mathcal{U}(\mathfrak{g}_\hbar)_\pm$
- Define the corresponding induced left $\mathcal{U}\mathfrak{g}_\hbar$-modules:

$$M(\hbar)_\pm = \mathcal{U}\mathfrak{g}_\hbar \otimes_{\mathcal{U}(\mathfrak{g}_\hbar)_\pm} W_\pm(\hbar) = \cdots = \mathcal{U}(\mathfrak{g}_\hbar)_\mp \cdot 1_\pm$$

  where $1_\pm = (1 \otimes_{\mathcal{U}(\mathfrak{g}_\hbar)_\pm} e_\pm)$, 1 is the unit of $\mathcal{U}\mathfrak{g}_\hbar$.
  As $\mathbb{K}_\hbar$-modules $M(\hbar)_\pm$ is then the $\mathbb{K}_\hbar$-module $\mathcal{U}(\mathfrak{g}_\hbar)_\mp$.

Lemma 2.1.6. $\widehat{M(\hbar)}_\pm \in Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}}$.

Theorem 2.1.7. *There exists a unique $\mathfrak{g}_\hbar$-module morphism*

$$i_\pm : M(\hbar)_\pm \longrightarrow M(\hbar)_\pm \otimes_{\mathbf{K}_\hbar} M(\hbar)_\pm$$

such that

$$i_\pm(1_\pm) = 1_\pm \otimes_{\mathbf{K}_\hbar} 1_\pm.$$

It is continuous for the $(\hbar)$-adic topology and extends in a unique way to a $\mathfrak{g}_\hbar$-module morphism

$$i_\pm : \widehat{M(\hbar)}_\pm \longrightarrow \widehat{M(\hbar)}_\pm \widehat{\otimes}_{\mathbf{K}_\hbar} \widehat{M(\hbar)}_\pm.$$

Theorem 2.1.8. *Define the following $\mathfrak{g}_\hbar$-module morphism*

$$\phi : \mathcal{U}(\mathfrak{g}_\hbar) \longrightarrow M(\hbar)_+ \otimes_{\mathbf{K}_\hbar} M(\hbar)_-$$

by $\phi(1) = 1_+ \otimes 1_-$.
Then ϕ is an isomorphism of $\mathfrak{g}_\hbar$-modules.

Proof. If $\phi(1) = 1_+ \otimes 1_-$ and ϕ is a $\mathfrak{g}_\hbar$-module morphism, the construction of ϕ is unique.

ϕ preserves the standard filtration, then it defines a map $grad\,\phi$ on the associated graded objects. This map $grad\,\phi$ is bijective and then (Bourbaki) ϕ is bijective. $\square$

2.1.9 Natural isomorphism of functors J

Definition 2.1.9. *A tensor structure on the functor $\mathcal{F} : \mathcal{M}_{\mathfrak{g}_\hbar} \longrightarrow \mathcal{A}$ is a natural isomorphism of functors*

$$J : \mathcal{F}(\cdot) \otimes \mathcal{F}(\cdot) \longrightarrow \mathcal{F}(\cdot \otimes \cdot)$$

We define this tensor structure following the same pattern that in Etingof-Kazhdan.

For $V_\hbar = V[[\hbar]]$, $W_\hbar = W[[\hbar]] \in Ob_{\mathcal{M}_{\mathfrak{g}_\hbar}}$, define

$$\begin{aligned} J_{V_\hbar, W_\hbar} : \mathcal{F}(V_\hbar) \hat{\otimes}_{\mathbf{K}_\hbar} \mathcal{F}(W_\hbar) \ &\longrightarrow \ \mathcal{F}(V_\hbar \hat{\otimes} W_\hbar) \\ v_\hbar \otimes w_\hbar \ &\longrightarrow \ J_{V_\hbar, W_\hbar}(v_\hbar \otimes w_\hbar), \end{aligned}$$

where $v_\hbar \in \mathcal{F}(V_\hbar) = Hom_{\mathfrak{g}_\hbar}(\mathcal{U}\mathfrak{g}_\hbar, V_\hbar)$ and $w_\hbar \in \mathcal{F}(W_\hbar)$, then

$$\begin{aligned} J_{V_\hbar, W_\hbar}(v_\hbar \otimes w_\hbar) &= (v_\hbar \otimes w_\hbar) \circ (\phi^{-1} \otimes \phi^{-1}) \circ \Phi^{-1}_{M(\hbar)_+, M(\hbar)_-, M(\hbar)_+ \otimes M(\hbar)_-} \circ \\ &\circ (1 \otimes \Phi_{M(\hbar)_-, M(\hbar)_+, M(\hbar)_-}) \circ (1 \otimes (\sigma \circ e^{\frac{\hbar}{2}\Omega_{M(\hbar)_+, M(\hbar)_-}}) \otimes 1) \circ \\ &\circ (1 \otimes \Phi^{-1}_{M(\hbar)_+, M(\hbar)_-, M(\hbar)_-}) \circ \Phi_{M(\hbar)_+, M(\hbar)_+, M(\hbar)_- \otimes M(\hbar)_-} \circ (i_+ \otimes i_-) \circ \phi. \end{aligned}$$

2.1.10 Definition of J

Theorem 2.1.10. *The maps $J_{V_\hbar, W_\hbar}$ are isomorphisms and define a tensor structure on the functor $\mathcal{F}$.*

From this tensor structure we get as in the $\mathbb{K}$ case the following element in $\widehat{\mathcal{U}\mathfrak{g}_\hbar} \hat{\otimes} \widehat{\mathcal{U}\mathfrak{g}_\hbar}$

$$\begin{aligned} J = (\phi^{-1} \otimes \phi^{-1}) \Big[\Big(& \Phi^{-1}_{1,2,34} \cdot \Phi_{2,3,4} \cdot e^{\frac{\hbar}{2}\Omega_{23}} \cdot (\sigma_{23} \Phi^{-1}_{2,3,4}) \cdot \\ & \cdot (\sigma_{23} \Phi_{1,2,34}) \Big) (1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) \Big]. \end{aligned}$$

2.1.11 Quantization of the classical double

Theorem 2.1.11. *The set $(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, \cdot, 1, \Delta', \epsilon_{\mathfrak{g}_\hbar}, S', R')$, such that*

$$\begin{aligned} \Delta'(u_\hbar) &= J^{-1} \cdot \Delta_{\mathfrak{g}_\hbar}(u_\hbar) \cdot J \\ S'(u_\hbar) &= Q^{-1}(\hbar) \cdot S_{\mathfrak{g}_\hbar}(u_\hbar) \cdot Q(\hbar), \\ R' &= \sigma J^{-1} \cdot e^{\frac{\hbar}{2}\Omega} \cdot J, \end{aligned}$$

*where $Q(\hbar)$ is an element in $\widehat{\mathcal{U}\mathfrak{g}_\hbar}$ obtained from J and $S_{\mathfrak{g}_\hbar}$, is a quasi-triangular Hopf algebra, and it is just the algebra obtained **twisting** via the element J^{-1} the quasi-triangular quasi-Hopf QUE algebra obtained in Subsection 1.1.6.*

In particular,

$$\Phi \cdot (\Delta_{\mathfrak{g}_\hbar} \otimes 1) J \cdot (J \otimes 1) = (1 \otimes \Delta_{\mathfrak{g}_\hbar}) J \cdot (1 \otimes J),$$

$$J = 1 + \frac{\hbar}{2} r + O(\hbar^2)$$
$$R = 1 + \hbar r + O(\hbar^2)$$

We call the above quasi-triangular Hopf quantized universal enveloping algebra a quantization of the quasi-triangular Lie bialgebra over $\mathbb{K}_\hbar$

$$(\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*, [,]_{\mathfrak{g}_\hbar}, \varepsilon_{\mathfrak{g}_\hbar} = d_c(\hbar) r).$$

double of the Lie bialgebra $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar})$ over the ring $\mathbb{K}_\hbar$.

2.2 Quantization of Lie bialgebras

2.2.1 Quasi-triangular Lie bialgebras

If the $\mathbb{K}_\hbar$-Lie bialgebra is quasi-triangular, that is, if $\varepsilon_{\mathfrak{a}_\hbar}$ is an exact 1-cocycle

$$\varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r_1(\hbar), \quad r_1(\hbar) \in \mathfrak{a}_\hbar \otimes \mathfrak{a}_\hbar$$

and $[r_1(\hbar), r_1(\hbar)]_{\mathfrak{a}_\hbar} = 0$, we want to obtain a quantization of the $\mathbb{K}_\hbar$-Lie bialgebra

$$(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r_1(\hbar)).$$

Following Etingof and Kazhdan and also an idea by Reshetikhin and Semenov-Tian-Shansky, we will construct a Manin triple over $\mathbb{K}_\hbar$ and quantize it as we did before.

Lemma 2.2.1. *There exist basis $\{a_i(\hbar),\ i = 1, \dots, dim\ \mathfrak{a}\}$ and $\{b_j(\hbar),\ j = 1, \dots, dim\ \mathfrak{a}\}$ of the $\mathbb{K}_\hbar$-module $\mathfrak{a}_\hbar$ such that*

$$r_1(\hbar) = \sum_{i=1}^{l} a_i(\hbar) \otimes b_i(\hbar) \in \mathfrak{a}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{a}_\hbar. \tag{2.1}$$

Proof. As $\mathbb{K}_\hbar$ is a principal ideal domain (PID) and $\mathfrak{a}_\hbar$ is a free $\mathbb{K}_\hbar$-module, $r_1(\hbar)$ has a rang, l, and a theorem about matrices with entries in a PID-module asserts that basis verifying (2.1) exist. □

Let us define maps, as in Reshetikhin-Semenov,

$$\mu_{r_1(\hbar)}, \lambda_{r_1(\hbar)} : \mathfrak{a}_\hbar^* \longrightarrow \mathfrak{a}_\hbar$$

as

$$\lambda_{r_1(\hbar)}(f) = \sum a_i(\hbar).f(b_i(\hbar))$$
$$\mu_{r_1(\hbar)}(f) = \sum f(a_i(\hbar)).b_i(\hbar), \quad f \in \mathfrak{a}_\hbar^*$$

and write

$$(\mathfrak{a}_\hbar)_+ = Im\ \lambda_{r_1(\hbar)} \qquad (\mathfrak{a}_\hbar)_- = Im\ \mu_{r_1(\hbar)}.$$

We have

$$\dim_{\mathbb{K}_\hbar}(\mathfrak{a}_\hbar)_+ = \dim_{\mathbb{K}_\hbar}(\mathfrak{a}_\hbar)_- = l = \text{rang } r_1(\hbar)$$

We may prove as in the $\mathbb{K}$ case.

Lemma 2.2.2. *The mapping*

$$\begin{aligned} \chi_{r_1(\hbar)} : (\mathfrak{a}_\hbar)_+^* &\longrightarrow (\mathfrak{a}_\hbar)_- \\ g &\longmapsto \chi_{r_1(\hbar)}(g) = (g \otimes 1) r_1(\hbar) \end{aligned}$$

is a $\mathbb{K}_\hbar$-module morphism ($\mathbb{K}_\hbar$ is a PID and $\mathfrak{a}_\hbar$ is free over $\mathbb{K}_\hbar$).

$(\mathfrak{a}_\hbar)_+$ and $(\mathfrak{a}_\hbar)_-$ are Lie subalgebras of $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$

On the $\mathbb{K}_\hbar$-module $\overline{\mathfrak{g}_\hbar} = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-$ we may define a skew-symmetric, bilinear mapping such that $[,]_{\overline{\mathfrak{g}_\hbar}}|_{(\mathfrak{a}_\hbar)_\pm} = [,]_{(\mathfrak{a}_\hbar)_\pm}$.

Theorem 2.2.3. *Let π be defined as*

$$\begin{aligned} \pi : \overline{\mathfrak{g}_\hbar} = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_- &\longrightarrow \mathfrak{a}_\hbar \\ (x(\hbar); y(\hbar)) &\longrightarrow x(\hbar) + y(\hbar). \end{aligned}$$

Then,

(a) $\pi\left([(x(\hbar); y(\hbar)), (z(\hbar); u(\hbar))]_{(\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-}\right) =$
$= [\pi(x(\hbar); y(\hbar)), \pi(z(\hbar); u(\hbar))]_{\mathfrak{a}_\hbar};$

(b) $((\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-, [,]_{(\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-}, \mathbb{K}_\hbar)$ *is a Lie algebra over* $\mathbb{K}_\hbar$.

¿From (a) and (b), π is a Lie algebra morphism.

Theorem 2.2.4. *The set $(\overline{\mathfrak{g}_\hbar} = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-, [,]_{\overline{\mathfrak{g}_\hbar}}, \langle ; \rangle_{\overline{\mathfrak{g}_\hbar}})$ where*

$$\langle (x_+(\hbar); x_-(\hbar)); (y_+(\hbar); y_-(\hbar)) \rangle_{\overline{\mathfrak{g}_\hbar}} = \chi^{-1}_{r_1(\hbar)}(x_-(\hbar)) \,.y_+(\hbar) + \chi^{-1}_{r_1(\hbar)}(y_-(\hbar)) \,.x_+(\hbar),$$

$x_+(\hbar), y_+(\hbar) \in (\mathfrak{a}_\hbar)_+, x_-(\hbar), y_-(\hbar) \in (\mathfrak{a}_\hbar)_-$, *is a Manin triple.*

In particular, the 2-form $\langle ; \rangle_{\overline{\mathfrak{g}_\hbar}}$ is $ad_{\overline{\mathfrak{g}_\hbar}}$-invariant.

Because it is a Manin triple the set

$$((\mathfrak{a}_\hbar)_+, [,]_{(\mathfrak{a}_\hbar)_+}, \varepsilon)$$

where $\varepsilon : (\mathfrak{a}_\hbar)_+ \longrightarrow (\mathfrak{a}_\hbar)_+ \otimes (\mathfrak{a}_\hbar)_+$ is the transpose of the Lie bracket on $(\mathfrak{a}_\hbar)_+^*$ defined as

$$[\xi(\hbar), \eta(\hbar)]_{(\mathfrak{a}_\hbar)_+^*} \quad = \quad \chi^{-1}_{r_1(\hbar)}\left([\chi_{r_1(\hbar)}(\xi(\hbar)), \chi_{r_1(\hbar)}(\eta(\hbar))]_{(\mathfrak{a}_\hbar)_-}\right),$$

is a Lie bialgebra.

Definition 2.2.5. *Let* $\{e_i(\hbar), i = 1, \dots, l\}$ *a basis of* $(\mathfrak{a}_\hbar)_+$ *and* $\{e^i(\hbar), i = 1, \dots, l\}$ *its dual basis on* $(\mathfrak{a}_\hbar)^*_+$. *Let*

$$r(\hbar) = (e_i(\hbar); 0) \otimes (0; e^i(\hbar))$$

the canonical element in the $\mathbb{K}_\hbar$*-module* $(\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)^*_+$. *Define*

$$\overline{r}(\hbar) = (1 \oplus \chi_{r_1(\hbar)}) \otimes (1 \oplus \chi_{r_1(\hbar)}).r(\hbar) \in (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-$$

We prove

Theorem 2.2.6. *The set* $(\overline{\mathfrak{g}_\hbar} = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_-, [,]_{\overline{\mathfrak{g}_\hbar}}, \varepsilon_{\overline{\mathfrak{g}_\hbar}} = d_c(\hbar)\overline{r}(\hbar))$ *is a quasi-triangular Lie bialgebra which is isomorphic to the quasi-triangular Lie bialgebra* $(\mathfrak{g}_\hbar = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)^*_+, [,]_{\mathfrak{g}_\hbar}, \varepsilon_{\mathfrak{g}_\hbar} = d_c(\hbar)r(\hbar))$, *double of the Lie bialgebra* $((\mathfrak{a}_\hbar)_+, [,]_{(\mathfrak{a}_\hbar)_+}, \varepsilon)$.

Theorem 2.2.7. *Let the mapping* $\tilde{\pi}$ *be defined by the commutativity of the diagram*

$$\begin{tikzcd}
\overline{\mathfrak{g}_\hbar} = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)_- \arrow[r, "\pi"] & \mathfrak{a}_\hbar \\
(\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)^*_+ \arrow[u, "(1 \oplus \chi_{r_1(\hbar)})"] \arrow[ur, "\tilde{\pi}"'] &
\end{tikzcd}$$

So, $\tilde{\pi} = \pi \circ (1 \oplus \chi_{r_1(\hbar)})$.

Then, $\tilde{\pi}$ *is a Lie bialgebra homomorphism verifying*

$$(\tilde{\pi} \otimes \tilde{\pi})r(\hbar) = r_1(\hbar).$$

2.2.2 Quantization of the quasi-triangular Lie bialgebra

$$(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = \mathbf{d_c}(\hbar)\mathbf{r_1}(\hbar), [\mathbf{r_1}(\hbar), \mathbf{r_1}(\hbar)]_{\mathfrak{a}_\hbar} = \mathbf{0}, \mathbf{r_1}(\hbar) \in \mathfrak{a}_\hbar \otimes \mathfrak{a}_\hbar)$$

We may project using the Lie bialgebra morphism

$$\tilde{\pi} : \mathfrak{g}_\hbar = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)^*_+ \longrightarrow \mathfrak{a}_\hbar$$

what we have done, about quantization of $\mathfrak{g}_\hbar$. We will get a quantization of $\mathfrak{a}_\hbar$. Precisely,

Theorem 2.2.8. *Let* $(\widehat{\mathcal{U}\mathfrak{a}_\hbar}, \cdot, 1, \Delta_{\mathfrak{a}_\hbar}, \epsilon_{\mathfrak{a}_\hbar}, S_{\mathfrak{a}_\hbar})$ *be the usual Hopf algebra. Let*

$$(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, \cdot, 1, \Delta_{\mathfrak{g}_\hbar}, \epsilon_{\mathfrak{g}_\hbar}, \Phi_{\mathfrak{g}_\hbar}, S_{\mathfrak{g}_\hbar}, R_{\mathfrak{g}_\hbar} = e^{\frac{\hbar}{2}\Omega})$$

be the quasi-triangular quasi-Hopf algebra in Subsection 1.1.6.

Let $\tilde{\Phi}_{\mathfrak{a}_\hbar} = (\tilde{\pi} \otimes \tilde{\pi} \otimes \tilde{\pi})\Phi_{\mathfrak{g}_\hbar}$ *and* $R_{\mathfrak{a}_\hbar} = (\tilde{\pi} \otimes \tilde{\pi})\, R_{\mathfrak{g}_\hbar}$, *then*

- $(\tilde{\pi} \otimes \tilde{\pi}) \circ \Delta_{\mathfrak{g}_\hbar} = \Delta_{\mathfrak{a}_\hbar} \circ \tilde{\pi}$
- $\tilde{\pi} \circ S_{\mathfrak{g}_\hbar} = S_{\mathfrak{a}_\hbar} \circ \tilde{\pi}$
- *The set*
$$(\widehat{\mathcal{U}\mathfrak{a}_\hbar}, \cdot, 1, \Delta_{\mathfrak{a}_\hbar}, \epsilon_{\mathfrak{a}_\hbar}, \tilde{\Phi}_{\mathfrak{a}_\hbar}, S_{\mathfrak{a}_\hbar}, R_{\mathfrak{a}_\hbar})$$
is a quasi-triangular quasi-Hopf QUE algebra. We call it a quantization (quasi-Hopf one) of the Lie bialgebra $(\mathfrak{a}_\hbar, r_1(\hbar) + \sigma r_1(\hbar))$.

Let $(\widehat{\mathcal{U}\mathfrak{g}_\hbar}, \cdot, 1, \Delta'_{\mathfrak{g}_\hbar}, \epsilon_{\mathfrak{g}_\hbar}, S'_{\mathfrak{g}_\hbar}, R'_{\mathfrak{g}_\hbar})$ where

- $\Delta'_{\mathfrak{g}_\hbar}(u_\hbar) = J^{-1}_{\mathfrak{g}_\hbar} \cdot \Delta_{\mathfrak{g}_\hbar}(u_\hbar) \cdot J_{\mathfrak{g}_\hbar}$
- $S'_{\mathfrak{g}_\hbar}(u_\hbar) = Q^{-1}(\hbar) \cdot S_{\mathfrak{g}_\hbar}(u_\hbar) \cdot Q(\hbar)$
- $R'_{\mathfrak{g}_\hbar} = \sigma J^{-1}_{\mathfrak{g}_\hbar} \cdot e^{\frac{\hbar}{2}\Omega} \cdot J_{\mathfrak{g}_\hbar}$

be the quasi-triangular Hopf QUE algebra obtained before following E-K scheme of quantization of the classical double

$$(\mathfrak{g}_\hbar = (\mathfrak{a}_\hbar)_+ \oplus (\mathfrak{a}_\hbar)^*_+, [,]_{\mathfrak{g}_\hbar}, \varepsilon_{\mathfrak{g}_\hbar} = d_c(\hbar) r).$$

Let us define

- $\tilde{J}_{\mathfrak{a}_\hbar} = (\tilde{\pi} \otimes \tilde{\pi}) J_{\mathfrak{g}_\hbar}; \quad \tilde{\Delta}_{\mathfrak{a}_\hbar}(a_\hbar) = \tilde{J}^{-1}_{\mathfrak{a}_\hbar} \cdot \Delta_{\mathfrak{a}_\hbar}(a_\hbar) \cdot \tilde{J}_{\mathfrak{a}_\hbar}$
- $\tilde{R}_{\mathfrak{a}_\hbar} = (\tilde{\pi} \otimes \tilde{\pi}) R'_{\mathfrak{g}_\hbar}; \quad \tilde{\epsilon}_{\mathfrak{a}_\hbar}$ the usual counit in $\mathcal{U}(\mathfrak{a}_\hbar)$
- $\tilde{S}_{\mathfrak{a}_\hbar}(a_\hbar) = \tilde{Q}^{-1}(\hbar) \cdot S_{\mathfrak{a}_\hbar}(a_\hbar) \cdot \tilde{Q}(\hbar)$ where $\tilde{Q} = \sum S_{\mathfrak{a}_\hbar}(r_i(\hbar)).s_i(\hbar)$ and $\tilde{J}_{\mathfrak{a}_\hbar} = \sum r_i(\hbar) \otimes s_i(\hbar)$.

Theorem 2.2.9. *Then the set*

$$(\widehat{\mathcal{U}(\mathfrak{a}_\hbar)}, \cdot, 1, \tilde{\Delta}_{\mathfrak{a}_\hbar}, \tilde{\epsilon}_{\mathfrak{a}_\hbar}, \tilde{S}_{\mathfrak{a}_\hbar}, \tilde{R}_{\mathfrak{a}_\hbar})$$

is a quasi-triangular Hopf QUE algebra and we call it a quantization (Hopf one) of the Lie bialgebra $(\mathfrak{a}_\hbar, r_1(\hbar))$. *We moreover see that it is obtained by a twist via the element*

$$\tilde{J}^{-1}_{\mathfrak{a}_\hbar}$$

from the quasi-triangular quasi-Hopf QUE algebra in Subsection 1.1.6.

In particular, we have

$$\tilde{\Phi}_{\mathfrak{a}_\hbar} \cdot (\tilde{J}_{\mathfrak{a}_\hbar})_{12,3} \cdot (\tilde{J}_{\mathfrak{a}_\hbar})_{12} = (\tilde{J}_{\mathfrak{a}_\hbar})_{1,23} \cdot (\tilde{J}_{\mathfrak{a}_\hbar})_{23},$$

where the product $\cdot$ is in $\mathcal{U}\mathfrak{a}_\hbar$.

2.2.3 Non-degenerate triangular Lie bialgebras

This is the case $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r_1(\hbar))$ where

- $r_1(\hbar)$ is non-degenerate $\Longleftrightarrow$ invertible , det $r_1(\hbar)$ is a unit in $\mathbb{K}_\hbar$ ($\mathbb{K}_\hbar$ is a PID)
- $r_1(\hbar)$ is skew-symmetric

Also,

- $(\mathfrak{a}_\hbar)_+ = \mathfrak{a}_\hbar, \qquad (\mathfrak{a}_\hbar)_- = \mathfrak{a}_\hbar$
- $\mathfrak{g}_\hbar = \mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar *$

In this case, last theorem is

Theorem 2.2.10. *Consider the non-degenerate triangular Lie bialgebra over* $\mathbb{K}_\hbar$

$$(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar) r_1(\hbar), r_1(\hbar) \in \mathfrak{a}_\hbar \otimes \mathfrak{a}_\hbar, \mathit{det}\ r_1(\hbar)\ \textit{ a unit in } \mathbb{K}_\hbar,$$
$$[r_1(\hbar), r_1(\hbar)]_{\mathfrak{a}_\hbar} = 0),$$

the set $(\widehat{\mathcal{U}\mathfrak{a}_\hbar}, \cdot, 1, \tilde{\Delta}_{\mathfrak{a}_\hbar}, \epsilon_{\mathfrak{a}_\hbar}, \tilde{S}_{\mathfrak{a}_\hbar}, \tilde{R}_{\mathfrak{a}_\hbar})$ *where*

- $\tilde{\Delta}_{\mathfrak{a}_\hbar}(a_\hbar) = \tilde{J}^{-1}_{\mathfrak{a}_\hbar} \cdot \Delta_{\mathfrak{a}_\hbar}(a_\hbar) \cdot \tilde{J}_{\mathfrak{a}_\hbar}$
- $\tilde{S}_{\mathfrak{a}_\hbar}(a_\hbar) = \tilde{Q}^{-1} \cdot S_{\mathfrak{a}_\hbar}(a_\hbar) \cdot \tilde{Q}$
- $\tilde{R}_{\mathfrak{a}_\hbar} = \sigma \tilde{J}^{-1}_{\mathfrak{a}_\hbar} \cdot \tilde{J}_{\mathfrak{a}_\hbar}, \quad (\tilde{\pi} \otimes \tilde{\pi})\Omega = 0!!!$

is a triangular Hopf QUE algebra. We have also the equality

$$(\tilde{J}_{\mathfrak{a}_\hbar})_{12,3} \cdot (\tilde{J}_{\mathfrak{a}_\hbar})_{12} = (\tilde{J}_{\mathfrak{a}_\hbar})_{1,23} \cdot (\tilde{J}_{\mathfrak{a}_\hbar})_{23}.$$

We denote it as

$$A_{\mathfrak{a}_\hbar, \tilde{J}^{-1}_{\mathfrak{a}_\hbar}}$$

meaning that is obtained by a twist via $\tilde{J}^{-1}_{\mathfrak{a}_\hbar}$ from the usual trivial Hopf triangular algebra

$$(\widehat{\mathcal{U}\mathfrak{a}_\hbar}, \cdot, 1, \Delta_{\mathfrak{a}_\hbar}, \epsilon_{\mathfrak{a}_\hbar}, S_{\mathfrak{a}_\hbar}, R_{\mathfrak{a}_\hbar} = 1 \otimes 1).$$

Informally, we could say that, in the Hoschild cohomology of $\mathcal{U}\mathfrak{a}_\hbar$, $\tilde{J}_{\mathfrak{a}_\hbar}$ *is an invariant star product on the formal "Lie group" on the ring* $\mathbb{K}_\hbar$ *whose Lie algebra is the Lie algebra* $\mathfrak{a}_\hbar$ *over the ring* $\mathbb{K}_\hbar$.

Lemma 2.2.11. *Let* $r_1(\hbar) \in \mathfrak{a}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{a}_\hbar$ *as before, that is skew-symmetric and invertible. Let* $\beta_1(\hbar) \in \mathfrak{a}^*_\hbar \wedge_{\mathbf{K}_\hbar} \mathfrak{a}^*_\hbar$ *defined as*

$$\beta_1(\hbar) = (\beta_1(\hbar))_{ab}\, e^a \otimes e^b$$

where

$$r_1(\hbar)^{ab}. (\beta_1(\hbar))_{ac} = \delta^b_c$$

and then $(\beta_1(\hbar))_{ca} .r_1(\hbar)^{ba} = \delta^b_c$, $r_1(\hbar)^{ba}. (\beta_1(\hbar))_{ca} = \delta^b_c$. *Then,*

- $[r_1(\hbar), r_1(\hbar)]_{\mathfrak{a}_\hbar} = 0 \Longleftrightarrow d_{ch}\beta_1(\hbar) = 0$
- $\mu_1^{-1} \circ d_{ch} \circ \mu_1(X(\hbar)) \Longleftrightarrow [r_1^{12}(\hbar), 1 \otimes X(\hbar)]_{\mathfrak{a}_\hbar} - [X(\hbar) \otimes 1, r_1^{12}(\hbar)]_{\mathfrak{a}_\hbar}, \quad X(\hbar) \in \mathfrak{a}_\hbar$
- *Let* $\alpha(\hbar) \in \mathfrak{a}_\hbar^*$. *Then*
  $$\begin{aligned} d_{ch}\alpha(\hbar) = 0 \Longleftrightarrow 0 = [r_1^{12}(\hbar), 1 \otimes \mu_1^{-1}(\alpha(\hbar))]_{\mathfrak{a}_\hbar} - \\ -[\mu_1^{-1}(\alpha(\hbar)) \otimes 1, r_1^{12}(\hbar)]_{\mathfrak{a}_\hbar} \end{aligned}$$

Remember: Poisson coboundary ∂ can be defined in this case as:

$$-\partial = \mu_r^{-1} \circ d_{ch} \circ \mu_r$$

$$\begin{tikzcd} \bigwedge^r(\mathfrak{a}_\hbar) \arrow[r, "-\partial"] \arrow[d, "\mu_r"'] & \bigwedge^{r+1}(\mathfrak{a}_\hbar) \arrow[d, "\mu_r"] \\ \bigwedge_r(\mathfrak{a}_\hbar) \arrow[r, "d_{ch}"'] & \bigwedge_{r+1}(\mathfrak{a}_\hbar) \end{tikzcd}$$

and

$$\begin{aligned} \mu_r^{-1} : \wedge_r(\mathfrak{a}_\hbar) &\longrightarrow \wedge^r(\mathfrak{a}_\hbar) \\ \alpha &\longmapsto \mu_r^{-1}(\alpha) \end{aligned}$$

where $\left(\mu_r^{-1}(\alpha)\right)^{i_1\ldots i_r} = r_1^{j_1 i_1}(\hbar) \cdots r_1^{j_r i_r}(\hbar)\alpha_{j_1\ldots j_r}(\hbar)$

What we want is to compare the two E-K quantizations we have obtained before of two different triangular non-degenerate Lie bialgebras defined by different elements $r_1(\hbar), r_1'(\hbar) \in \mathfrak{a}_\hbar \otimes \mathfrak{a}_\hbar$, $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon_{\mathfrak{a}_\hbar} = d_c(\hbar)r_1(\hbar))$ and $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, \varepsilon'_{\mathfrak{a}_\hbar} = d_c(\hbar)r_1'(\hbar))$.

The Lie bracket in $\mathfrak{a}_\hbar$ is the same for both of them. The Lie bracket of the dual $\mathbb{K}_\hbar$-module $\mathfrak{a}_\hbar^*$ is different:

$$\begin{aligned} [\xi(\hbar), \eta(\hbar)]_{\mathfrak{a}_\hbar^*} &= (d_c(\hbar)r_1(\hbar))^{\mathsf{t}}(\xi(\hbar) \otimes \eta(\hbar)); \\ [\xi(\hbar), \eta(\hbar)]'_{\mathfrak{a}_\hbar^*} &= \left(d_c(\hbar)r_1'(\hbar)\right)^{\mathsf{t}}(\xi(\hbar) \otimes \eta(\hbar)). \end{aligned}$$

And therefore their doubles:

$$\begin{aligned} &(\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*, [,]_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*}, \varepsilon_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*} = d_c(\hbar)r) \\ &(\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*, [,]'_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*}, \varepsilon'_{\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*} = d_c'(\hbar)r) \end{aligned}$$

although r is the same in both cases, $d_c(\hbar)$ and $d_c'(\hbar)$ are different because are defined through the Lie bracket structures of $\mathfrak{a}_\hbar \oplus \mathfrak{a}_\hbar^*$.

The main point in this comparison is the following classical fact:

Theorem 2.2.12. *Let G be a Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. Let Ad be the adjoint representation of G (on $\mathfrak{g}$). Let $Ad^\wedge$ be the contragradient representation of G (on $\mathfrak{g}^*$), $Ad^\wedge = (Ad\, g^{-1})^T$. Then:*

- *$Ad^\wedge$ induces a representation of G on the exterior algebra $\wedge\mathfrak{g}^*$;*
- *$Ad^\wedge$, $g \in G$, commutes with the Chevalley-Eilenberg diferencial, d_{ch}, on $\mathfrak{g}^*$;*
- *$Ad^\wedge$ induces a representation, $Ad^\sharp$, of G on the Chevalley cohomological vector space $H^*_{ch}(\mathfrak{g})$;*
- *The representation $Ad^\sharp$ is trivial, that is $Ad^\sharp g = Id_{H^*_{ch}(\mathfrak{g})}$, $\forall g \in G$.*

We prove a theorem of this type for a Lie algebra $\mathfrak{a}_\hbar$ over $\mathbb{K}_\hbar$ and mappings $\mathrm{Ad}(\exp x(\hbar)) = \exp(ad\, x(\hbar)) : \mathfrak{a}_\hbar \longrightarrow \mathfrak{a}_\hbar$, $x(\hbar) \in \mathfrak{a}_\hbar$.

2.2.4 Interior isomorphisms of $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$

Theorem 2.2.13. *Let $X_\hbar \in \mathfrak{a}_\hbar$ and let*

$$\varphi^1_\hbar : \mathfrak{a}_\hbar \longrightarrow \mathfrak{a}_\hbar$$

be defined as

$$\varphi^1_\hbar . Y_\hbar = \exp(\hbar\, ad_{X_\hbar}).Y_\hbar.$$

Then

- *$\varphi^1_\hbar$ is well defined, that is $Im\, \varphi^1_\hbar \subset \mathfrak{a}_\hbar$;*
- *$\varphi^1_\hbar$ is invertible;*
- *$\varphi^1_\hbar$ is an isomorphism of Lie algebras.*

Our interest is in the contragradient mapping

$$\varphi^2_\hbar = \left(\left(\varphi^1_\hbar\right)^{\mathsf{t}}\right)^{-1} = \left(\exp\left(\hbar\, ad^{\mathsf{t}}_{X_\hbar}\right)\right)^{-1} = \exp(-\hbar\, ad^{\mathsf{t}}_{X_\hbar}) = \exp(\hbar\, ad^*_{X_\hbar})$$

$\mathbb{K}_\hbar$-module isomorphism of $\mathfrak{a}_\hbar$. We have

$$\begin{aligned}
\varphi^2_\hbar \otimes \varphi^2_\hbar &= \exp(\hbar\, ad^*_{X_\hbar}) \otimes \exp(\hbar\, ad^*_{X_\hbar}) \\
&= \left(\exp(\hbar\, ad^*_{X_\hbar}) \otimes Id_{\mathfrak{a}^*_\hbar}\right) \circ \left(Id_{\mathfrak{a}^*_\hbar} \otimes \exp(\hbar\, ad^*_{X_\hbar})\right) \\
&= \exp\left(ad^*_{\hbar X_\hbar} \otimes Id_{\mathfrak{a}^*_\hbar} + Id_{\mathfrak{a}^*_\hbar} \otimes ad^*_{\hbar X_\hbar}\right)
\end{aligned}$$

because $ad^*_{\hbar X_\hbar} \otimes Id_{\mathfrak{a}^*_\hbar}$ and $Id_{\mathfrak{a}^*_\hbar} \otimes ad^*_{\hbar X_\hbar}$ commute.

Theorem 2.2.14. *Let $\beta_\hbar \in \mathfrak{a}_\hbar^* \wedge \mathfrak{a}_\hbar^*$ be a Chevalley 2-cocycle, that is,*

$$d_{ch}\beta_\hbar = 0,$$

relatively to the Lie algebra structure of $\mathfrak{a}_\hbar$ and the trivial representation of $\mathfrak{a}_\hbar$ on $\mathbb{K}_\hbar$.

Then, we have

(i) $$(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar = \exp\left(ad^*_{\hbar X_\hbar} \otimes Id_{\mathfrak{a}_\hbar^*} + Id_{\mathfrak{a}_\hbar^*} \otimes ad^*_{\hbar X_\hbar}\right)\beta_\hbar = \beta_\hbar + d_{ch}\gamma_\hbar, \quad \text{where } \gamma_\hbar \in \mathfrak{a}_\hbar^*;$$

(ii) Since $\beta_\hbar$ is closed, $(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar$ is closed;

(iii) If $\beta_\hbar$ is exact, $(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar$ is exact;

(iv) $(\varphi_\hbar^2 \otimes \varphi_\hbar^2)$ acts on $H^2_{ch}(\mathfrak{a}_\hbar)$ and this action is the identity.

Proof. It is enough to prove *(i)*.

We have for $n = 1$ and any $e_i, e_j \in \mathfrak{a}_\hbar$ elements in a basis,

$$\begin{aligned}
&\left\langle (Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)\ (\beta_\hbar); e_i \otimes e_j \right\rangle = \\
&= -\left\langle \beta_\hbar; (Id \otimes ad_{\hbar X_\hbar} + ad_{\hbar X_\hbar} \otimes Id) e_i \otimes e_j \right\rangle \\
&= -\left\langle \beta_\hbar; e_i \otimes [\hbar X_\hbar, e_j]_{\mathfrak{a}_\hbar} + [\hbar X_\hbar, e_i]_{\mathfrak{a}_\hbar} \otimes e_j \right\rangle \\
&= -\hbar \left\langle \beta_\hbar; e_i \otimes X_\hbar^k C^l_{kj}(\hbar)\, e_l + X_\hbar^k C^l_{ki}(\hbar)\, e_l \otimes e_j \right\rangle \\
&= -\hbar X_\hbar^k C^l_{kj}(\hbar)(\beta_\hbar)_{il} - \hbar X_\hbar^k C^l_{ki}(\hbar)(\beta_\hbar)_{lj} \\
&= -\hbar X_\hbar^k \left(C^l_{jk}(\hbar)(\beta_\hbar)_{li} + C^l_{ki}(\hbar)(\beta_\hbar)_{lj} \right) \\
&= -\hbar X_\hbar^k \left(-C^l_{ij}(\hbar)\ (\beta_\hbar)_{lk} \right) \\
&= \hbar\, X_\hbar^k\, \beta_\hbar \left([e_i, e_j]_{\mathfrak{a}_\hbar}, e_k \right) \\
&= \hbar \beta_\hbar \left([e_i, e_j]_{\mathfrak{a}_\hbar}, X_\hbar \right) \\
&= -\hbar \beta_\hbar \left(X_\hbar, [e_i, e_j]_{\mathfrak{a}_\hbar} \right) \\
&= -\left(i(\hbar X_\hbar)\beta_\hbar \right) [e_i, e_j]_{\mathfrak{a}_\hbar} \\
&= d_{ch} \left(i(\hbar X_\hbar)\beta_\hbar \right) (e_i, e_j),
\end{aligned}$$

where we used that $\beta_\hbar$ is a 2-cocycle ($\beta_\hbar([x,y],z) + \beta_\hbar([y,z],x) + \beta_\hbar([z,x],y) = 0$, $x, y, z \in \mathfrak{a}_\hbar$), $C^k_{ij}(\hbar)$ are the structure constants of the Lie algebra $\mathfrak{a}_\hbar$ in a basis $\{e_i\}$ and $d_{ch}\alpha(e_a \otimes e_b) = -\alpha([e_a, e_b])$.

So, we obtain

$$(Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)(\beta_\hbar) = d_{ch}(i_{\hbar X_\hbar}\beta_\hbar),$$

for any cocycle $\beta_\hbar$ and $X_\hbar \in \mathfrak{a}_\hbar$.

For any elements e_i, e_j in a basis of $\mathfrak{a}_\hbar$, we have $(n = 2)$

$$
\begin{aligned}
&\left\langle (Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)^2 \beta_\hbar, e_i \otimes e_j \right\rangle = \\
&\quad = \left\langle (Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id) \cdot d_{ch}(i_{\hbar X_\hbar} \beta_\hbar); e_i \otimes e_j \right\rangle \\
&\quad = - \langle d_{ch}(i_{\hbar X_\hbar} \beta_\hbar); (Id \otimes ad_{\hbar X_\hbar} + ad_{\hbar X_\hbar} \otimes Id)(e_i \otimes e_j) \rangle \\
&\quad = - \langle d_{ch}(i_{\hbar X_\hbar} \beta_\hbar); e_i \otimes [\hbar X_\hbar, e_j]_{\mathfrak{a}_\hbar} + [\hbar X_\hbar, e_i]_{\mathfrak{a}_\hbar} \otimes e_j \rangle \\
&\quad = (i_{\hbar X_\hbar} \beta_\hbar)([e_i, [\hbar X_\hbar, e_j]_{\mathfrak{a}_\hbar}]_{\mathfrak{a}_\hbar} + [[\hbar X_\hbar, e_i]_{\mathfrak{a}_\hbar}, e_j]_{\mathfrak{a}_\hbar}) \\
&\quad = (i_{\hbar X_\hbar} \beta_\hbar)([[e_j, \hbar X_\hbar]_{\mathfrak{a}_\hbar}, e_i]_{\mathfrak{a}_\hbar} + [[\hbar X_\hbar, e_i]_{\mathfrak{a}_\hbar}, e_j]_{\mathfrak{a}_\hbar}) \\
&\quad = (i_{\hbar X_\hbar} \beta_\hbar)(-[[e_i, e_j]_{\mathfrak{a}_\hbar}, \hbar X_\hbar]_{\mathfrak{a}_\hbar}) \\
&\quad = (i_{\hbar X_\hbar} \beta_\hbar)([\hbar X_\hbar, [e_i, e_j]_{\mathfrak{a}_\hbar}]_{\mathfrak{a}_\hbar}) \\
&\quad = (i_{\hbar X_\hbar} \beta_\hbar) \circ ad_{\hbar X_\hbar}([e_i, e_j]_{\mathfrak{a}_\hbar}) \\
&\quad = \langle d_{ch}(-(i_{\hbar X_\hbar} \beta_\hbar) \circ ad_{\hbar X_\hbar}), e_i \otimes e_j \rangle .
\end{aligned}
$$

Thus, we get

$$
(Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)^2 (\beta_\hbar) = -d_{ch} \left((i(\hbar X_\hbar) \beta_\hbar) \circ ad_{\hbar X_\hbar} \right).
$$

For $n = 3$, we have

$$
\begin{aligned}
&\left\langle (Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)^3 \beta_\hbar, e_a \otimes e_b \right\rangle = \\
&\quad = \left\langle -(Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)\, d_{ch} \left((i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar) \right); e_a \otimes e_b \right\rangle \\
&\quad = \langle d_{ch} \left((i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar) \right); (Id \otimes ad_{\hbar X_\hbar} + ad_{\hbar X_\hbar} \otimes Id)(e_a \otimes e_b) \rangle \\
&\quad = \langle d_{ch} ((i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar)); e_a \otimes [\hbar X_\hbar, e_b]_{\mathfrak{a}_\hbar} + [\hbar X_\hbar, e_a]_{\mathfrak{a}_\hbar} \otimes e_b \rangle \\
&\quad = - \langle (i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar); [e_a, [\hbar X_\hbar, e_b]_{\mathfrak{a}_\hbar}]_{\mathfrak{a}_\hbar} + [[\hbar X_\hbar, e_a]_{\mathfrak{a}_\hbar}, e_b]_{\mathfrak{a}_\hbar} \rangle \\
&\quad = - \langle (i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar); [[e_b, \hbar X_\hbar]_{\mathfrak{a}_\hbar}, e_a]_{\mathfrak{a}_\hbar} + [[\hbar X_\hbar, e_a]_{\mathfrak{a}_\hbar}, e_b]_{\mathfrak{a}_\hbar} \rangle \\
&\quad = \langle (i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar); [[e_a, e_b]_{\mathfrak{a}_\hbar}, \hbar X_\hbar]_{\mathfrak{a}_\hbar} \rangle \\
&\quad = - \langle (i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar) \circ ad(\hbar X_\hbar); [e_a, e_b]_{\mathfrak{a}_\hbar} \rangle \\
&\quad = \langle d_{ch} \left((i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar) \circ ad(\hbar X_\hbar) \right); e_a \otimes e_b \rangle .
\end{aligned}
$$

Then,

$$
(Id \otimes ad^*_{\hbar X_\hbar} + ad^*_{\hbar X_\hbar} \otimes Id)^3 \beta_\hbar = d_{ch} \left((i_{\hbar X_\hbar} \beta_\hbar) \circ ad(\hbar X_\hbar) \circ ad(\hbar X_\hbar) \right).
$$

We have obtained

$$
\begin{aligned}
(\varphi^2_\hbar \otimes \varphi^2_\hbar) \beta_\hbar = \beta_\hbar + \hbar\, d_{ch}(i(X_\hbar) \beta_\hbar) + \\
&+ \hbar^2 d_{ch} \left(-\frac{1}{2!} (i(X_\hbar) \beta_\hbar) \circ ad_{X_\hbar} \right) + \\
&+ \hbar^3 d_{ch} \left(\frac{1}{3!} (i(X_\hbar) \beta_\hbar) \circ ad_{X_\hbar} \circ ad_{X_\hbar} \right) + \ldots
\end{aligned}
$$

Let us write

$$\begin{aligned}\gamma_1(\hbar) &= (i(X_\hbar)\beta_\hbar) \in \mathfrak{a}_\hbar^*\\ \gamma_2(\hbar) &= \left(-\frac{1}{2!}(i(X_\hbar)\beta_\hbar) \circ ad_{X_\hbar}\right) \in \mathfrak{a}_\hbar^*\\ \gamma_3(\hbar) &= \left(\frac{1}{3!}(i(X_\hbar)\beta_\hbar) \circ ad_{X_\hbar} \circ ad_{X_\hbar}\right) \in \mathfrak{a}_\hbar^*.\end{aligned}$$

Then, we have

$$(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar = \beta_\hbar + d_{ch}(\hbar\gamma_1(\hbar) + \hbar^2\gamma_2(\hbar) + \hbar^3\gamma_3(\hbar) + \dots)$$

and it is easy to get a general formula for $\gamma_k(\hbar) \in \mathfrak{a}_\hbar^*$, $k \in \mathbb{N}$. □

Let us compute the first terms in powers of $\hbar$ of $\gamma_1(\hbar)$, $\gamma_2(\hbar)$, $\gamma_3(\hbar)$, etc.

$$\begin{aligned}\gamma_1(\hbar) = i(X_\hbar)\beta_\hbar &= i(X_a\,\hbar^{a-1})(\beta_b\,\hbar^{b-1}) = (i(X_a)\beta_b)\,\hbar^{a+b-2}\\ &= \sum_{R\geq 0}\left(\sum_{a+b=R+2} i(X_a)\beta_b\right)\hbar^R\\ &= i(X_1)\beta_1 + [i(X_2)\beta_1 + i(X_1)\beta_2]\,\hbar +\\ &\quad + [i(X_1)\beta_3 + i(X_2)\beta_2 + i(X_3)\beta_1]\,\hbar^2 +\\ &\quad + [\quad \dots \quad]\hbar^3 + \dots,\end{aligned}$$

where $X_a \in \mathfrak{a}$, $\beta_b \in \mathfrak{a}^* \otimes_{\mathbf{K}} \mathfrak{a}^*$ and therefore $i(X_a)\beta_b \in \mathfrak{a}^*$.

$$\begin{aligned}\gamma_2(\hbar) &= -\frac{1}{2!}\left((i(X_\hbar)\beta_\hbar) \circ ad_{X_\hbar}\right) = -\frac{1}{2!}\left((i(X_m\,\hbar^{m-1})(\beta_l\,\hbar^{l-1}) \circ ad_{X_p\,\hbar^{p-1}}\right)\\ &= -\frac{1}{2!}\left((i(X_m)\beta_l) \circ ad_{X_p}\right)\hbar^{m+l+p-3}.\end{aligned}$$

As $X_m \in \mathfrak{a}$, $\beta_l \in \mathfrak{a}^* \otimes_{\mathbf{K}} \mathfrak{a}^*$, the map $i(X_m)\beta_l$ sends $\mathfrak{a}$ to $\mathbb{K}$, that is, $i(X_m)\beta_l \in \mathfrak{a}^*$. But relatively to ad_{X_p}, even being $X_p \in \mathfrak{a}$, this map doesn't send $\mathfrak{a}$ in $\mathfrak{a}$ but $\mathfrak{a}$ into $\mathfrak{a}_\hbar$.

Let $\{e_i\}$ be a basis of $(\mathfrak{a}, \mathbb{K})$, then

$$\begin{aligned}ad_{X_p}e_i = [X_p, e_i]_{\mathfrak{a}_\hbar} &= [X_p^l e_l, e_i]_{\mathfrak{a}_\hbar} = X_p^l[e_l, e_i]_{\mathfrak{a}_\hbar} = X_p^l C_{li}^k(\hbar)e_k\\ &= X_p^l\left(C_{li}^k(s)\hbar^{s-1}\right)e_k = \left(X_p^l C_{li}^k(s)e_k\right)\hbar^{s-1}\end{aligned} \tag{2.2}$$

where we have used

$$C_{li}^k(\hbar) = \sum_{s\geq 1} C_{li}^k(s)\hbar^{s-1}, \quad C_{li}^k(s) \in \mathfrak{a}.$$

Lemma 2.2.15. *Let us consider the following mappings*

$$\begin{aligned}B_s : \mathfrak{a} \times \mathfrak{a} &\longrightarrow \mathfrak{a}\\ (y, z) &\longmapsto B_s(y, z)\end{aligned}$$

where

$$B_s(y,z) = B_s(y^l e_l, z^i e_i) = y^l z^i B_s(e_l, e_i) = y^l z^i C_{li}^k(s) e_k.$$

Then, we have

(i) The definition of B_s is independent of the basis $\{e_i\}$ on $(\mathfrak{a}, \mathbb{K})$;

(ii) B_s is a $\mathbb{K}$-bilinear mapping on $\mathfrak{a}$ with values in $\mathfrak{a}$;

(iii) $[e_l, e_i]_{\mathfrak{a}_\hbar} = B_s(e_l, e_i)\hbar^{s-1}$.

We can write (2.2) as

$$\begin{aligned} ad_{X_p} e_i &= (X_p^l C_{li}^k(s) e_k)\hbar^{s-1} \\ &= X_p^l B_s(e_l, e_i)\hbar^{s-1} = B_s(X_p^l e_l, e_i)\hbar^{s-1} = B_s(X_p, e_i)\hbar^{s-1} \\ &= ((i(X_p)B_s)e_i)\hbar^{s-1} = ((i(X_p)B_s)\hbar^{s-1})e_i. \end{aligned}$$

Therefore

$$ad_{X_p} = (i(X_p)B_s)\,\hbar^{s-1},\ s = 1, 2, 3, \ldots$$

Here $i(X_p)B_s \in Hom_{\mathbf{K}}(\mathfrak{a}, \mathfrak{a})$ and doesn't contain $\hbar$.

Returning to the expression of $\gamma_2(\hbar)$, we can write now

$$\begin{aligned} \gamma_2(\hbar) &= -\frac{1}{2!}\left((i(X_m)\beta_l) \circ ad_{X_p}\right)\hbar^{m+l+p-3} \\ &= -\frac{1}{2!}\left((i(X_m)\beta_l) \circ \left((i(X_p)B_s)\hbar^{s-1}\right)\right)\hbar^{m+l+p-3} \\ &= -\frac{1}{2!}\left((i(X_m)\beta_l) \circ (i(X_p)B_s)\right)\hbar^{m+l+p+s-4} \\ &= \sum_{R\geq 0}\left(\sum_{\substack{m+l+p+s=R+4 \\ m,l,p,s\geq 1}} -\frac{1}{2!}\left((i(X_m)\beta_l) \circ (i(X_p)B_s)\right)\right)\hbar^R \\ &= -\frac{1}{2!}\left[(i(X_1)\beta_1) \circ (i(X_1)B_1)\right] + \\ &\quad -\frac{1}{2!}\left[(i(X_2)\beta_1) \circ (i(X_1)B_1) + (i(X_1)\beta_2) \circ (i(X_1)B_1) + \right. \\ &\quad \left. + (i(X_1)\beta_1) \circ (i(X_2)B_1) + (i(X_1)\beta_1) \circ (i(X_1)B_2)\right]\hbar + \\ &\quad + [\ldots]\,\hbar^2 + \ldots \end{aligned}$$

We have also

$$
\begin{aligned}
\gamma_3(\hbar) &= \frac{1}{3!}\left((i(X_\hbar)\beta_\hbar) \circ ad_{X_\hbar} \circ ad_{X_\hbar}\right) \\
&= \frac{1}{3!}\Big(i(X_p\hbar^{p-1})(\beta_a\hbar^{a-1}) \circ (ad_{X_b}\hbar^{b-1}) \circ (ad_{X_q}\hbar^{q-1})\Big) \\
&= \frac{1}{3!}\left((i(X_p)\beta_a) \circ (i(X_b)B_s) \circ (i(X_q)B_r)\right)\hbar^{p+a+b+q+s+r-6} \\
&= \sum_{M\geq 0}\left(\sum_{\substack{p+a+b+q+s+r=M+6\\ p,a,b,q,s,r\geq 1}} \frac{1}{3!}\left((i(X_p)\beta_a) \circ (i(X_b)B_s) \circ (i(X_q)B_r)\right)\right)\hbar^M \\
&= \frac{1}{3!}\left[(i(X_1)\beta_1) \circ (i(X_1)B_1) \circ (i(X_1)B_1)\right] + \\
&\quad + \frac{1}{3!}\left[(i(X_2)\beta_1) \circ (i(X_1)B_1) \circ (i(X_1)B_1) + \right. \\
&\quad + (i(X_1)\beta_2) \circ (i(X_1)B_1) \circ (i(X_1)B_1) + \\
&\quad + (i(X_1)\beta_1) \circ (i(X_2)B_1) \circ (i(X_1)B_1) + \\
&\quad + (i(X_1)\beta_1) \circ (i(X_1)B_2) \circ (i(X_1)B_1) \\
&\quad + (i(X_1)\beta_1) \circ (i(X_1)B_1) \circ (i(X_2)B_1) + \\
&\quad \left. +(i(X_1)\beta_1) \circ (i(X_1)B_1) \circ (i(X_1)B_2)\right]\hbar + \\
&\quad + [\dots]\,\hbar^2 + \dots
\end{aligned}
$$

Then we get

$$
\begin{aligned}
(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar &= \beta_\hbar + d_{ch}(\hbar\gamma_1(\hbar) + \hbar^2\gamma_2(\hbar) + \hbar^3\gamma_3(\hbar) + \dots) \\
&= \beta_\hbar + d_{ch}\left[i(X_1)\beta_1\right]\hbar + \\
&\quad + d_{ch}\left[i(X_2)\beta_1 + i(X_1)\beta_2 - \frac{1}{2!}(i(X_1)\beta_1) \circ (i(X_1)\beta_1)\right]\hbar^2 \\
&\quad + d_{ch}\left[i(X_1)\beta_3 + i(X_2)\beta_2 + i(X_3)\beta_1 + \right. \\
&\quad - \frac{1}{2!}(i(X_2)\beta_1 \circ i(X_1)B_1 + i(X_1)\beta_2 \circ i(X_1)B_1 + \\
&\quad + i(X_1)\beta_1 \circ i(X_2)B_1 + i(X_1)\beta_1 \circ i(X_1)B_2) + \\
&\quad \left. + \frac{1}{3!}i(X_1)\beta_1 \circ i(X_1)B_1 \circ i(X_1)B_1\right]\hbar^3 + \dots
\end{aligned}
$$

Let us define the following elements in $\mathfrak{a}^*$,

$$
\begin{aligned}
&\alpha_1 = i(X_1)\beta_1 \\
&\alpha_2 = i(X_2)\beta_1 + i(X_1)\beta_2 - \frac{1}{2!}(i(X_1)\beta_1) \circ (i(X_1)\beta_1 \qquad (2.3)\\
&\alpha_3 = i(X_1)\beta_3 + i(X_2)\beta_2 + i(X_3)\beta_1 - \frac{1}{2!}(i(X_2)\beta_1 \circ i(X_1)B_1 + \\
&\quad + i(X_1)\beta_2 \circ i(X_1)B_1 + i(X_1)\beta_1 \circ i(X_2)B_1 + i(X_1)\beta_1 \circ i(X_1)B_2) + \\
&\quad + \frac{1}{3!} i(X_1)\beta_1 \circ i(X_1)B_1 \circ i(X_1)B_1 \\
&\quad \vdots \\
&\alpha_k = \sum_{j=1}^{k} \left(\sum_{i=1}^{k-j+1} \frac{(-1)^{i+1}}{i!} \sum_{\substack{a_1+a_2+b_2 \cdots + a_i + b_i = i+k-j \\ a_1, a_2, b_2, \ldots, a_i, b_i \geq 1}} \Big((i_{X_{a_1}} \beta_j) \circ i_{X_{a_2}} B_{b_2} \cdots \circ i_{X_{a_i}} B_{b_i} \Big) \right)
\end{aligned}
$$

then

$$(\varphi_\hbar^2 \otimes \varphi_\hbar^2)\beta_\hbar = \beta_\hbar + \sum_{k \geq 1} d_{ch}(\alpha_k)\hbar^k, \ \alpha_k \in \mathfrak{a}^*.$$

We then prove a converse of last theorem.

Theorem 2.2.16. *Let*

$$\alpha(\hbar) = \alpha_k \hbar^k \in \mathfrak{a}_\hbar^*, \quad \alpha_k \in \mathfrak{a}^*, \ k = 1, 2, 3, \ldots,$$

there exists a unique element

$$X_\hbar = X_l \hbar^{l-1}, \quad X_l \in \mathfrak{a}$$

verifying the equality

$$\left(exp(ad^*_{\hbar X_\hbar}) \otimes \exp(ad^*_{\hbar X_\hbar})\right) \beta_\hbar = \beta_\hbar + d_{ch}\alpha(\hbar).$$

Proof. If $X_\hbar = X_l \hbar^{l-1}$ exists verifying last equality, it must be obtained from the equalities (2.3).

Due to the invertibility of β_1, from the first equality we determine X_1. Then, because we know at this step $X_1, \beta_1, \beta_2, \alpha_2$ and β_1 is invertible, from the second equality we determine X_2. X_3 can be computed from the third equality because β_1 is invertible and at this step we know X_1, X_2, B_1, B_2, β_1, β_2 and β_3, etc, etc.. □

Recall

- $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$ is a finite-dimensional Lie algebra over $\mathbb{K}_\hbar$ which is a deformation Lie algebra of the Lie algebra $(\mathfrak{a}, [,]_\mathfrak{a})$ over $\mathbb{K}$.

- $r_1(\hbar), r_1'(\hbar)$ are two elements of $\mathfrak{a}_\hbar \otimes_{\mathbf{K}_\hbar} \mathfrak{a}_\hbar$ which are non degenerate skew-symmetric and solutions of YBE relatively to the Lie algebra $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$ and such that the terms in power $\hbar^0$ coincide and are equal to $r_1 \in \mathfrak{a} \otimes_{\mathbf{K}} \mathfrak{a}$.

- $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, r_1(\hbar))$, $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar}, r_1'(\hbar))$ are the corresponding finite dimensional Lie bialgebras over $\mathbb{K}_\hbar$, both deformation Lie bialgebras for the Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, r_1)$ over $\mathbb{K}$.

- Let $\varphi_\hbar^1 : \mathfrak{a}_\hbar \longrightarrow \mathfrak{a}_\hbar$ be a Lie algebra automorphism of $(\mathfrak{a}_\hbar, [,]_{\mathfrak{a}_\hbar})$.

- Let $\psi_\hbar : \mathfrak{a}^*_{r_1(\hbar)} \longrightarrow \mathfrak{a}^*_{r_1'(\hbar)}$ be a Lie algebra isomorphism from the Lie algebra $(\mathfrak{a}^*_{r_1(\hbar)}, [,]_{\mathfrak{a}^*_{r_1(\hbar)}})$ to the Lie algebra $(\mathfrak{a}^*_{r_1'(\hbar)}, [,]_{\mathfrak{a}^*_{r_1'(\hbar)}})$.

- Let $(\varphi_\hbar^1, \psi_\hbar) : \mathfrak{a}_\hbar \oplus \mathfrak{a}^*_{r_1(\hbar)} \longrightarrow \mathfrak{a}_\hbar \oplus \mathfrak{a}^*_{r_1'(\hbar)}$ be a Lie algebra isomorphism between the doubles.

- Let $\tilde{\varphi}_\hbar^1 : \mathcal{U}(\mathfrak{a}_\hbar) \longrightarrow \mathcal{U}(\mathfrak{a}_\hbar)$ and $\tilde{\psi}_\hbar : \mathcal{U}(\mathfrak{a}^*_{r_1(\hbar)}) \longrightarrow \mathcal{U}(\mathfrak{a}^*_{r_1'(\hbar)})$ be the (which are continuous in the $(\hbar)$-adic topology) associative algebra isomorphisms extensions respectively of $\varphi_\hbar^1$ and $\psi_\hbar$.

Theorem 2.2.17. *Let $\tilde{J}^{r_1(\hbar)}_{\mathfrak{a}_\hbar}$ and $\tilde{J}^{r_1'(\hbar)}_{\mathfrak{a}_\hbar}$ be invariant star products obtained in Theorem 3.10. Suppose $\beta_1(\hbar) \in \mathfrak{a}^*_\hbar \wedge_{\mathbf{K}_\hbar} \mathfrak{a}^*_\hbar$ is defined as $\beta_1(\hbar) = (\beta_1(\hbar))_{ab}\, e^a \otimes e^b$ where $r_1(\hbar)^{ab}.\,(\beta_1(\hbar))_{ac} = \delta^b_c$ and $\beta_1'(\hbar) \in \mathfrak{a}^*_\hbar \wedge_{\mathbf{K}_\hbar} \mathfrak{a}^*_\hbar$ is defined in a similar way from $r_1'(\hbar)$.*

Then, $\tilde{J}^{r_1(\hbar)}_{\mathfrak{a}_\hbar}$ and $\tilde{J}^{r_1'(\hbar)}_{\mathfrak{a}_\hbar}$ are equivalent star products if, and only if, $\beta_1(\hbar)$ and $\beta_1'(\hbar)$ belong to the same cohomological class. In other words, $\tilde{J}^{r_1(\hbar)}_{\mathfrak{a}_\hbar}$ and $\tilde{J}^{r_1'(\hbar)}_{\mathfrak{a}_\hbar}$ are equivalent star products if, and only if, there exists a 1-cochain $\alpha(\hbar) \in \mathfrak{a}^*_\hbar$ such that

$$\beta_1'(\hbar) = \beta_1(\hbar) + d_{ch}\alpha(\hbar).$$

Sketch of the proof $(\Longleftarrow)$

- $[\beta_1(\hbar)] = [\beta_1'(\hbar)]$, there exists $X_\hbar \in \mathfrak{a}_\hbar$ such that

$$\exp(ad^*_{\hbar X_\hbar})^{\otimes^2} \beta_1(\hbar) = \beta_1'(\hbar).$$

- Then $\varphi_\hbar^1 = \exp(ad_{\hbar X_\hbar})$ is a Lie algebra isomorphism $\mathfrak{a}_\hbar \longrightarrow \mathfrak{a}_\hbar$ and $(\varphi_\hbar^1 \otimes \varphi_\hbar^1) r_1(\hbar) = r_1'(\hbar)$.

- The map $(\varphi_\hbar^1, ((\varphi_\hbar^1)^{\mathfrak{t}})^{-1})$ is a Lie bialgebra isomorphism between $\mathfrak{a}_\hbar \oplus \mathfrak{a}^*_{r_1(\hbar)}$ and $\mathfrak{a}_\hbar \oplus \mathfrak{a}^*_{r_1'(\hbar)}$.

- We prove that $\tilde{J}^{r_1'\hbar}_{\mathfrak{a}_\hbar} = (\tilde{\varphi}_\hbar^1 \otimes \tilde{\varphi}_\hbar^1)\tilde{J}^{r_1\hbar}_{\mathfrak{a}_\hbar}$.

- Using a theorem by Drinfeld, there exists an element $u = \exp(\hbar X_\hbar)$ such that

$$\tilde{J}^{r_1'\hbar}_{\mathfrak{a}_\hbar} = \Delta_{\mathfrak{a}}(u) \cdot \tilde{J}^{r_1\hbar}_{\mathfrak{a}_\hbar} \cdot (u^{-1} \otimes u^{-1})$$

and u^{-1} defines the equivalence.

Sketch of the proof ($\Longrightarrow$)

- Hochschid cohomology on $\mathcal{U}(\mathfrak{a}_\hbar)$ appears.
- We construct an element $\alpha(\hbar) \in \mathfrak{a}_\hbar^*$ such that

$$\beta_1'(\hbar) = \beta_1(\hbar) + d_{ch}\alpha(\hbar)$$

  where $\alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \dots$.

2.2.5 Hochschild cohomology on the coalgebra $\mathcal{U}\mathfrak{a}_\hbar$

From a theorem by Cartier, we get

$$H^*_{Hoch}(\Gamma(\mathfrak{a}_\hbar)) = \wedge(\mathfrak{a}_\hbar)$$

where $\Gamma(\mathfrak{a}_\hbar)$ is the coalgebra of divided powers (see also Bourbaki). But from Cartier and Bourbaki, we know

$$\Gamma(\mathfrak{a}_\hbar) \simeq TS(\mathfrak{a}_\hbar) \simeq S(\mathfrak{a}_\hbar)$$

as $\mathbb{K}_\hbar$ bialgebras (making a proof similar to the classical one for Lie algebras over a field of characteristic 0)

We have also an isomorphism

$$S(\mathfrak{a}_\hbar) \simeq \mathcal{U}(\mathfrak{a}_\hbar)$$

as coalgebras. From these isomorphisms, we get what we want

$$H^*_{Hoch}(\mathcal{U}(\mathfrak{a}_\hbar)) \simeq \wedge(\mathfrak{a}_\hbar).$$

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On central extension of Leibniz n–algebras

J. M. Casas[*]

Abstract

The study of central extensions of Leibniz n-algebras by means of homological methods is the main goal of the paper. Thus induced abelian extensions are introduced and the classification of various classes of central extensions depending on the character of the homomorphism $\theta_*(E)$ in the five-term exact sequence

$$_nHL_1(\mathcal{K}) \to {_nHL_1(\mathcal{L})} \stackrel{\theta_*(E)}{\to} \mathrm{M} \to {_nHL_0(\mathcal{K})} \to {_nHL_0(\mathcal{L})} \to 0$$

associated to the abelian extension $E : 0 \to \mathrm{M} \stackrel{\kappa}{\to} \mathcal{K} \stackrel{\pi}{\to} \mathcal{L} \to 0$ is done. Homological characterizations of this various classes of central extensions are given. The universal central extension corresponding to a perfect Leibniz n-algebra is constructed and characterized. The endofunctor $\mathfrak{uce}$ which assigns to a perfect Leibniz n-algebra its universal central extension is described. Functorial properties are obtained and several results related with the classification in isogeny classes are achieved. Finally, for a covering $f : \mathcal{L}' \twoheadrightarrow \mathcal{L}$ (a central extension with $\mathcal{L}'$ a perfect Leibniz n-algebra), the conditions under which an automorphism or a derivation of $\mathcal{L}$ can be lifted to an automorphism or a derivation of $\mathcal{L}'$ are obtained.

1 Introduction

The state of a classical dynamic system is described in Hamiltonian mechanics by means of N coordinates $q_1, \ldots, q_N$ and N momenta $p_1, \ldots, p_N$. The 2N variables $\{q_1, \ldots, p_N\}$ are referred as canonical variables of the system. Other physically important quantities as energy and momentum are functions $F = F(q, p)$ of the canonical variables. These functions, called observables, form an infinite dimensional Lie algebra with respect to the Poisson bracket $\{F, G\} = \sum_{i=1}^{N} (\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i})$. The equation of motion are $\dot{q_i} = \frac{\partial H}{\partial p_i}$ and $\dot{p_i} = -\frac{\partial H}{\partial q_i}$, where the Hamiltonian operator of the system H is the total energy. These equations may be written in terms of Poisson brackets as $\dot{q_i} = \{q_i, H\}; \dot{p_i} = \{p_i, H\}$. In general the time evolution of an observable F is given by $\dot{F} = \{F, H\}$.

The simplest phase space for Hamiltonian mechanics is $\mathbb{R}^2$ with coordinates x, y and canonical Poisson bracket $\{f_1, f_2\} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} = \frac{\partial(f_1,f_2)}{\partial(x,y)}$. This bracket satisfies the

[*]Dpto. Matemática Aplicada I, Universidad de Vigo, E. U. I. T. Forestal, Campus Universitario A Xunqueira, 36005 Pontevedra, Spain. Email:jmcasas@uvigo.es

Jacobi identity $\{f_1,\{f_2,f_3\}\}+\{f_3,\{f_1,f_2\}\}+\{f_2,\{f_3,f_1\}\}=0$ and give rise to the Hamilton equations of motion $\frac{df}{dt}=\{H,f\}$.

In 1973, Nambu [23] proposed the generalization of last example defining for a tern of classical observables on the three dimensional space $\mathbb{R}^3$ with coordinates x,y,z the canonical bracket given by $\{f_1,f_2,f_3\}=\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$ where the right hand side is the Jacobian of the application $f=(f_1,f_2,f_3):\mathbb{R}^3\to\mathbb{R}^3$. This formula naturally generalizes the usual Poisson bracket from a binary to a ternary operation on the classical observables. The Nambu-Hamilton generalized motion equations include two Hamiltonian operators H_1 and H_2 and have the form $\frac{df}{dt}=\{H_1,H_2,f\}$. For the canonical Nambu bracket the following fundamental identity holds

$$\{\{f_1,f_2,f_3\},f_4,f_5\}+\{f_3,\{f_1,f_2,f_4\},f_5\}+\{f_3,f_4,\{f_1,f_2,f_5\}\}=$$
$$\{f_1,f_2,\{f_3,f_4,f_5\}\}$$

This formula can be considered as the most natural generalization, at least from the dynamical viewpoint, of the Jacobi identity. Within the framework of Nambu mechanics, the evolution of a physical system is determined by $n-1$ functions $H_1,\dots,H_{n-1}\in C^\infty(\mathrm{M})$ and the equation of motion of an observable $f\in C^\infty(\mathrm{M})$ is given by $df/dt=\{H_1,\dots,H_{n-1},f\}$.

These ideas inspired novel mathematical structures by extending the binary Lie bracket to a n-ary bracket. The study of this kind of structures and its application in different areas as Geometry and Mathematical Physics is the subject of a lot of papers, for example see [8], [10], [11], [12], [13], [14], [16], [22], [24], [25], [26], [27], [28], [29], [30] and references given there.

The aim of this paper is to continue with the development of the Leibniz n-algebras theory. Concretely, using homological machinery developed in [3], [4], [5], [8] we board an extensive study of central extensions of Leibniz n-algebras. Thus in Section 3 we deal with induced abelian extensions and Section 4 is devoted to the classification of various classes of central extensions depending on the character of the homomorphism $\theta_*[E]$ in the five-term exact sequence

$${}_nHL_1(\mathcal{K})\to {}_nHL_1(\mathcal{L})\stackrel{\theta_*(E)}{\to}\mathrm{M}\to {}_nHL_0(\mathcal{K})\to {}_nHL_0(\mathcal{L})\to 0$$

associated to the abelian extension $E:0\to\mathrm{M}\stackrel{\kappa}{\to}\mathcal{K}\stackrel{\pi}{\to}\mathcal{L}\to 0$. Homological characterizations of this various classes of central extensions are given. When we restrict to the case $n=2$ we recover results on central extensions of Leibniz algebras in [2], [6], [7], [9].

Sections 5 and 6 are devoted to the construction and characterization of universal central extensions of perfect Leibniz n-algebras. We construct an endofunctor $\mathfrak{uce}$ which assigns to a perfect Leibniz n-algebra its universal central extension. Functorial properties are obtained and several results related with the classification in isogeny classes are achieved. Finally, in Section 7 we analyze the conditions to lift an automorphism or a derivation of $\mathcal{L}$ to $\mathcal{L}'$ in a covering (central extension where $\mathcal{L}'$ is a perfect Leibniz n-algebra) $f:\mathcal{L}\twoheadrightarrow\mathcal{L}'$.

2 Preliminaries on Leibniz n-algebras

A *Leibniz n-algebra* is a $\mathbb{K}$-vector space $\mathcal{L}$ equipped with a n-linear bracket $[-,\dots,-] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the following fundamental identity

$$[[x_1,x_2,\dots,x_n],y_1,y_2,\dots,y_{n-1}] = \sum_{i=1}^{n}[x_1,\dots,x_{i-1},[x_i,y_1,y_2,\dots,y_{n-1}],x_{i+1},\dots,x_n] \tag{1}$$

A morphism of Leibniz n-algebras is a linear map preserving the n-bracket. Thus we have defined the category of Leibniz n-algebras, denoted by ${}_n\mathbf{Leib}$. In case $n = 2$ the identity (1) is the Leibniz identity, so a Leibniz 2-algebra is a Leibniz algebra [18, 19, 20], and we use **Leib** instead of ${}_2\mathbf{Leib}$.

Leibniz $(n+1)$-algebras and Leibniz algebras are related by means of the Daletskii's functor [10] which assigns to a Leibniz $(n+1)$-algebra $\mathcal{L}$ the Leibniz algebra $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ with bracket

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, b_1,\dots,b_n] \otimes \cdots \otimes a_n \tag{2}$$

Conversely, if $\mathcal{L}$ is a Leibniz algebra, then also it is a Leibniz n-algebra under the following n-bracket [8]

$$[x_1,x_2,\dots,x_n] := [x_1,[x_2,\dots,[x_{n-1},x_n]]] \tag{3}$$

Examples:

1. Examples of Leibniz algebras in [1], [19] provides examples of Leibniz n-algebras with the bracket defined by equation (3).

2. A Lie triple system [17] is a vector space equipped with a ternary bracket $[-,-,-]$ that satisfies the same identity (1) (particular case $n = 3$) and, instead of skew-symmetry, satisfies the conditions $[x,y,z] + [y,z,x] + [z,x,y] = 0$ and $[x,y,y] = 0$. It is an easy exercise to verify that Lie triple systems are non-Lie Leibniz 3-algebras.

3. $\mathbb{R}^{n+1}$ is a Leibniz n-algebra with the bracket given by $[\overrightarrow{x_1},\overrightarrow{x_2},\dots,\overrightarrow{x_n}] := det(A)$, where A is the following matrix

$$\begin{pmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & \dots & \overrightarrow{e_{n+1}} \\ x_{11} & x_{21} & \dots & x_{(n+1)1} \\ x_{12} & x_{22} & \dots & x_{(n+1)2} \\ \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & \dots & x_{(n+1)n} \end{pmatrix}$$

Here $\overrightarrow{x_i}= x_{1i}\,\overrightarrow{e_1} + x_{2i}\,\overrightarrow{e_2} + \cdots + x_{(n+1)i}\,\overrightarrow{e_{n+1}}$ and $\{\overrightarrow{e_1},\overrightarrow{e_2},\dots,\overrightarrow{e_{n+1}}\}$ is the canonical basis of $\mathbb{R}^{n+1}$.

4. An associative trialgebra is a $\mathbb{K}$-vector space A equipped with three binary operations: $\dashv, \perp, \vdash$ (called left, middle and right, respectively), satisfying eleven associative relations [21]. Then A can be endowed with a structure of Leibniz 3-algebra with respect to the bracket

$$[x, y, z] = x \dashv (y \perp z) - (y \perp z) \vdash x - x \dashv (z \perp y) + (z \perp y) \vdash x$$

$$= x \dashv (y \perp z - z \perp y) - (y \perp z - z \perp y) \vdash x$$

for all $x, y, z \in$ A.

5. Let $\mathcal{C}^\infty(\mathbb{R}^n)$ be the algebra of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ and $x_1, \dots, x_n$ be the coordinates on $\mathbb{R}^n$. Then $\mathcal{C}^\infty(\mathbb{R}^n)$ equipped with the bracket $[f_1, \dots, f_n] = det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\dots,n}$ is a n-Lie algebra [12], so also it is a Leibniz n-algebra.

Let $\mathcal{L}$ be a Leibniz n-algebra. A subalgebra $\mathcal{K}$ of $\mathcal{L}$ is called *n-sided ideal* if $[l_1, l_2, \dots, l_n] \in \mathcal{K}$ as soon as $l_i \in \mathcal{K}$ and $l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_n \in \mathcal{L}$, for all $i = 1, 2, \dots, n$. This definition guarantees that the quotient $\mathcal{L}/\mathcal{K}$ is endowed with a well defined bracket induced naturally by the bracket in $\mathcal{L}$.

Let $\mathcal{M}$ and $\mathcal{P}$ be n-sided ideals of a Leibniz n-algebra $\mathcal{L}$. The *commutator ideal* of $\mathcal{M}$ and $\mathcal{P}$, denoted by $[\mathcal{M}, \mathcal{P}, \mathcal{L}^{n-2}]$, is the n-sided ideal of $\mathcal{L}$ spanned by the brackets $[l_1, \dots, l_i, \dots, l_j, \dots, l_n]$ as soon as $l_i \in \mathcal{M}, l_j \in \mathcal{P}$ and $l_k \in \mathcal{L}$ for all $k \neq i, k \neq j; i, j, k \in \{1, 2, \dots, n\}$. Obviously $[\mathcal{M}, \mathcal{P}, \mathcal{L}^{n-2}] \subset \mathcal{M} \cap \mathcal{P}$. In the particular case $\mathcal{M} = \mathcal{P} = \mathcal{L}$ we obtain the definition of *derived algebra* of a Leibniz n-algebra $\mathcal{L}$. If $\mathcal{L} = [\mathcal{L}, \overset{n}{\dots}, \mathcal{L}] = [\mathcal{L}^n]$, then the Leibniz n-algebra is called *perfect*.

For a Leibniz n-algebra $\mathcal{L}$, we define its *centre* as the n-sided ideal

$$\mathrm{Z}(\mathcal{L}) = \{l \in \mathcal{L} \mid [l_1, \dots, l_{i-1}, l, l_{i+1}, \dots, l_n] = 0, \forall l_i \in \mathcal{L}, i = 1, \dots, \hat{i}, \dots, n\}$$

An *abelian* Leibniz n-algebra is a Leibniz n-algebra with trivial bracket, that is, the commutator n-sided ideal $[\mathcal{L}^n] = [\mathcal{L}, \dots, \mathcal{L}] = 0$. It is clear that a Leibniz n-algebra $\mathcal{L}$ is abelian if and only if $\mathcal{L} = Z(\mathcal{L})$. To any Leibniz n-algebra $\mathcal{L}$ we can associate its *largest abelian quotient* $\mathcal{L}_{ab}$. It is easy to verify that $\mathcal{L}_{ab} \cong \mathcal{L}/[\mathcal{L}^n]$.

A *representation* of a Leibniz n-algebra $\mathcal{L}$ is a $\mathbb{K}$-vector space M equipped with n actions $[-, \dots, -] : \mathcal{L}^{\otimes i} \otimes$ M $\otimes \mathcal{L}^{\otimes(n-1-i)} \to$ M, $0 \leq i \leq n-1$, satisfying $(2n-1)$ axioms which are obtained from (1) by letting exactly one of the variables $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ be in M and all the others in $\mathcal{L}$.

If we define the multilinear applications $\rho_i : \mathcal{L}^{\otimes n-1} \to \mathrm{End}_{\mathbb{K}}(\mathrm{M})$ by

$$\rho_i(l_1, \dots, l_{n-1})(m) = [l_1, \dots, l_{i-1}, m, l_i, \dots, l_{n-1}], 1 \leq i \leq n-1$$

$$\rho_n(l_1, \dots, l_{n-1})(m) = [l_1, \dots, l_{n-1}, m]$$

then the axioms of representation can be expressed by means of the following identities [3]:

1. For $2 \leq k \leq n$,
$$\rho_k([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2}) =$$
$$\sum_{i=1}^{n} \rho_i(l_1, \ldots, \hat{l_i}, \ldots, l_n) \cdot \rho_k(l_i, l_{n+1}, \ldots, l_{2n-2})$$

2. For $1 \leq k \leq n$,
$$[\rho_1(l_n, \ldots, l_{2n-2}), \rho_k(l_1, \ldots, l_{n-1})] =$$
$$\sum_{i=1}^{n-1} \rho_k(l_1, \ldots, l_{i-1}, [l_i, l_n, \ldots, l_{2n-2}], l_{i+1}, \ldots, l_{n-1})$$

   being the bracket on $\mathrm{End}_{\mathbb{K}}(\mathrm{M})$ the usual one for associative algebras.

A particular instance of representation is the case M $=\mathcal{L}$, where the applications ρ_i are the adjoint representations

$$ad_i(l_1, \ldots, l_{n-1})(l) = [l_1, \ldots, l_{i-1}, l, l_i, \ldots, l_{n-1}], 1 \leq i \leq n-1$$
$$ad_n(l_1, \ldots, l_{n-1})(l) = [l_1, \ldots, l_{n-1}, l]$$

If the components of the representation $ad : \mathcal{L}^{\otimes n-1} \to \mathrm{End}_{\mathbb{K}}(\mathcal{L})$ are $ad = (ad_1, \ldots, ad_n)$, then Ker $ad = \{l \in \mathcal{L} \mid ad_i(l_1, \ldots, l_{n-1})(l) = 0, \forall (l_1, \ldots, l_{n-1}) \in \mathcal{L}^{\otimes n-1}, 1 \leq i \leq n\}$, that is, Ker ad is the centre of $\mathcal{L}$.

Now we briefly recall the (co)homology theory for Leibniz n-algebras developed in [3, 8].

Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. Then $Hom(\mathcal{L}, M)$ is a $\mathcal{D}_{n-1}(\mathcal{L})$-representation as Leibniz algebras [8]. One defines the cochain complex ${}_nCL^*(\mathcal{L}, M)$ to be $CL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. We also put ${}_nHL^*(\mathcal{L}, M) := H^*({}_nCL^*(\mathcal{L}, M))$. Thus, by definition ${}_nHL^*(\mathcal{L}, M) \cong HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. Here $CL^\star$ denotes the Leibniz complex and $HL^\star$ its homology, called Leibniz cohomology (see [19, 20] for more information). In case $n = 2$, this cohomology theory gives ${}_2HL^m(\mathcal{L}, M) \cong HL^{m+1}(\mathcal{L}, M), \ \ m \geq 1$ and ${}_2HL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M)$. On the other hand, ${}_nHL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M)$ and ${}_nHL^1(\mathcal{L}, M)$ $\cong \mathrm{Ext}(\mathcal{L}, M)$, where $\mathrm{Ext}(\mathcal{L}, M)$ denotes the set of isomorphism classes of abelian extensions of $\mathcal{L}$ by M [8].

Homology with trivial coefficients of a Leibniz n-algebra $\mathcal{L}$ is defined in [3] as the homology of the Leibniz complex ${}_nCL_\star(\mathcal{L}) := CL_\star(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$, where $\mathcal{L}$ is endowed with a structure of $\mathcal{D}_{n-1}(\mathcal{L})$ symmetric corepresentation [20]. We denote the homology groups of this complex by ${}_nHL_\star(\mathcal{L})$. When $\mathcal{L}$ is a Leibniz 2-algebra, that is a Leibniz algebra, then we have that ${}_2HL_k(\mathcal{L}) \cong HL_{k+1}(\mathcal{L}), k \geq 1$. Particularly, ${}_2HL_0(\mathcal{L}) \cong HL_1(\mathcal{L}) \cong \mathcal{L}/[\mathcal{L}, \mathcal{L}] = \mathcal{L}_{ab}$. On the other hand, ${}_nHL_0(\mathcal{L}) = \mathcal{L}_{ab}$ and ${}_nHL_1(\mathcal{L}) \cong (\mathcal{R} \cap [\mathcal{F}^n])/[\mathcal{R}, \mathcal{F}^{n-1}]$ for a free presentation $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{L} \to 0$.

Moreover, to a short exact sequence $0 \to \mathcal{M} \to \mathcal{K} \to \mathcal{L} \to 0$ of Leibniz n-algebras we can associate the following five-term natural exact sequences [3]:

$$0 \to {}_nHL^0(\mathcal{L}, A) \to {}_nHL^0(\mathcal{K}, A) \to Hom_{\mathcal{L}}(\mathcal{M}/[\mathcal{M}, \mathcal{M}, \mathcal{K}^{n-2}], A) \to$$

$$_nHL^1(\mathcal{L}, A) \to {_nHL^1}(\mathcal{K}, A) \tag{4}$$

for every $\mathcal{L}$-representation A, and

$$_nHL_1(\mathcal{K}) \to {_nHL_1}(\mathcal{L}) \to \mathcal{M}/[\mathcal{M}, \mathcal{K}^{n-1}] \to {_nHL_0}(\mathcal{K}) \to {_nHL_0}(\mathcal{L}) \to 0 \tag{5}$$

3 Induced abelian extensions

An *abelian extension* of Leibniz n-algebras is an exact sequence $E : 0 \to \mathrm{M} \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0$ of Leibniz n-algebras such that $[k_1, \dots, k_n] = 0$ as soon as $k_i \in \mathrm{M}$ and $k_j \in \mathrm{M}$ for some $1 \leq i, j \leq n$ (i. e., $[\mathrm{M}, \mathrm{M}, \mathcal{K}^{n-2}] = 0$). Here $k_1, \dots, k_n \in \mathcal{K}$. Clearly then M is an *abelian* Leibniz n-algebra. Let us observe that the converse is true only for $n = 2$. Two such extensions E and E' are isomorphic when there exists a Leibniz n-algebra homomorphism from $\mathcal{K}$ to $\mathcal{K}'$ which is compatible with the identity on M and on $\mathcal{L}$. One denotes by $\mathrm{Ext}(\mathcal{L}, M)$ the set of isomorphism classes of extensions of $\mathcal{L}$ by M.

If E is an abelian extension of Leibniz n-algebras, then M is equipped with a $\mathcal{L}$-representation structure given by

$$[l_1, \dots, l_{i-1}, m, l_{i+1}, \dots, l_n] = [k_1, \dots, k_{i-1}, \kappa(m), k_{i+1}, \dots, k_n]$$

such that $\pi(k_j) = l_j, j = 1, \dots, i-1, i+1, \dots, n, i = 1, 2, \dots, n$.

The abelian extensions of Leibniz n-algebras are the objects of a category whose morphisms are the commutative diagrams of the form:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{M}_1 \arrow[r, "\kappa_1"] \arrow[d, "\alpha"'] & \mathcal{K}_1 \arrow[r, "\pi_1"] \arrow[d, "\beta"'] & \mathcal{L}_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathrm{M}_2 \arrow[r, "\kappa_2"] & \mathcal{K}_2 \arrow[r, "\pi_2"] & \mathcal{L}_2 \arrow[r] & 0
\end{tikzcd}$$

We denote such morphism as $(\alpha, \beta, \gamma) : (E_1) \to (E_2)$. It is evident that α and γ satisfy the following identities

$$\alpha([l_1, \dots, l_{i-1}, m, l_{i+1}, \dots, l_n]) = [\gamma(l_1), \dots, \gamma(l_{i-1}), \alpha(m), \gamma(l_{i+1}), \dots, \gamma(l_n)]$$

$i = 1, 2, \dots, n$, provided than M_2 is considered as $\mathcal{L}_1$-representation via γ. That is, α is a morphism of $\mathcal{L}_1$-representations.

Given an abelian extension E and a homomorphism of Leibniz n-algebras $\gamma : \mathcal{L}_1 \to \mathcal{L}$ we obtain by pulling back along γ an extension E_γ of M by $\mathcal{L}_1$, where $\mathcal{K}_\gamma = \mathcal{K} \times_{\mathcal{L}} \mathcal{L}_1$, together with a morphism of extensions $(1, \gamma', \gamma) : E_\gamma \to E$. We call to the the extension (E_γ) the *backward induced extension* of E.

Proposition 1. *Every morphism $(\alpha, \beta, \gamma) : E_1 \to E$ of abelian extensions of Leibniz n-algebras admits a unique factorization of the form*

$$E_1 \overset{(\alpha,\eta,1)}{\to} E_\gamma \overset{(1,\gamma',\gamma)}{\to} E$$

Given a homomorphism of $\mathcal{L}$-representations $\alpha : \mathrm{M} \to \mathrm{M}_0$ we obtain the extension ${}^{\alpha}E : 0 \to \mathrm{M}_0 \overset{\kappa_0}{\to} {}^{\alpha}\mathcal{K} \overset{\pi_0}{\to} \mathcal{L} \to 0$ by putting ${}^{\alpha}\mathcal{K} = (M_0 \rtimes \mathcal{K})/S$, where $S = \{(\alpha(m), -\kappa(m)) \mid m \in M\}$. We call to the the extension ${}^{\alpha}E$ the *forward induced extension* of E.

Proposition 2. *Every morphism* $(\alpha, \beta, \gamma) : E \to E_0$ *of abelian extensions of Leibniz* n*-algebras admits a unique factorization of the form*

$$E \overset{(\alpha,\alpha',1)}{\to} ({}^{\alpha}E) \overset{(1,\xi,\gamma)}{\to} (E_0)$$

through the forward induced extension determined by α.

4 Various classes of central extensions

Let $E : 0 \to \mathrm{M} \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0 \in Ext(\mathcal{L}, \mathrm{M})$ be. Since M is a $\mathcal{L}$-representation, we have associated to it the exact sequence (4)

$$0 \to Der(\mathcal{L}, \mathrm{M}) \overset{Der(\pi)}{\to} Der(\mathcal{K}, \mathrm{M}) \overset{\rho}{\to} Hom_{\mathcal{L}}(\mathrm{M}, \mathrm{M}) \overset{\theta^*(E)}{\to} {}_nHL^1(\mathcal{L}, \mathrm{M}) \overset{\pi^*}{\to} {}_nHL^1(\mathcal{K}, \mathrm{M})$$

Then we define $\Delta : Ext(\mathcal{L}, \mathrm{M}) \to {}_nHL^1(\mathcal{L}, \mathrm{M}), \Delta([E]) = \theta^*(E)(1_{\mathrm{M}})$. The naturality of the sequence (4) implies the well definition of Δ. Now we fix a free presentation $0 \to \mathcal{R} \overset{\chi}{\to} \mathcal{F} \overset{\epsilon}{\to} \mathcal{L} \to 0$, then there exists a homomorphism $f : \mathcal{F} \to \mathcal{K}$ such that $\pi.f = \epsilon$, which restricts to $\overline{f} : \mathcal{R} \to \mathrm{M}$. Moreover $\overline{f}$ induces a $\mathcal{L}$-representation homomorphism $\varphi : \mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}^{n-2}] \to \mathrm{M}$ where the action from $\mathcal{L}$ on $\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}^{n-2}]$ is given via ϵ, that is,

$$[l_1, \ldots, l_{i-1}, \overline{r}, l_{i+1}, \ldots, l_n] = [x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n] + [\mathcal{R}, \mathcal{R}, \mathcal{F}^{n-2}]$$

where $\epsilon(x_j) = l_j, j \in \{1, \ldots, i-1, i+1, \ldots, n\}, i \in \{1, \ldots, n\}$. The naturality of sequence (4) induces the following commutative diagram

$$\begin{tikzcd}
Der(\mathcal{K}, \mathrm{M}) \arrow[r] \arrow[d, "f^*"'] & Hom_{\mathcal{L}}(\mathrm{M}, \mathrm{M}) \arrow[r, "\theta^*(E)"] \arrow[d, "\varphi^*"'] & {}_nHL^1(\mathcal{L}, \mathrm{M}) \arrow[r] \arrow[d, equal] & {}_nHL^1(\mathcal{K}, \mathrm{M}) \arrow[d, "f^*"] \\
Der(\mathcal{F}, \mathrm{M}) \arrow[r, "\tau^*"] & Hom_{\mathcal{L}}(\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}^{n-2}], \mathrm{M}) \arrow[r, "\sigma^*"] & {}_nHL^1(\mathcal{L}, \mathrm{M}) \arrow[r] & {}_nHL^1(\mathcal{F}, \mathrm{M})
\end{tikzcd}$$

Having in mind that ${}_nHL^1(\mathcal{F}, \mathrm{M}) = 0$ [8], then $\Delta[E] = \theta^*(E)(1_{\mathrm{M}}) = \sigma^*\varphi^*(1_{\mathrm{M}}) = \sigma^*(\varphi)$.

Proposition 3. $\Delta : Ext(\mathcal{L}, \mathrm{M}) \to {}_nHL^1(\mathcal{L}, \mathrm{M})$ *is an isomorphism.*

Proof. It is a tedious but straightforward adaptation of the Theorem 3.3, p 207 in [15]. $\diamond$

Definition 1. *Let* $E : 0 \to M \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0$ *be an extension of Leibniz* n*-algebras. We call* E *central if* $[\mathrm{M}, \mathcal{K}^{n-1}] = 0$.

Associated to E we have the isomorphism ${}_nHL^k(\mathcal{L},\mathrm{M}) \overset{\theta_*}{\cong} Hom({}_nHL_k(\mathcal{L}),\mathrm{M})$ (see Theorem 3 in [3]). Let us observe that M is a trivial $\mathcal{L}$-representation since E is a central extension. On the other hand, $\Delta[E] \in {}_nHL^1(\mathcal{L},\mathrm{M})$, then $\theta_*\Delta[E] \in Hom({}_nHL_1(\mathcal{L}),\mathrm{M})$. Moreover $\theta_*\Delta[E] = \theta_*(E)$, being $\theta_*(E)$ the homomorphism given by the exact sequence (5):

$${}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L}) \overset{\theta_*(E)}{\to} \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0$$

When $0 \to M \to \mathcal{K} \to \mathcal{L} \to 0$ is a central extension, the sequence (5) can be enlarged with a new term as follows (see [5]):

$$\oplus_{i=1}^{n-1} J_i \to {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L}) \to \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0 \qquad (6)$$

where $J_i = (\mathrm{M}\otimes \overset{n-i}{\cdots} \otimes \mathrm{M} \otimes \mathcal{K}_{\mathrm{ab}}\otimes \overset{i}{\cdots} \otimes\mathcal{K}_{\mathrm{ab}}) \oplus (\mathrm{M}\otimes \overset{n-i-1}{\cdots} \otimes\mathrm{M} \otimes \mathcal{K}_{\mathrm{ab}} \otimes \mathrm{M} \otimes \mathcal{K}_{\mathrm{ab}}\otimes \overset{i-1}{\cdots} \otimes\mathcal{K}_{\mathrm{ab}}) \oplus \cdots \oplus (\mathcal{K}_{\mathrm{ab}}\otimes \overset{i}{\cdots} \otimes\mathcal{K}_{\mathrm{ab}} \otimes \mathrm{M}\otimes \overset{n-i}{\cdots} \otimes\mathrm{M})$.

According to the character of the homomorphism $\theta_*\Delta[E]$ we can classify the central extensions of Leibniz n-algebras. Thus we have the following

Definition 2. *The central extension $E : 0 \to M \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0$ is called:*

1. *Commutator extension if $\theta_*\Delta[E] = 0$.*
2. *Quasi-commutator extension if $\theta_*\Delta[E]$ is a monomorphism.*
3. *Stem extension if $\theta_*\Delta[E]$ is an epimorphism.*
4. *Stem cover if $\theta_*\Delta[E]$ is an isomorphism.*

Let us observe that Definition 2 in case $n = 2$ agrees with the definitions in [2] for Leibniz algebras. It is clear, by naturality of sequence (5), that the property of a central extension which belongs to any of the described classes only depends on the isomorphism class.

Following, we characterize the various classes defined in terms of homological properties.

Proposition 4. *The following statements are equivalent:*

1. *E is a commutator extension.*
2. *$0 \to \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0$ is exact.*
3. *$\pi_* : [\mathcal{K}^n] \overset{\sim}{\to} [\mathcal{L}^n]$.*
4. *$\mathrm{M} \cap [\mathcal{K}^n] = 0$.*

Proof. E is a commutator extension $\Leftrightarrow \theta_*(E) = 0 \Leftrightarrow 0 \to \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0$ is exact (use sequence (5)) $\Leftrightarrow \mathrm{M} \cap [\mathcal{K}^n] = 0 \Leftrightarrow [\mathcal{K}^n] \xrightarrow{\sim} [\mathcal{L}^n]$ (use next diagram)

$$\begin{tikzcd} \mathrm{M} \cap [\mathcal{K}^n] \arrow[r, tail] \arrow[d, tail] & {[\mathcal{K}^n]} \arrow[r, two heads, "\pi_*"] \arrow[d, tail] & {[\mathcal{L}^n]} \arrow[d, tail] \\ \mathrm{M} \arrow[r, tail] \arrow[d, two heads] & \mathcal{K} \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \mathcal{L} \arrow[d, two heads] \\ \frac{\mathrm{M}}{\mathrm{M} \cap [\mathcal{K}^n]} \arrow[r, tail] & \mathcal{K}_{ab} \arrow[r, two heads, "\overline{\pi}"] & \mathcal{L}_{ab} \end{tikzcd} \tag{7}$$

$\diamond$

Proposition 5. *The following statements are equivalent:*

1. *E is a quasi-commutator extension.*
2. *${}_nHL_1(\pi) : {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L})$ is the zero map.*
3. *$0 \to {}_nHL_1(\mathcal{L}) \to \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0$ is exact.*
4. *${}_nHL_1(\mathcal{L}) \cong \mathrm{M} \cap [\mathcal{K}^n]$.*

Proof. $\theta_*(E)$ is a monomorphism $\Leftrightarrow$ Ker $\theta_*(E) = \mathrm{Im}\ \pi_* = 0 \Leftrightarrow \pi_* : {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L})$ is the zero map $\Leftrightarrow 0 \to {}_nHL_1(\mathcal{L}) \to \mathrm{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0$ is exact (by exactness in sequence (5)).

For the equivalence of last statement we must use that the monomorphism $\theta_*(E)$ factors as ${}_nHL_1(\mathcal{L}) \twoheadrightarrow \mathrm{M} \cap [\mathcal{K}^n] \hookrightarrow \mathrm{M}$. $\diamond$

Corollary 1. *If E is a quasi-commutator extension with ${}_nHL_1(\mathcal{L}) = 0$, then E is a commutator extension.*

Proof. $\mathrm{M} \cap [\mathcal{K}^n] \cong {}_nHL_1(\mathcal{L}) = 0$. $\diamond$

Corollary 2. *Let E be a central extension with $\mathcal{K}$ a free Leibniz n-algebra, then E is a quasi-commutator extension.*

Proof. ${}_nHL_1(\mathcal{K}) = 0$ [8] and use sequence (5). $\diamond$

Proposition 6. *The following statements are equivalent:*

1. *E is a stem extension.*
2. *$\kappa_* : \mathrm{M} \to {}_nHL_0(\mathcal{K})$ is the zero map.*
3. *$\overline{\pi} : {}_nHL_0(\mathcal{K}) \xrightarrow{\sim} {}_nHL_0(\mathcal{L})$.*
4. *$\mathrm{M} \subseteq [\mathcal{K}^n]$.*

Proof. In exact sequence (5), $\theta_*(E)$ is an epimorphism $\Leftrightarrow \kappa_* = 0 \Leftrightarrow \pi_*$ is an isomorphism. In diagram (7), $\mathcal{K}_{ab} \cong \mathcal{L}_{ab} \Leftrightarrow \frac{\mathrm{M}}{\mathrm{M} \cap [\mathcal{K}^n]} = 0 \Leftrightarrow \mathrm{M} \subseteq [\mathcal{K}^n]$. $\diamond$

Proposition 7. *Every class of central extensions of a $\mathcal{L}$-trivial representation* M *is forward induced from a stem extension.*

Proof. Pick any central extension class $E : 0 \to \mathrm{M} \to \mathcal{K} \to \mathcal{L} \to 0$, then $\theta_\star(E) : {}_nHL_1(\mathcal{L}) \to \mathrm{M}$ factors as $i.\tau : {}_nHL_1(\mathcal{L}) \twoheadrightarrow \mathrm{M} \cap [\mathcal{K}^n] = \mathrm{M}_1 \hookrightarrow \mathrm{M}$. As M_1 is a trivial $\mathcal{L}$-representation, then given τ there exists a central extension $E_1 \in {}_nHL^1(\mathcal{L}, \mathrm{M}_1)$ such that $\theta_\star(E_1) = \tau$. Moreover, E_1 is a stem extension since $\theta_\star(E_1)$ is an epimorphism. By naturality of sequence (5) on the forward construction $E_1 \to {}^i(E_1)$, we have that $\theta_\star{}^i(E_1) = i\theta_*(E_1) = i\tau = \theta_\star(E)$, i. e., ${}^i(E_1) = E$, and so E is forward induced by E_1, which is a stem extension. $\diamond$

Proposition 8. *Let $\mathcal{L}$ be a Leibniz n-algebra and let U be a subspace of ${}_nHL_1(\mathcal{L})$, then there exists a stem extension E with $U = Ker\ \theta_\star\Delta[E]$.*

Proof. We consider the quotient vector space $\mathrm{M} = {}_nHL_1(\mathcal{L})/U$ as a $\mathcal{L}$-trivial representation. We consider the central extension $E : 0 \to \mathrm{M} \to \mathcal{K} \to \mathcal{L} \to 0 \in {}_nHL^1(\mathcal{L}, \mathrm{M})$. Thus $\theta_\star\Delta[E] = \theta_\star(E) \in Hom({}_nHL_1(\mathcal{L}), \mathrm{M})$. If $\theta_\star(E) : {}_nHL^1(\mathcal{L}) \to \mathrm{M} = {}_nHL^1(\mathcal{L})/U$ is the canonical projection, then there exists a central extension $E : 0 \to \mathrm{M} \to \mathcal{K} \to \mathcal{L} \to 0$ such that $\theta_\star\Delta[E] = \theta_\star(E)$ is the canonical projection. Associated to E we have the exact sequence (5), in which $U = \mathrm{Ker}\ \theta_\star(E) = \mathrm{Ker}\ \theta_\star\Delta[E]$. Moreover E is a stem extension since $\theta_\star\Delta[E] = \theta_\star(E)$ is an epimorphism. $\diamond$

Proposition 9. *The following statements are equivalent:*

1. *E is a stem cover.*

2. *$\overline{\pi} : {}_nHL_0(\mathcal{K}) \xrightarrow{\sim} {}_nHL_0(\mathcal{L})$ and ${}_nHL_1(\pi) : {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L})$ is the zero map.*

Proof. $\theta_\star(E)$ is an isomorphism in sequence (6). $\diamond$

Corollary 3. *A stem extension is a stem cover if and only if $U = 0$.*

Proof. $U = 0 \Leftrightarrow \mathrm{Ker}\ \theta_\star(E) = 0 \Leftrightarrow \theta_\star(E) : {}_nHL_1(\mathcal{L}) \to \mathrm{M}$ is an isomorphism. $\diamond$

5 Universal central extensions

Definition 3. *A central extension $E : 0 \to \mathrm{M} \xrightarrow{\kappa} \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ is called universal if for every central extension $E' : 0 \to M' \to \mathcal{K}' \xrightarrow{\pi'} \mathcal{L} \to 0$ there exists a unique homomorphism $h : \mathcal{K} \to \mathcal{K}'$ such that $\pi' h = \pi$.*

Lemma 1. *Let $0 \to \mathrm{N} \to \mathcal{H} \xrightarrow{\pi} \mathcal{L} \to 0$ be a central extension of Leibniz n-algebras, being $\mathcal{H}$ a perfect Leibniz n-algebra. Let $0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\sigma} \mathcal{L} \to 0$ be another central extension of Leibniz n-algebras. If there exists a homomorphism of Leibniz n-algebras $\phi : \mathcal{H} \to \mathcal{K}$ such that $\sigma\phi = \pi$, then ϕ is unique.*

Proof. See Lemma 5 in [4]. $\diamond$

Lemma 2. *If* $0 \to \mathrm{N} \to \mathcal{H} \xrightarrow{\rho} \mathcal{K} \to 0$ *and* $0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *are central extensions with* $\mathcal{K}$ *a perfect Leibniz* n*-algebra, then* $0 \to L = Ker\, \pi \circ \rho \to \mathcal{H} \xrightarrow{\pi\circ\rho} \mathcal{L} \to 0$ *is a central extension. Moreover, if* $\mathcal{K}$ *is a universal central extension of* $\mathcal{L}$*, then* $0 \to \mathrm{N} \to \mathcal{H} \xrightarrow{\rho} \mathcal{K} \to 0$ *splits.*

Proof. Since $\mathcal{K}$ is a perfect Leibniz n-algebra, then ρ restricts to the epimorphism $\rho' : \mathrm{Z}(\mathcal{H}) \twoheadrightarrow \mathrm{Z}(\mathcal{K})$. From this argument and using classical techniques it is an easy task to end the proof. $\diamond$

Lemma 3. *If* $\mathcal{K}$ *is a perfect Leibniz* n*-algebra and* $\pi : \mathcal{K} \twoheadrightarrow \mathcal{L}$ *is an epimorphism, then* $\mathcal{L}$ *is a perfect Leibniz* n*-algebra.*

Lemma 4. *If* $E : 0 \to \mathrm{M} \xrightarrow{i} \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *is a universal central extension, then* $\mathcal{K}$ *and* $\mathcal{L}$ *are perfect Leibniz* n*-algebra.*

Proof. Assume that $\mathcal{K}$ is not a perfect Leibniz n-algebra, then $\mathcal{K}_{ab}$ is an abelian Leibniz n-algebra and, consequently, is a trivial $\mathcal{L}$-representation. We consider the central extension $E : 0 \to \mathcal{K}_{ab} \to \mathcal{K}_{ab} \times \mathcal{L} \xrightarrow{pr} \mathcal{L} \to 0$, then the homomorphisms of Leibniz n-algebras $\varphi, \psi : \mathcal{K} \to \mathcal{K}_{ab} \times \mathcal{L}, \varphi(k) = (\overline{k}, \pi(k)); \psi(k) = (0, \pi(k)), k \in \mathcal{K}$ verify that $pr \circ \varphi = \pi = pr \circ \psi$, so E can not be a universal central extension. Lemma 3 ends the proof. $\diamond$

Theorem 1.

1. *If* $E : 0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *is a central extension with* $\mathcal{K}$ *a perfect Leibniz* n*-algebra and every central extension of* $\mathcal{K}$ *splits, if and only if* E *is universal.*
2. *A Leibniz* n*-algebra* $\mathcal{L}$ *admits a universal central extension if and only if* $\mathcal{L}$ *is perfect.*
3. *The kernel of the universal central extension is canonically isomorphic to* ${}_nHL_1(\mathcal{L}, \mathbb{K})$*.*

Proof. See Theorem 5 in [3]. The equivalence of statement (1) is due to Lemma 2. $\diamond$

Corollary 4. *The central extension* $0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *is universal if and only if* ${}_nHL_0(\mathcal{K}) = {}_nHL_1(\mathcal{K}) = 0$ *(that is,* $\mathcal{K}$ *is a superpefect Leibniz* n*-algebra).*

Proof. By Theorem 1 (1) $\mathcal{K}$ is a perfect Leibniz n-algebra, so ${}_nHL_0(\mathcal{K}) = \mathcal{K}_{ab} = 0$. On the other hand, the splitting of any central extension by $\mathcal{K}$ is equivalent to ${}_nHL_1(\mathcal{K}) = 0$, since $0 \to 0 \to \mathcal{K} \xrightarrow{\sim} \mathcal{K} \to 0$ is the universal central extension of $\mathcal{K}$ (that is, $\mathcal{K}$ is centrally closed) . $\diamond$

Corollary 5. *Let* $E : 0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *be a central extension with* $\mathcal{L}$ *a perfect Leibniz* n*-algebra.* E *is a stem cover if and only if* ${}_nHL_0(\mathcal{K}) = {}_nHL_1(\mathcal{K}) = 0$*.*

Proof. From the exact sequence (5) associated to E and Proposition 9. $\diamond$

Remark. Let us observe that the stem cover of a perfect Leibniz n-algebra $\mathcal{L}$ is isomorphic to the universal central extension of $\mathcal{L}$.

Proposition 10. *Let $E : 0 \to \mathrm{M} \to \mathcal{K} \stackrel{\pi}{\to} \mathcal{L} \to 0$ be a central extension of Leibniz n-algebras. If $\mathcal{K}$ is a perfect Leibniz n-algebra, then the sequence $0 \to {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L}) \to M \to 0$ is exact and ${}_nHL_0(\mathcal{L}) = 0$.*

Proof. $\mathcal{L}$ is perfect by Lemma 3. Exact sequence (6) associated to E ends the proof. $\diamond$

Following we achieve a construction of the universal central extension of a perfect Leibniz n-algebra slightly different to the construction given in [4]. This approach permits us to obtain new results concerning to the endofunctor $\mathfrak{uce}$ which assigns to a perfect Leibniz n-algebra its universal central extension. To do this, we recall that the computation of the homology with trivial coefficients of a Leibniz n-algebra $\mathcal{L}$ (see [3]) uses the chain complex

$$\cdots \to {}_nCL_2(\mathcal{L}) = CL_2(\mathcal{L}^{\otimes n-1}, \mathcal{L}) = \mathcal{L}^{\otimes 2n-1} \stackrel{d_2}{\to} {}_nCL_1(\mathcal{L}) = CL_1(\mathcal{L}^{\otimes n-1}, \mathcal{L}) = \mathcal{L}^{\otimes n} \stackrel{d_1}{\to}$$

$$\stackrel{d_1}{\to} {}_nCL_0(\mathcal{L}) = CL_0(\mathcal{L}^{\otimes n-1}, \mathcal{L}) = \mathcal{L} \to 0$$

where

$$d_2(x_1 \otimes \cdots \otimes x_{2n-1}) = [x_1, x_2, \ldots, x_n] \otimes x_{n+1} \otimes \cdots \otimes x_{2n-1} -$$
$$\sum_{i=1}^{n} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_{n+1}, \ldots, x_{2n-1}] \otimes x_{i+1} \otimes \cdots \otimes x_n$$

and

$$d_1(x_1 \otimes \cdots \otimes x_n) = [x_1, x_2, \ldots, x_n]$$

Let $\mathcal{L}$ be a perfect Leibniz n-algebra. As $\mathbb{K}$-vector spaces, we consider the submodule I of $\mathcal{L}^{\otimes n}$ spanned by the elements of the form

$$[x_1, x_2, \ldots, x_n] \otimes x_{n+1} \otimes \cdots \otimes x_{2n-1} -$$

$$\sum_{i=1}^{n} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_{n+1}, \ldots, x_{2n-1}] \otimes x_{i+1} \otimes \cdots \otimes x_n$$

for all $x_1, \ldots, x_{2n-1} \in \mathcal{L}$. Let us observe that $I = \mathrm{Im}\, d_2$. Then we construct $\mathfrak{uce}(\mathcal{L}) = \mathcal{L}^{\otimes n}/I$. We denote by $\{x_1, \ldots, x_n\}$ the element $x_1 \otimes \cdots \otimes x_n + I$ of $\mathfrak{uce}(\mathcal{L})$. By construction the following identity holds in $\mathfrak{uce}(\mathcal{L})$

$$\{[x_1, x_2, \ldots, x_n], x_{n+1}, \ldots, x_{2n-1}\} =$$
$$\sum_{i=1}^{n} \{x_1, \ldots, x_{i-1}, [x_i, x_{n+1}, \ldots, x_{2n-1}], x_{i+1}, \ldots, x_n\} \tag{8}$$

The linear map d_1 vanishes on the elements of I, so it induces an epimorphism $\overline{d} : \mathfrak{uce}(\mathcal{L}) \to \mathcal{L}$ which is defined by $\overline{d}(\{x_1, x_2, \dots, x_n\}) = [x_1, x_2, \dots, x_n]$. Let us observe that Ker $\overline{d}$ = Ker d_1/I = Ker d_1/Im $d_2 = {}_nHL_1(\mathcal{L})$. On the other hand, the $\mathbb{K}$-vector space $\mathfrak{uce}(\mathcal{L})$ is endowed with a structure of Leibniz n-algebra by means of the following n-ary bracket:

$$[x_1, x_2, \dots, x_n] := \{\overline{d}(x_1), \overline{d}(x_2), \dots, \overline{d}(x_n)\}$$

for all $x_1, \dots, x_n \in \mathfrak{uce}(\mathcal{L})$. In this way $\overline{d}$ becomes into a Leibniz n-algebras homomorphism. Particularly, the following identity holds:

$$[\{x_{11}, \dots, x_{1n}\}, \{x_{21}, \dots, x_{2n}\}, \dots, \{x_{n1}, \dots, x_{nn}\}] =$$

$$\{[x_{11}, \dots, x_{1n}], [x_{21}, \dots, x_{2n}], \dots, [x_{n1}, \dots, x_{nn}]\}$$

Consequently, $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}$ is an epimorphism of Leibniz n-algebras. Actually $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow [\mathcal{L}^n]$, but if $\mathcal{L}$ is perfect, then $\mathcal{L} = [\mathcal{L}^n]$. It is an easy task, using identity (8) and Lemma 1, to verify that the epimorphism $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow [\mathcal{L}^n]$ is the universal central extension of $\mathcal{L}$ when $\mathcal{L}$ is a perfect Leibniz n-algebra.

By the uniqueness and having in mind the constructions of the universal central extension given in [4] we derive that $\mathfrak{uce}(\mathcal{L}) \cong [\mathcal{F}^n]/[\mathcal{R}, \mathcal{F}^{n-1}] \cong \mathcal{L} \star \overset{n}{\dots} \star \mathcal{L}$, where $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{L} \to 0$ is a free presentation of the perfect Leibniz n-algebra $\mathcal{L}$ and $\star$ denotes a non-abelian tensor product of Leibniz n-algebras introduced in [4].

As we can observe, last construction does not depend on the perfectness of the Leibniz n-algebra $\mathcal{L}$, that is, in general case, we have constructed the universal central extension of $[\mathcal{L}^n]$. Following we explore the functorial properties of this construction. So we consider a homomorphism of perfect Leibniz n-algebras $f : \mathcal{L}' \to \mathcal{L}$. Let $I_{\mathcal{L}'}$ and $I_{\mathcal{L}}$ be as the submodule defined previously. The canonical application $f^{\otimes n} : {}_nCL_1(\mathcal{L}') = \mathcal{L}'^{\otimes n} \to {}_nCL_1(\mathcal{L}) = \mathcal{L}^{\otimes n}, f^{\otimes n}(x_1 \otimes \cdots \otimes x_n) = f(x_1) \otimes \cdots \otimes f(x_n)$ maps $I_{\mathcal{L}'}$ into $I_{\mathcal{L}}$, thus it induces a linear map $\mathfrak{uce}(f) : \mathfrak{uce}(\mathcal{L}') \to \mathfrak{uce}(\mathcal{L}), \mathfrak{uce}(f)\{x_1, \dots, x_n\} = \{f(x_1), \dots, f(x_n)\}$. Moreover $\mathfrak{uce}(f)$ is a homomorphism of Leibniz n-algebras.

On the other hand, one verifies by construction that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{uce}(\mathcal{L}') \arrow[r, "\mathfrak{uce}(f)"] \arrow[d, "\overline{d'}"] & \mathfrak{uce}(\mathcal{L}) \arrow[d, "\overline{d}"] \\ \mathcal{L}' \arrow[r, "f"] & \mathcal{L} \end{tikzcd} \tag{9}$$

Thus we have a right exact covariant functor $\mathfrak{uce} : {}_{\mathbf{n}}\mathbf{Leib} \to {}_{\mathbf{n}}\mathbf{Leib}$ and, consequently, an automorphism f of $\mathcal{L}$ gives rise to an automorphism $\mathfrak{uce}(f)$ of $\mathfrak{uce}(\mathcal{L})$. The commutativity of last diagram implies that $\mathfrak{uce}(f)$ leaves ${}_nHL_1(\mathcal{L})$ invariant. Thus, we obtain the group homomorphism

$$Aut(\mathcal{L}) \to \{g \in Aut(\mathfrak{uce}(\mathcal{L})) : g({}_nHL_1(\mathcal{L})) = {}_nHL_1(\mathcal{L})\} : f \mapsto \mathfrak{uce}(f)$$

6 Isogeny classes in ${}_{\mathrm{n}}\mathbf{Leib}$

Definition 4. *A Leibniz n-algebra $\mathcal{L}$ is said to be unicentral if every central extension $\pi : \mathcal{K} \to \mathcal{L}$ maps $\mathrm{Z}(\mathcal{K})$ onto $\mathrm{Z}(\mathcal{L})$:*

$$\begin{tikzcd}
 & \mathrm{Z}(\mathcal{K}) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Z}(\mathcal{L}) \arrow[d, tail] \\
\mathrm{Ker}(\pi) \arrow[ur, tail] \arrow[r, tail] & \mathcal{K} \arrow[r, two heads, "\pi"] & \mathcal{L}
\end{tikzcd}$$

Proposition 11. *Let $\mathcal{L}$ be a perfect Leibniz n-algebra, then $\mathcal{L}$ is unicentral.*

Proof. $\mathcal{L}_{ab} = 0$, so the Ganea map $C : \oplus_{i=1}^{n} J_i \to {}_nHL_1(\mathcal{L})$ given by the exact sequence (6) associated to the central extension $0 \to \mathrm{Z}(\mathcal{L}) \to \mathcal{L} \to \mathcal{L}/\mathrm{Z}\mathcal{L} \to 0$ is the zero map, so Corollary 3.6 in [5] ends the proof. $\diamond$

Under certain hypothesis the composition of universal central extensions is again a universal central extension:

Corollary 6. *Let $0 \to \mathrm{N} \to \mathcal{H} \stackrel{\tau}{\to} \mathcal{K} \to 0$ and $0 \to \mathrm{M} \to \mathcal{K} \stackrel{\pi}{\to} \mathcal{L} \to 0$ be two central extensions of Leibniz n-algebras. Then $\pi \circ \tau : \mathcal{H} \twoheadrightarrow \mathcal{L}$ is a universal central extension if and only if $\tau : \mathcal{H} \twoheadrightarrow \mathcal{K}$ is a universal central extension.*

Proof. If $\pi \circ \tau : \mathcal{H} \twoheadrightarrow \mathcal{L}$ is a universal central extension, then ${}_nHL_0(\mathcal{H}) = {}_nHL_1(\mathcal{H}) = 0$ by Corollary 4, this Corollary also implies that $0 \to \mathrm{N} \to \mathcal{H} \stackrel{\tau}{\to} \mathcal{K} \to 0$ is a universal central extension.

Conversely, if $0 \to \mathrm{N} \to \mathcal{H} \stackrel{\tau}{\to} \mathcal{K} \to 0$ is a universal central extension, then $\mathcal{K}$ is perfect by Theorem 1, (2), then $\pi \circ \tau : \mathcal{H} \twoheadrightarrow \mathcal{L}$ is a central extension by Lemma 2. Now Corollary 4 ends the proof. $\diamond$

Proposition 12. *For every perfect Leibniz n-algebra $\mathcal{L}$ there is the isomorphism $\frac{\mathcal{L}}{\mathrm{Z}(\mathcal{L})} \cong \frac{\mathfrak{uce}(\mathcal{L})}{\mathrm{Z}(\mathfrak{uce}(\mathcal{L}))}$.*

Proof. If $\mathcal{L}$ is perfect, then $\mathcal{L}$ is unicentral by Proposition 11, so in the following diagram

$$\begin{tikzcd}
\mathrm{Z}(\mathfrak{uce}(\mathcal{L})) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Z}(\mathcal{L}) \arrow[d, tail] \\
\mathfrak{uce}(\mathcal{L}) \arrow[r, two heads] & \mathcal{L}
\end{tikzcd}$$

the kernels of the horizontal morphisms coincide, and then the cokernels of the vertical morphisms are isomorphic. $\diamond$

Remark: For a Leibniz n-algebra $\mathcal{L}$ satisfying that $\mathcal{L}/\mathrm{Z}(\mathcal{L})$ is unicentral, then the central extension $0 \to \mathrm{Z}(\mathcal{L}) \to \mathcal{L} \to \mathcal{L}/\mathrm{Z}(\mathcal{L}) \to 0$ implies that $\mathrm{Z}(\mathcal{L}/\mathrm{Z}(\mathcal{L})) = 0$ (see Corollary 3.8 in [5]). Hence for a perfect Leibniz n-algebra $\mathcal{L}$, $\mathcal{L}/\mathrm{Z}(\mathcal{L})$ is perfect, so it is unicentral and by Proposition 12 we have that $\frac{\mathcal{L}}{\mathrm{Z}(\mathcal{L})} \cong \frac{\mathfrak{uce}(\mathcal{L})}{\mathrm{Z}(\mathfrak{uce}(\mathcal{L}))}$ are centerless, that is its center is trivial.

Corollary 7. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two perfect Leibniz n-algebras. Then $\mathfrak{uce}(\mathcal{L}) \cong \mathfrak{uce}(\mathcal{L}')$ if and only if $\mathcal{L}/\mathrm{Z}(\mathcal{L}) \cong \mathcal{L}'/\mathrm{Z}(\mathcal{L}')$.*

Proof. If $\mathfrak{uce}(\mathcal{L}) \cong \mathfrak{uce}(\mathcal{L}')$, then, by Proposition 12, $\mathcal{L}/\mathrm{Z}(\mathcal{L}) \cong \mathfrak{uce}(\mathcal{L})/\mathrm{Z}(\mathfrak{uce}(\mathcal{L})) \cong \mathfrak{uce}(\mathcal{L}')/\mathrm{Z}(\mathfrak{uce}(\mathcal{L}')) \cong \mathcal{L}'/\mathrm{Z}(\mathcal{L}')$.

Conversely, if $\mathcal{L}/\mathrm{Z}(\mathcal{L}) \cong \mathcal{L}'/\mathrm{Z}(\mathcal{L}')$ then $\mathfrak{uce}(\mathcal{L}/\mathrm{Z}(\mathcal{L})) \cong \mathfrak{uce}(\mathcal{L}'/\mathrm{Z}(\mathcal{L}'))$. Corollary 6 applied to the central extensions $\mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}$ and $\mathcal{L} \twoheadrightarrow \mathcal{L}/\mathrm{Z}(\mathcal{L})$ yields the isomorphism $\mathfrak{uce}(\mathcal{L}/\mathrm{Z}(\mathcal{L})) \cong \mathfrak{uce}(\mathcal{L})$, from which the result is derived. $\diamond$

Definition 5. *We say that the perfect Leibniz n-algebras $\mathcal{L}$ and $\mathcal{L}'$ are isogenous if $\mathfrak{uce}(\mathcal{L}) \cong \mathfrak{uce}(\mathcal{L}')$.*

Remarks:

i) Isogeny classes are in an obvious bijection with centerless perfect Leibniz n-algebras: by Corollary 7 two different centerless perfect Leibniz n-algebras have different universal central extensions and they are non-isogenous, and in each isogeny class, namely the class of a perfect Leibniz n-algebra $\mathcal{L}$ there is always an isogenous centerless perfect Leibniz n-algebra $\mathcal{L}/\mathrm{Z}(\mathcal{L})$.

ii) Isogeny classes are also in bijection with superperfect Leibniz n-algebras: in each isogeny class, namely the class of a perfect Leibniz n-algebra $\mathcal{L}$ there is always an isogenous superperfect Leibniz n-algebra $\mathfrak{uce}(\mathcal{L})$, and two different superperfect Leibniz n-algebras have different universal central extensions and are non-isogenous since a superperfect Leibniz n-algebra $\mathcal{L}$ is centrally closed.

iii) Remark the fact that each isogeny class C is the set of central factors of its superperfect representant $\mathcal{K}$, and then C is an ordered set. In this ordered set there is a maximal element, which is the superperfect representant, and a minimal element, which is the centerless representant.

iv) A Leibniz n-algebra $\mathcal{L}$ is called *capable* if there exists a Leibniz n-algebra $\mathcal{K}$ such that $\mathcal{L} \cong \mathcal{K}/\mathrm{Z}(\mathcal{K})$. Capable Leibniz n-algebras are characterized in [5]. From Proposition 12 and Corollaries 3.2 and 3.8 in [5] we derive that centerless perfect Leibniz n-algebras are equivalent to capable perfect Leibniz n-algebras.

The following holds for the centerless representant $\mathcal{L}$ and the superperfect representant $\mathfrak{uce}(\mathcal{L})$ of each isogeny class:

Corollary 8. *Let $\mathcal{L}$ be a centerless perfect Leibniz n-algebra. Then $\mathrm{Z}(\mathfrak{uce}(\mathcal{L})) = {}_nHL_1(\mathcal{L})$ and the universal central extension of $\mathcal{L}$ is $0 \to \mathrm{Z}(\mathfrak{uce}(\mathcal{L})) \to \mathfrak{uce}(\mathcal{L}) \to \mathcal{L} \to 0$*

Proof. Since $\mathcal{L}$ is centerless then Proposition 12 implies that $0 \to \mathrm{Z}(\mathfrak{uce}(\mathcal{L})) \to \mathfrak{uce}(\mathcal{L}) \to \mathcal{L} \to 0$ is isomorphic to the universal central extension of $\mathcal{L}$. $\diamond$

7 Lifting automorphisms and derivations

Lifting automorphisms: Let $f : \mathcal{L}' \twoheadrightarrow \mathcal{L}$ be a covering (that means a central extension with $\mathcal{L}'$ a perfect Leibniz n-algebra). Consequently, $\mathcal{L}$ also is a perfect Leibniz n-algebra by Lemma 3, so diagram (9) becomes into

$$\begin{tikzcd}
{}_nHL_1(\mathcal{L}') \arrow[d, rightarrowtail] & {}_nHL_1(\mathcal{L}) \arrow[d, rightarrowtail] \\
\mathfrak{uce}(\mathcal{L}') \arrow[r, "\mathfrak{uce}(f)"] \arrow[d, twoheadrightarrow, "\overline{d'}"] & \mathfrak{uce}(\mathcal{L}) \arrow[d, twoheadrightarrow, "\overline{d}"] \\
\mathcal{L}' \arrow[r, twoheadrightarrow, "f"] & \mathcal{L}
\end{tikzcd} \tag{10}$$

By Corollary 6, the central extension $f.\overline{d'} : \mathfrak{uce}(\mathcal{L}') \twoheadrightarrow \mathcal{L}$ is universal. Moreover $\mathfrak{uce}(f)$ is a homomorphism from this universal central extension to the universal central extension $\bar{d} : \mathfrak{uce}(\mathcal{L}) \rightarrow \mathcal{L}$. Consequently $\mathfrak{uce}(f)$ is an isomorphism (two universal central extensions of $\mathcal{L}$ are isomorphic). So we obtain a covering $\overline{d'}.\mathfrak{uce}(f)^{-1} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}'$ with kernel

$$C := \text{Ker } (\overline{d'}.\mathfrak{uce}(f)^{-1}) = \mathfrak{uce}(f)(\text{Ker } \overline{d'}) = \mathfrak{uce}(f)({}_nHL_1(\mathcal{L}'))$$

Theorem 2. (lifting of automorphisms) *Let $f : \mathcal{L}' \twoheadrightarrow \mathcal{L}$ be a covering.*

a) Let be $h \in Aut(\mathcal{L})$. Then there exists $h' \in Aut(\mathcal{L}')$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{L}' \arrow[r, twoheadrightarrow, "f"] \arrow[d, rightarrowtail, twoheadrightarrow, "h'"] & \mathcal{L} \arrow[d, rightarrowtail, twoheadrightarrow, "h"] \\
\mathcal{L}' \arrow[r, twoheadrightarrow, "f"] & \mathcal{L}
\end{tikzcd} \tag{11}$$

if and only if the automorphism $\mathfrak{uce}(h)$ of $\mathfrak{uce}(\mathcal{L})$ satisfies $\mathfrak{uce}(h)(C) = C$.
In this case, h' is uniquely determined by (11) and $h'(\text{Ker } f) = \text{Ker } f$.

b) With the notation in statement a), the map $h \mapsto h'$ is a group isomorphism

$$\{h \in Aut(\mathcal{L}) : \mathfrak{uce}(h)(C) = C\} \rightarrow \{g \in Aut(\mathcal{L}') : g(\text{Ker } f) = \text{Ker}(f)\}$$

Proof. a) If h' exists, then it is a homomorphism from the covering $h.f$ to the covering f, so h' is unique by Lemma 1. By applying the functor $\mathfrak{uce}(-)$ to the diagram (11) we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{uce}(\mathcal{L}') \arrow[r, twoheadrightarrow, "\mathfrak{uce}(f)"] \arrow[d, rightarrowtail, "\mathfrak{uce}(h')"] & \mathfrak{uce}(\mathcal{L}) \arrow[d, rightarrowtail, "\mathfrak{uce}(h)"] \\
\mathfrak{uce}(\mathcal{L}') \arrow[r, twoheadrightarrow, "\mathfrak{uce}(f)"] & \mathfrak{uce}(\mathcal{L})
\end{tikzcd}$$

Hence

$$\mathfrak{uce}(h)(C) = \mathfrak{uce}(h).\mathfrak{uce}(f)({}_nHL_1(\mathcal{L}')) =$$

$$\mathfrak{uce}(f).\mathfrak{uce}(h')({}_nHL_1(\mathcal{L}')) = \mathfrak{uce}(f)({}_nHL_1(\mathcal{L}')) = C$$

Conversely, from diagram (10) one derives that $\overline{d} = f.\overline{d'}.\mathfrak{uce}(f)^{-1}$, and hence one obtains the following diagram:

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathfrak{uce}(\mathcal{L}) \arrow[r, "\overline{d'}.\mathfrak{uce}(f)^{-1}"] \arrow[d, "\mathfrak{uce}(h)"] & \mathcal{L}' \arrow[r, two heads, "f"] \arrow[d, "h'"] & \mathcal{L} \arrow[d, "h"] \\
C \arrow[r, hook] & \mathfrak{uce}(\mathcal{L}) \arrow[r, "\overline{d'}.\mathfrak{uce}(f)^{-1}"] & \mathcal{L}' \arrow[r, two heads, "f"] & \mathcal{L}
\end{tikzcd}$$

If $\mathfrak{uce}(h)(C) = C$, then $\overline{d'}.\mathfrak{uce}(f)^{-1}.\mathfrak{uce}(h)(C) = \overline{d'}.\mathfrak{uce}(f)^{-1}(C) = 0$, then there exists a unique $h' : \mathcal{L}' \to \mathcal{L}'$ such that $h'.\overline{d'}.\mathfrak{uce}(f)^{-1} = \overline{d'}.\mathfrak{uce}(f)^{-1}.\mathfrak{uce}(h)$. On the other hand, $h.f.\overline{d'}.\mathfrak{uce}(f)^{-1} = f.\overline{d'}.\mathfrak{uce}(f)^{-1}.\mathfrak{uce}(h) = f.h'.\overline{d'}.\mathfrak{uce}(f)^{-1}$, so $h.f = f.h'$.

Commutativity of (10) implies that $h'(\mathrm{Ker}f) = \mathrm{Ker}f$.

b) By a), the map is well-defined. It is a monomorphism by uniqueness in (a) and it is an epimorphism, since every $g \in Aut(\mathcal{L}')$ with $g(\mathrm{Ker}f) = \mathrm{Ker}f$ induces an automorphism $h : \mathcal{L} \to \mathcal{L}$ such that $h.f = f.g$. Hence, by a), $g = h'$ and $\mathfrak{uce}(h)(C) = C$. $\diamond$

Corollary 9. *If $\mathcal{L}$ is a perfect Leibniz n-algebra, then the map*

$$Aut(\mathcal{L}) \to \{g \in Aut(\mathfrak{uce}(\mathcal{L})) : g({}_nHL_1(\mathcal{L})) = {}_nHL_1(\mathcal{L})\} : f \mapsto \mathfrak{uce}(f)$$

is a group isomorphism. In particular, $Aut(\mathcal{L}) \cong Aut(\mathfrak{uce}(\mathcal{L}))$ if $\mathcal{L}$ is centerless.

Proof. We apply statement b) in Theorem 2 to the covering $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}$. In this case $C = 0$ and $\mathfrak{uce}(f)(0) = 0$.

If A is centerless, then ${}_nHL_1(\mathcal{L}) = \mathrm{Z}(\mathfrak{uce}(\mathcal{L}))$ by Corollary 8. Since any automorphism leaves the center invariant, then the second claim is a consequence of the first one. $\diamond$

Lifting derivations: Let $\mathcal{L}$ be a Leibniz n-algebra and $d \in Der(\mathcal{L})$. The linear map $\varphi : \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}, \varphi(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes x_{i-1} \otimes d(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n$, leaves invariant the submodule I and hence it induces a linear map $\mathfrak{uce}(d) : \mathfrak{uce}(\mathcal{L}) \longrightarrow \mathfrak{uce}(\mathcal{L}), \{X_1, \dots, X_n\} \mapsto \{d(X_1), X_2, \dots, X_n\} + \{X_1, d(X_2), \dots, X_n\} + \cdots + \{X_1, X_2, \dots, d(X_n)\}$ which commutes the following diagram:

$$\begin{tikzcd}
\mathfrak{uce}(\mathcal{L}) \arrow[r, "\mathfrak{uce}(d)"] \arrow[d, two heads, "\overline{d}"'] & \mathfrak{uce}(\mathcal{L}) \arrow[d, two heads, "\overline{d}"'] \\
\mathcal{L} \arrow[r, "d"'] & \mathcal{L}
\end{tikzcd}$$

In particular, $\mathfrak{uce}(d)$ leaves Ker $(\overline{d})$ invariant. Moreover, a tedious but straightforward verification stands that $\mathfrak{uce}(d)$ is a derivations of $\mathfrak{uce}(\mathcal{L})$. On the other hand,

$$\mathfrak{uce} : \mathrm{Der}(\mathcal{L}) \to \{\delta \in \mathrm{Der}(\mathfrak{uce}(\mathcal{L})) : \delta({}_nHL_1(\mathcal{L})) \subseteq {}_nHL_1(\mathcal{L})\} : d \mapsto \mathfrak{uce}(d)$$

is a homomorphism of Lie algebras ($\mathrm{Der}(\mathcal{L})$ is a Lie algebra, see p. 193 in [8]). Its kernel is contained in the subalgebra of those derivations vanishing on $[\mathcal{L}^n]$. It is also verified that

$$\mathfrak{uce}(ad_{[x_{11},\ldots,x_{n1}]\otimes\cdots\otimes[x_{1,n-1},\ldots,x_{n,n-1}]}) = ad_{\{x_{11},\ldots,x_{n1}\}\otimes\cdots\otimes\{x_{1,n-1},\ldots,x_{n,n-1}\}}$$

where $ad_{X_1\otimes\cdots\otimes X_{n-1}} : \mathcal{L} \to \mathcal{L}, X \mapsto [X, X_1, \ldots, X_{n-1}]$ and

$$\begin{array}{rcl} ad_{\{x_{11},\ldots,x_{n1}\}\otimes\cdots\otimes\{x_{1,n-1},\ldots,x_{n,n-1}\}} : \mathfrak{uce}(\mathcal{L}) & \to & \mathfrak{uce}(\mathcal{L}) \\ \{z_1, \ldots, z_n\} & \mapsto & [\{z_1, \ldots, z_n\}, \{x_{11}, \ldots, x_{n1}\}, \ldots, \\ & & \{x_{1,n-1}, \ldots, x_{n,n-1}\}]. \end{array}$$

Hence $\mathfrak{uce}(ad_{\overline{d}(X_1)\otimes\cdots\otimes\overline{d}(X_{n-1})}) = ad_{X_1\otimes\cdots\otimes X_{n-1}}$, being $X_i = \{x_{i1}, \ldots, x_{in}\}, i = 1, \ldots, n-1$, and $\mathfrak{uce}(ad_{[\mathcal{L}^n]\otimes\overset{n-1}{\cdots}\otimes[\mathcal{L}^n]}) = \mathrm{IDer}(\mathfrak{uce}(\mathcal{L}))$, where $\mathrm{IDer}(\mathfrak{uce}(\mathcal{L}))$ are the inner derivations

$$d_{X_1\otimes\cdots\otimes X_{n-1}} : \mathfrak{uce}(\mathcal{L}) \to \mathfrak{uce}(\mathcal{L}), \{z_1, \ldots, z_n\} \mapsto [\{z_1, \ldots, z_n\}, X_1, \ldots, X_{n-1}],$$

with $X_i = \{x_{i1}, \ldots, x_{in}\} \in \mathfrak{uce}(\mathcal{L}), i = 1, \ldots, n-1$.

The functorial properties of the functor $\mathfrak{uce}(-)$ concerning derivations are described in the following

Lemma 5. *Let $f : \mathcal{L}' \to \mathcal{L}$ be a homomorphism of Leibniz n-algebras, let $d \in \mathrm{Der}(\mathcal{L})$ and $d' \in \mathrm{Der}(\mathcal{L}')$ be such that $f.d' = d.f$, then $\mathfrak{uce}(f).\mathfrak{uce}(d') = \mathfrak{uce}(d).\mathfrak{uce}(f)$*

Proof. A straightforward computation on the typical elements $\{x_1, \ldots, x_n\}$ of $\mathfrak{uce}(\mathcal{L})$ shows the commutativity. $\diamond$

Theorem 3. (lifting of derivations) *Let $f : \mathcal{L}' \twoheadrightarrow \mathcal{L}$ be a covering. We denote $C = \mathfrak{uce}(f)({}_nHL_1(\mathcal{L}')) \subseteqq {}_nHL_1(\mathcal{L})$.*

a) A derivation d of $\mathcal{L}$ lifts to a derivation d' of $\mathcal{L}'$ satisfying $d.f = f.d'$ if and only if the derivation $\mathfrak{uce}(d)$ of $\mathfrak{uce}(\mathcal{L})$ satisfies $\mathfrak{uce}(d)(C) \subseteqq C$. In this case, d' is uniquely determined and leaves Ker f invariant.

b) The map

$$\{d \in \mathrm{Der}(\mathcal{L}) : \mathfrak{uce}(d)(C) \subseteqq C\} \to \{\delta \in \mathrm{Der}(\mathcal{L}') : \delta(\mathrm{Ker}(f)) \subseteqq \mathrm{Ker}(f)\}$$

$$d \longmapsto d'$$

is an isomorphism of Lie algebras mapping $\mathrm{IDer}(\mathcal{L})$ onto $\mathrm{IDer}(\mathcal{L}')$.

c) For the covering $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}$, the map

$$\mathfrak{uce} : \mathrm{Der}(\mathcal{L}) \to \{\delta \in \mathrm{Der}(\mathfrak{uce}(\mathcal{L})) : \delta({}_nHL_1(\mathcal{L})) \subseteqq {}_nHL_1(\mathcal{L})\}$$

is an isomorphism preserving inner derivations. If $\mathcal{L}$ is centerless, then $\mathrm{Der}(\mathcal{L}) \cong \mathrm{Der}(\mathfrak{uce}(\mathcal{L}))$.

Proof. a) Assume the existence of a derivation d' such that $f.d' = d.f$, then by Lemma 5 we have that

$$\mathfrak{uce}(d)(C) = \mathfrak{uce}(d).\mathfrak{uce}(f)({}_nHL_1(\mathcal{L}')) =$$

$$\mathfrak{uce}(f).\mathfrak{uce}(d')({}_nHL_1(\mathcal{L}')) \subseteqq \mathfrak{uce}(f)({}_nHL_1(\mathcal{L}')) = C$$

The converse is parallel to the proof of Theorem 2 a) having in mind that d is a derivation if and only if $Id + d$ is an automorphism.
b) The $\mathbb{K}$-vector space $\mathrm{Der}(\mathcal{L})$ is endowed with a structure of Lie algebra by means of the bracket $[d_1, d_2] = d_1.d_2 - d_2.d_1$. Now the proof easily follows from a).

c) Apply b) to the covering $\overline{d} : \mathfrak{uce}(\mathcal{L}) \twoheadrightarrow \mathcal{L}$. Observe that $C = 0$ in this situation.

If $\mathcal{L}$ is centerless, then by Corollary 8 we have that ${}_nHL_1(\mathcal{L}) = \mathrm{Z}(\mathfrak{uce}(\mathcal{L}))$ and a derivation of $\mathcal{L}$ leaves the center invariant, so the isomorphism follows from b). $\diamond$

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Fine gradings on some exceptional algebras

Cristina Draper[*] Cándido Martín[†]

Abstract

We describe the fine gradings, up to equivalence, on the exceptional Lie algebras of least dimensions, $\mathfrak{g}_2$ and $\mathfrak{f}_4$.

1 Introduction

The research activity around gradings on Lie algebras has grown in the last years. Many works on the subject could be mentioned but, for briefness, we shall cite [2] and [5]. The most known fine grading on a simple Lie algebra, that is, the decomposition in root spaces, has shown to have many applications to the Lie algebras theory and to representation theory. So, it seems that other fine gradings could give light about different aspects of these algebras.

This paper is based in [3] and [4]. In the first one we classify up to equivalence all the gradings on $\mathfrak{g}_2$, and, in the second one all the nontoral gradings on $\mathfrak{f}_4$. In this last algebra we rule out the study of the toral gradings because they provide essentially the same perspective of the algebra than the root space decomposition. The gradings which summarize the information about all the ways in which an algebra can be divided are the fine gradings, because any grading is obtained by joining homogeneous spaces of a fine grading. Our purpose in these pages is provide a complete description of them, in the cases of $\mathfrak{g}_2$ and $\mathfrak{f}_4$. This objective does not need so technical tools as those used in the above works. These technicalities are only needed to show that, effectively, the described gradings will cover all the possible cases.

Besides we will describe the fine gradings on the Cayley algebra and on the Albert algebra, motivated by their close relationship to $\mathfrak{g}_2$ and $\mathfrak{f}_4$ respectively.

2 About gradings and automorphisms

Let F be an algebraically closed field of characteristic zero, which will be used all through this work. If V is an F-algebra and G an abelian group, we shall say that the decomposition $V =$

[*]Departamento de Matemática Aplicada, Campus de El Ejido, S/N, 29071 Málaga, Spain.

[†]Departamento de Álgebra, Geometría y Topología, Campus de Teatinos, S/N. Facultad de Ciencias, Ap. 59, 29080 Málaga, Spain. Supported by the Spanish MCYT projects MTM2004-06580-C02-02 and MTM2004-08115-C04-04, and by the Junta de Andalucía PAI projects FQM-336 and FQM-1215

$\oplus_{g\in G}V_g$ is a G-*grading* on V whenever for all $g, h \in G$, $V_g V_h \subset V_{gh}$. The set $\{g \in G \mid V_g \neq 0\}$ is called the *support* of the grading and denoted by $\mathrm{Supp}(G)$. We shall always suppose that G is generated by $\mathrm{Supp}(G)$. We say that two gradings $V = \oplus_{g\in G}X_g = \oplus_{h\in H}Y_h$ are *equivalent* if the sets of homogeneous subspaces are the same up to isomorphism, that is, there are an automorphism $f \in \mathrm{aut}(V)$ and a bijection between the supports $\alpha\colon \mathrm{Supp}(G) \to \mathrm{Supp}(H)$ such that $f(X_g) = Y_{\alpha(g)}$ for any $g \in \mathrm{Supp}(G)$. A convenient invariant for equivalence is that of *type*. Suppose we have a grading on a finite-dimensional algebra, then for each positive integer i we will denote by h_i the number of homogeneous components of dimension i. Besides we shall say that the grading is of type $(h_1, h_2, \dots, h_l)$, for l the greatest index such that $h_l \neq 0$. Of course the number $\sum_i ih_i$ agrees with the dimension of the algebra.

There is a close relationship between group gradings and automorphisms. More precisely, if $\{f_1, \dots, f_n\} \subset \mathrm{aut}(V)$ is a set of commuting semisimple automorphisms, the simultaneous diagonalization becomes a group grading, and conversely, given $V = \oplus_{g\in G}V_g$ a G-grading, the set of automorphisms of V such that every V_g is contained in some eigenspace is an abelian group formed by semisimple automorphisms.

Consider an F-algebra V, a G-grading $V = \oplus_{g\in G}X_g$ and an H-grading $V = \oplus_{h\in H}Y_h$. We shall say that the H-grading is a *coarsening* of the G-grading if and only if each nonzero homogeneous component Y_h with $h \in H$ is a direct sum of some homogeneous components X_g. In this case we shall also say that the G-grading is a *refinement* of the H-grading. A group grading is *fine* if its unique refinement is the given grading. In such a case the group of automorphisms above mentioned is a maximal abelian subgroup of semisimple elements, usually called a *MAD* ("maximal abelian diagonalizable"). It is convenient to observe that the number of conjugacy classes of MADs groups of $\mathrm{aut}(V)$ agrees with the number of equivalence classes of fine gradings on V. Our objective is to describe the fine gradings, up to equivalence, on the exceptional Lie algebras $\mathfrak{g}_2$ and $\mathfrak{f}_4$.

3 Gradings on C and $\mathrm{Der}(C)$

Under the hypotesis about the ground field there is only one isomorphy class of Cayley algebras so that we take one and forever any representative C of the class. Consider also $\mathfrak{g}_2 := \mathrm{Der}(C)$. The Lie algebra $\mathfrak{g}_2$ is generated by the set of derivations $\{D_{x,y} \mid x, y \in C\}$, where

$$D_{x,y} = [l_x, l_y] + [l_x, r_y] + [r_x, r_y] \in \mathrm{Der}(C)$$

for l_x and r_x the left and right multiplication operators in C.

In our context a grading on an algebra is always induced by a set of commuting diagonalizable automorphisms of the algebra. Thus, an important tool for translating gradings on C

to gradings on $\mathfrak{g}_2$ (and conversely) is given by the isomorphism of algebraic groups

$$\begin{array}{rcl} \mathrm{Ad}\colon \mathrm{aut}(C) & \to & \mathrm{aut}(\mathfrak{g}_2) \\ f & \mapsto & \mathrm{Ad}(f); \quad \mathrm{Ad}(f)(d) := fdf^{-1}. \end{array}$$

Since $\mathrm{Ad}(f)(D_{x,y}) = D_{f(x),f(y)}$, the grading induced on $\mathfrak{g}_2$ by the grading $C = \oplus_{g\in G} C_g$ is given by $\mathfrak{g}_2 = L = \oplus_{g\in G} L_g$ with $L_g = \sum_{g_1+g_2=g} D_{C_{g_1},C_{g_2}}$. As pointed out in [3] this translation procedure has the drawback that equivalent gradings on C are not necessarily transformed into equivalent gradings on $\mathfrak{g}_2$. But of course isomorphic gradings are indeed transformed into isomorphic ones and reciprocally. Moreover two fine equivalent gradings on C are transformed into fine equivalent gradings on $\mathfrak{g}_2$ and reciprocally.

3.1 Gradings on the Cayley algebra

Next we fix a basis of C given by:

$$B = (e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3).$$

This is called the *standard basis* of the Cayley algebra C, and is defined for instance in [3, Section 3] by the following relations

$$\begin{array}{lllllllll} e_1 u_j & = u_j & = u_j e_2, & u_i u_j & = v_k & = -u_j u_i, & u_i v_i & = e_1, \\ e_2 v_j & = v_j & = v_j e_1, & -v_i v_j & = u_k & = v_j v_i, & v_i u_i & = e_2, \end{array}$$

where e_1 and e_2 are orthogonal idempotents, (i, j, k) is any cyclic permutation of $(1, 2, 3)$, and the remaining relations are null. This algebra is isomorphic to the Zorn matrices algebra. For further reference we recall here that the *standard involution* $x \mapsto \bar{x}$ of C is the one permuting e_1 and e_2 and making $\bar{x} = -x$ for $x = u_i$ or v_j $(i, j = 1, 2, 3)$. This enable us to define the *norm* $n\colon C \to F$ by $n(x) := x\bar{x}$, the *trace map* $\mathrm{tr}\colon C \to F$ by $\mathrm{tr}(x) := x + \bar{x}$, and the subspace C_0 of all trace zero elements. This is f-invariant for any $f \in \mathrm{aut}(C)$.

The gradings on C are computed in [6] in a more general context. In particular, up to equivalence, there are only two fine gradings on C. These are:

a) The $\mathbb{Z}^2$-toral grading given by

$$\begin{array}{lll} & C_{0,0} = \langle e_1, e_2\rangle & \\ C_{0,1} = \langle u_1\rangle & C_{1,1} = \langle u_2\rangle & C_{-1,-2} = \langle u_3\rangle \\ C_{0,-1} = \langle v_1\rangle & C_{-1,-1} = \langle v_2\rangle & C_{1,2} = \langle v_3\rangle \end{array}$$

whose homogeneous elements out of the zero component $C_{0,0}$ have zero norm.

b) The $\mathbb{Z}_2^3$-nontoral grading given by

$$\begin{array}{ll} C_{000} = \langle e_1 + e_2\rangle & C_{001} = \langle e_1 - e_2\rangle \\ C_{100} = \langle u_1 + v_1\rangle & C_{010} = \langle u_2 + v_2\rangle \\ C_{101} = \langle u_1 - v_1\rangle & C_{011} = \langle u_2 - v_2\rangle \end{array}$$

$$C_{110} = \langle u_3 + v_3 \rangle \qquad C_{111} = \langle u_3 - v_3 \rangle \tag{1}$$

This grading verifies that every homogeneous element is invertible.

Any other group grading on C is obtained by coarsening of these fine gradings.

3.2 Gradings on $\mathfrak{g}_2$

Applying the previous procedure for translating gradings from C to $\mathfrak{g}_2 = L$ we can obtain the fine gradings on this algebra. These are the following:

a) The $\mathbb{Z}^2$-toral grading given by

$$\begin{array}{lll} & L_{0,0} = \langle D_{u_1,v_1}, D_{u_2,v_2}, D_{u_3,v_3} \rangle & \\ L_{1,0} = \langle D_{v_1,u_2} \rangle & L_{0,1} = \langle D_{v_2,v_3} \rangle & L_{1,1} = \langle D_{v_1,v_3} \rangle \\ L_{1,2} = \langle D_{u_1,u_2} \rangle & L_{1,3} = \langle D_{u_1,v_3} \rangle & L_{2,3} = \langle D_{u_2,v_3} \rangle \\ L_{-1,0} = \langle D_{u_1,v_2} \rangle & L_{0,-1} = \langle D_{u_2,u_3} \rangle & L_{-1,-1} = \langle D_{u_1,u_3} \rangle \\ L_{-1,-2} = \langle D_{v_1,v_2} \rangle & L_{-1,-3} = \langle D_{v_1,u_3} \rangle & L_{-2,-3} = \langle D_{v_2,u_3} \rangle \end{array}$$

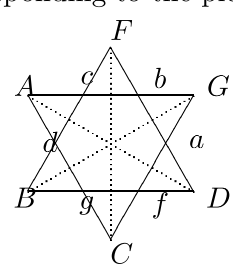
This is of course the root decomposition relative to the Cartan subalgebra $\mathfrak{h} = L_{0,0}$. Moreover, if Φ is the root system relative to $\mathfrak{h}$, and we take $\Delta = \{\alpha_1, \alpha_2\}$ the roots related to D_{v_1,u_2} and D_{v_2,v_3} respectively, it is clear that Δ is a basis of Φ such that $L_{n_1,n_2} = L_{n_1\alpha_1+n_2\alpha_2}$.

All the homogeneous elements, except the ones belonging to $L_{0,0}$, are nilpotent.

b) The $\mathbb{Z}_2^3$-nontoral grading is given by

$$\begin{array}{ll} L_{0,0,0} = 0 & L_{0,0,1} = \mathfrak{h} \\ L_{0,1,0} = \langle c+f, B+G \rangle & L_{0,1,1} = \langle -c+f, B-G \rangle \\ L_{1,1,0} = \langle A+D, b+g \rangle & L_{1,1,1} = \langle -A+D, -b+g \rangle \\ L_{1,0,0} = \langle a+d, C+F \rangle & L_{1,0,1} = \langle -a+d, C-F \rangle \end{array}$$

if we denote by $A := D_{v_1,u_2}$, $a := D_{v_2,v_3}$, $c := D_{v_1,v_3}$, $b := D_{u_1,u_2}$, $G := D_{u_1,v_3}$, $F := D_{u_2,v_3}$, $D := D_{u_1,v_2}$, $d := D_{u_2,u_3}$, $f := D_{u_1,u_3}$, $g := D_{v_1,v_2}$, $B := D_{v_1,u_3}$ and $C := D_{v_2,u_3}$, that is, a collection of root vectors corresponding to the picture



Notice that each of the nonzero homogeneous components is a Cartan subalgebra, that is, every homogeneous element is semisimple.

4 Gradings on J and $\mathrm{Der}(J)$

The Albert algebra is the exceptional Jordan algebra of dimension 27, that is,

$$J = H_3(C) = \{x = (x_{ij}) \in M_3(C) \mid x_{ij} = \overline{x_{ji}}\}$$

with product $x \cdot y := \frac{1}{2}(xy + yx)$, where juxtaposition stands for the usual matrix product. Denote, if $x \in C$, by

$$x^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} \quad x^{(2)} = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \quad x^{(3)} = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and denote by E_1, E_2 and E_3 the three orthogonal idempotents given by the elementary matrices e_{11}, e_{22} and e_{33} respectively. The multiplication table of the commutative algebra J may be summarized in the following relations:

$$\begin{array}{lll} E_i^2 = E_i, & E_i\, a^{(i)} = 0, & a^{(i)}b^{(i)} = \frac{1}{2}\,\mathrm{tr}(a\bar{b})(E_j + E_k), \\ E_iE_j = 0, & E_i\, a^{(j)} = \frac{1}{2}a^{(j)}, & a^{(i)}b^{(j)} = \frac{1}{2}(\bar{b}\bar{a})^{(k)}, \end{array}$$

where (i, j, k) is any cyclic permutation of $(1, 2, 3)$ and $a, b \in C$.

We fix for further reference our *standard basis* of the Albert algebra:

$$\mathcal{B} = (E_1, E_2, E_3, e_1^{(3)}, e_2^{(3)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)}, v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, e_2^{(2)}, e_1^{(2)}, -u_1^{(2)}, -u_2^{(2)},$$

$$-u_3^{(2)}, -v_1^{(2)}, -v_2^{(2)}, -v_3^{(2)}, e_1^{(1)}, e_2^{(1)}, u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}).$$

The Lie algebra $\mathfrak{f}_4 = \mathrm{Der}(J)$ is generated by the set of derivations $\{[R_x, R_y] \mid x, y \in C\}$, where R_x is the multiplication operator. As in the case of C and $\mathfrak{g}_2$ we also have an algebraic group isomorphism relating automorphisms of the Albert algebra J and of $\mathfrak{f}_4$. This is given by

$$\begin{array}{rcl} \mathrm{Ad}\colon \mathrm{aut}(J) & \to & \mathrm{aut}(\mathfrak{f}_4) \\ f & \mapsto & \mathrm{Ad}(f); \quad \mathrm{Ad}(f)(d) := fdf^{-1}. \end{array}$$

This provides also a mechanism to translate gradings from J to $\mathfrak{f}_4$ and conversely. Since $\mathrm{Ad}(f)([R_x, R_y]) = [R_{f(x)}, R_{f(y)}]$, the grading induced on $\mathfrak{f}_4$ by $J = \oplus_{g\in G} J_g$ is given by $\mathfrak{f}_4 = L = \oplus_{g\in G} L_g$ with $L_g = \sum_{g_1+g_2=g}[R_{J_{g_1}}, R_{J_{g_2}}]$.

4.1 Gradings on the Albert algebra

There are four fine gradings on J, all of them quite natural if we look at them from a suitable perspective.

a) The $\mathbb{Z}^4$-toral grading on J.

Define the maximal torus $\mathfrak{T}_0$ of F_4 whose elements are the automorphisms of J which are diagonal relative to $\mathcal{B}$. This is isomorphic to $(F^\times)^4$ and it is not difficult to check that the matrix of any such automorphism relative to $\mathcal{B}$ is

$$\mathrm{diag}\Big(1,1,1,\alpha,\frac{1}{\alpha},\beta,\gamma,\frac{\delta^2}{\alpha\,\beta\,\gamma},\frac{1}{\beta},\frac{1}{\gamma},\frac{\alpha\,\beta\,\gamma}{\delta^2},\delta,\frac{1}{\delta},\frac{\alpha\,\beta}{\delta},\frac{\alpha\,\gamma}{\delta},\frac{\delta}{\beta\,\gamma},\frac{\delta}{\alpha\,\beta},\frac{\delta}{\alpha\,\gamma},\frac{\beta\,\gamma}{\delta},\frac{\delta}{\alpha},$$
$$\frac{\alpha}{\delta},\frac{\beta}{\delta},\frac{\gamma}{\delta},\frac{\delta}{\alpha\,\beta\,\gamma},\frac{\delta}{\beta},\frac{\delta}{\gamma},\frac{\alpha\,\beta\,\gamma}{\delta}\Big)$$

for some α, β, γ, $\delta\in F^\times$. Define now $t_{\alpha,\beta,\gamma,\delta}$ as the automorphism in $\mathfrak{T}_0$ whose matrix relative to $\mathcal{B}$ is just the above one. Notice that we have a $\mathbb{Z}^4$-grading on J such that $t_{\alpha,\beta,\gamma,\delta}$ acts in $J_{(n_1,n_2,n_3,n_4)}$ with eigenvalue $\alpha^{n_1}\beta^{n_2}\gamma^{n_3}\delta^{n_4}$. This is just

$$\begin{array}{lll}
 & J_{0,0,0,0}=\langle E_1,E_2,E_3\rangle & \\
J_{1,0,0,0}=\langle e_1^{(3)}\rangle & J_{0,0,0,-1}=\langle e_1^{(2)}\rangle & J_{-1,0,0,1}=\langle e_1^{(1)}\rangle\\
J_{-1,0,0,0}=\langle e_2^{(3)}\rangle & J_{0,0,0,1}=\langle e_2^{(2)}\rangle & J_{1,0,0,-1}=\langle e_2^{(1)}\rangle\\
J_{0,1,0,0}=\langle u_1^{(3)}\rangle & J_{1,1,0,-1}=\langle u_1^{(2)}\rangle & J_{0,1,0,-1}=\langle u_1^{(1)}\rangle\\
J_{0,0,1,0}=\langle u_2^{(3)}\rangle & J_{1,0,1,-1}=\langle u_2^{(2)}\rangle & J_{0,0,1,-1}=\langle u_2^{(1)}\rangle\\
J_{-1,-1,-1,2}=\langle u_3^{(3)}\rangle & J_{0,-1,-1,1}=\langle u_3^{(2)}\rangle & J_{-1,-1,-1,1}=\langle u_3^{(1)}\rangle\\
J_{0,-1,0,0}=\langle v_1^{(3)}\rangle & J_{-1,-1,0,1}=\langle v_1^{(2)}\rangle, & J_{0,-1,0,1}=\langle v_1^{(1)}\rangle\\
J_{0,0,-1,0}=\langle v_2^{(3)}\rangle & J_{-1,0,-1,1}=\langle v_2^{(2)}\rangle & J_{0,0,-1,1}=\langle v_2^{(1)}\rangle\\
J_{1,1,1,-2}=\langle v_3^{(3)}\rangle & J_{0,1,1,-1}=\langle v_3^{(2)}\rangle & J_{1,1,1,-1}=\langle v_3^{(1)}\rangle.
\end{array}$$

Recall from Schafer ([9, (4.41), p. 109]) that any $x\in J$ satisfies a cubic equation $x^3-\mathrm{Tr}(x)x^2+\mathrm{Q}(x)x-\mathrm{N}(x)1=0$ where $\mathrm{Tr}(x),\mathrm{Q}(x),\mathrm{N}(x)\in F$. Notice that again in this grading every homogeneous element $b\notin J_{0,0,0,0}$ verifies that $N(b)=0$.

b) The $\mathbb{Z}_2^5$-nontoral grading on J.

Define $H\equiv H_3(F)=\{x\in M_3(F)\mid x=x^t\}$ and $K\equiv K_3(F)=\{x\in M_3(F)\mid x=-x^t\}$. There is a vector space isomorphism

$$J=H\oplus K\otimes C_0 \tag{2}$$

given by $E_i\mapsto E_i$, $1^{(i)}\mapsto 1^{(i)}\in H$ and for $x\in C_0$, $x^{(i)}\mapsto(e_{jk}-e_{kj})\otimes x\in K\otimes C_0$, being (i,j,k) any cyclic permutation of $(1,2,3)$ and $e_{ij}\in M_3(F)$ the elementary (i,j)-matrix. Thus J is a Jordan subalgebra of $M_3(F)\otimes C$ with the product $(c\otimes x)\cdot(d\otimes y)=\frac{1}{2}((c\otimes x)(d\otimes y)+(d\otimes y)(c\otimes x))$ for $(c\otimes x)(d\otimes y)=cd\otimes xy$.

This way of looking at J allows us to observe that the gradings on the Cayley algebra C, so as the gradings on the Jordan algebra $H_3(F)$, induce gradings on J, because $\mathrm{aut}(C)$ and $\mathrm{aut}(H_3(F))$ are subgroups of $\mathrm{aut}(J)$. Moreover, both kind of gradings are compatible. To be more precise, consider a G_1-grading on the Jordan algebra $H=\oplus_{g\in G_1}H_g$. This grading will come from a grading on $M_3(F)$ such that the Lie algebra K has also an induced grading ([7,

p. 184-185]). Taking a G_2-grading on the Cayley algebra $C = \oplus_{g\in G_2} C_g$, we get a $G_1 \times G_2$-grading on J given by

$$J_{g_1,e} = H_{g_1} \oplus K_{g_1} \otimes (C_0)_e, \qquad J_{g_1,g_2} = K_{g_1} \otimes (C_0)_{g_2}, \tag{3}$$

for $g_1 \in G_1, g_2 \in G_2$ and e the zero element in any group.

Take now as G_2-grading on $\mathfrak{g}_2$ the $\mathbb{Z}_2^3$-nontoral grading (1). Note that $(C_0)_e = 0$, so that $J_{g_1,e} = H_{g_1}$. And take as G_1-grading on $H_3(F)$, the $\mathbb{Z}_2^2$-grading given by

$$\begin{array}{ll} H_e = \langle E_1, E_2, E_3\rangle & H_{0,1} = \langle 1^{(1)}\rangle \\ H_{1,0} = \langle 1^{(2)}\rangle & H_{1,1} = \langle 1^{(3)}\rangle, \end{array}$$

which induces in K the $\mathbb{Z}_2^2$-grading such that

$$K_e = 0 \quad K_{0,1} = \langle e_{12} - e_{21}\rangle \quad K_{1,1} = \langle e_{23} - e_{32}\rangle \quad K_{1,0} = \langle e_{13} - e_{31}\rangle.$$

Combining them as above, we find a $\mathbb{Z}_2^5$-grading on J with dimensions

$$\begin{aligned} &\dim J_{e,e} = \dim H_e = 3,\\ &\dim J_{e,g_2} = 0,\\ &\dim J_{g_1,e} = \dim H_{g_1} = 1,\\ &\dim J_{g_1,g_2} = \dim K_{g_1} \otimes (C_0)_{g_2} = 1, \end{aligned}$$

which is of type $(24, 0, 1)$. The grading so obtained turns out to be, after using the isomorphism (2),

$$\begin{aligned} &J_{e,e} = \langle E_1, E_2, E_3\rangle\\ &J_{e,g} = 0\\ &J_{1,1,g} = (C_g)^{(1)}\\ &J_{1,0,g} = (C_g)^{(2)}\\ &J_{0,1,g} = (C_g)^{(3)}, \end{aligned}$$

for $g \in \mathbb{Z}_2^3$ and C_g again given by (1).

c) The $\mathbb{Z}_2^3 \times \mathbb{Z}$-nontoral grading on J.

If $p \in \mathrm{SO}(3, F)$, denote by $\mathrm{In}(p)$ the automorphism in $\mathrm{aut}(H_3(F))$ given by $\mathrm{In}(p)(x) = pxp^{-1}$. It is well known that a maximal torus of $\mathrm{SO}(3)$ is given by the matrices of the form

$$p_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & 0\\ 0 & \alpha & \beta\\ 0 & -\beta & \alpha \end{pmatrix}$$

with $\alpha, \beta \in F$ such that $\alpha^2 + \beta^2 = 1$. Thus, the set of all $\tau_{\alpha,\beta} = \mathrm{In}(p_{\alpha,\beta})$ is a maximal torus of $\mathrm{aut}(H_3(F))$ and the set of eigenvalues of $\tau_{\alpha,\beta}$ is $S_{\alpha,\beta} = \{(\alpha+i\beta)^n \mid n = 0, \pm1, \pm2\}$. Supposing

$|S_{\alpha,\beta}| = 5$, we find for $\tau_{\alpha,\beta}$ the following eigenspaces

$$\begin{aligned}
H_2 &= \langle -iE_2 + iE_3 + 1^{(1)} \rangle \\
H_1 &= \langle -i^{(3)} + 1^{(2)} \rangle \\
H_0 &= \langle E_1, E_2 + E_3 \rangle \\
H_{-1} &= \langle i^{(3)} + 1^{(2)} \rangle \\
H_{-2} &= \langle iE_2 - iE_3 + 1^{(1)} \rangle,
\end{aligned}$$

where the subindex n indicates that the eigenvalue of $\tau_{\alpha,\beta}$ is $(\alpha + i\beta)^n$. This gives a $\mathbb{Z}$-grading on $H_3(F)$, which induces the Cartan grading on $K = \mathrm{Der}(H_3(F))$, a Lie algebra of type $\mathfrak{a}_1$. An equivalent, but more comfortable, way of looking at these gradings, is:

$$\begin{array}{ll}
H_2 = \langle e_{23} \rangle & K_2 = 0 \\
H_1 = \langle e_{13} + e_{21} \rangle & K_1 = \langle e_{13} - e_{21} \rangle \\
H_e = \langle E_1, E_2 + E_3 \rangle & K_e = \langle E_2 - E_3 \rangle \\
H_{-1} = \langle e_{12} + e_{31} \rangle & K_{-1} = \langle e_{12} - e_{31} \rangle \\
H_{-2} = \langle e_{32} \rangle & K_{-2} = 0
\end{array} \tag{4}$$

When mixing them with the $\mathbb{Z}_2^3$-grading on C as explained in (3), we obtain a $\mathbb{Z} \times \mathbb{Z}_2^3$-fine grading on J, whose dimensions are

$$\begin{aligned}
&\dim J_{2,e} = \dim H_2 = 1 \\
&\dim J_{1,e} = \dim H_1 = 1 \\
&\dim J_{e,e} = \dim H_e = 2 \\
&\dim J_{2,g} = 0 \\
&\dim J_{1,g} = \dim K_1 \otimes (C_0)_g = 1 \\
&\dim J_{e,g} = \dim K_e \otimes (C_0)_g = 1,
\end{aligned}$$

for $g \in \mathbb{Z}_2^3$. It is obviously a grading of type $(25, 1)$. A detailed description of the components could be obtained directly by using (3).

d) The $\mathbb{Z}_3^3$-nontoral grading on J.

There is another way in which the Albert algebra can be constructed, the so called Tits construction described in [8, p. 525]. Let us start with the F-algebra $A = M_3(F)$ and denote by $\mathrm{Tr}_A, \mathrm{Q}_A, \mathrm{N}_A \colon A \to F$ the coefficients of the generic minimal polynomial such that $x^3 - \mathrm{Tr}_A(x)x^2 + \mathrm{Q}_A(x)x - \mathrm{N}_A(x)1 = 0$ for all $x \in A$. Define also the quadratic map $\sharp \colon A \to A$ by $x^\sharp := x^2 - \mathrm{Tr}_A(x)x + Q_A(x)1$. For any $x, y \in A$ denote $x \times y := (x+y)^\sharp - x^\sharp - y^\sharp$, and $x^* := \frac{1}{2}x \times 1 = \frac{1}{2}\mathrm{Tr}_A(x)1 - \frac{1}{2}x$. Finally consider the Jordan algebra A^+ whose underlying vector space agrees with that of A but whose product is $x \cdot y = \frac{1}{2}(xy + yx)$. Next, define in $A^3 := A \times A \times A$ the product

$$(a_1, b_1, c_1)(a_2, b_2, c_2) := \\
\big(a_1 \cdot a_2 + (b_1 c_2)^* + (b_2 c_1)^*, a_1^* b_2 + a_2^* b_1 + \tfrac{1}{2}(c_1 \times c_2), c_2 a_1^* + c_1 a_2^* + \tfrac{1}{2}(b_1 \times b_2)\big).$$

Then A^3 with this product is isomorphic to $J = H_3(C)$.

This construction allows us to extend any f automorphism of A to the automorphism $f^\bullet$ of $J = A^3$ given by $f^\bullet(x,y,z) := (f(x), f(y), f(z))$. As a further consequence we will be able to get gradings on J coming from gradings on the associative algebra A via this monomorphism of algebraic groups. Thus, consider the $\mathbb{Z}_3^2$-grading on A produced by the commuting automorphisms $f := \mathrm{In}(p)$ and $g := \mathrm{In}(q)$, for $p = \mathrm{diag}(1, \omega, \omega^2)$, being ω a primitive cubic root of the unit and

$$q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The simultaneous diagonalization of A relative to $\{f, g\}$ yields $A = \oplus_{i,j=0}^2 A_{i,j}$ where

$$\begin{array}{rrr} A_{00} = \langle 1_A \rangle, & A_{01} = \langle \omega^2 e_{11} - \omega e_{22} + e_{33} \rangle, & A_{02} = \langle -\omega e_{11} + \omega^2 e_{22} + e_{33} \rangle, \\ A_{10} = \langle e_{13} + e_{21} + e_{32} \rangle, & A_{11} = \langle \omega^2 e_{13} - \omega e_{21} + e_{32} \rangle, & A_{12} = \langle -\omega e_{13} + \omega^2 e_{21} + e_{32} \rangle, \\ A_{20} = \langle e_{12} + e_{23} + e_{31} \rangle, & A_{21} = \langle \omega^2 e_{12} - \omega e_{23} + e_{31} \rangle, & A_{22} = \langle -\omega e_{12} + \omega^2 e_{23} + e_{31} \rangle. \end{array}$$

If we make a simultaneous diagonalization of J relative to the automorphisms $\{f^\bullet, g^\bullet\}$ we get the toral $\mathbb{Z}_3^2$-grading $J = \oplus_{i,j=0}^2 A_{i,j}^3$. Now consider a third order three automorphism $\phi \in \mathrm{aut}(J)$ given by $\phi(a_0, a_1, a_2) = (a_0, \omega a_1, \omega^2 a_2)$. It is clear that $\{f^\bullet, g^\bullet, \phi\}$ is a commutative set of semisimple automorphisms of J, producing the simultaneous diagonalization $J = \oplus_{i,j,k=0}^2 J_{i,j,k}$ where

$$\begin{aligned} J_{i,j,0} &= A_{ij} \times 0 \times 0 \\ J_{i,j,1} &= 0 \times A_{ij} \times 0 \\ J_{i,j,2} &= 0 \times 0 \times A_{ij}, \end{aligned}$$

so that we have 27 one-dimensional homogeneous components. In particular this $\mathbb{Z}_3^3$-grading on J is fine and nontoral (otherwise $J_{0,0,0}$ would contain three orthogonal idempotents). Observe also that the homogeneous elements are invertible.

4.2 Gradings on $\mathfrak{f}_4$

Recall that the isomorphism $\mathrm{Ad}\colon \mathrm{aut}(J) \cong \mathrm{aut}(\mathfrak{f}_4)$, introduced at the beginning of Section 4, provides a mechanism for translating gradings from J to $\mathfrak{f}_4$ and conversely. Therefore, there will be four fine gradings on $\mathfrak{f}_4$ too, over $\mathbb{Z}^4$, $\mathbb{Z}_2^5$, $\mathbb{Z}_2^3 \times \mathbb{Z}$ and $\mathbb{Z}_3^3$.

a) The $\mathbb{Z}^4$-toral grading on $\mathfrak{f}_4$.

Denote by ω_i the i-th element in the basis $\mathcal{B}$. The $\mathbb{Z}^4$-toral grading on $J = \oplus_{g \in \mathbb{Z}^4} J_g$

induces a grading on $\mathfrak{f}_4 = L = \oplus_{g\in\mathbb{Z}^4} L_g$ by $L_g = \sum_{g_1+g_2=g} [R_{J_{g_1}}, R_{J_{g_2}}]$:

$$L_{0,0,0,0} = \langle [R_{\omega_4}, R_{\omega_5}], [R_{\omega_6}, R_{\omega_9}], [R_{\omega_7}, R_{\omega_{10}}], [R_{\omega_{12}}, R_{\omega_{13}}] \rangle$$

$L_{1,0,0,0} = \langle [R_{\omega_1}, R_{\omega_4}] \rangle$	$L_{0,0,0,1} = \langle [R_{\omega_1}, R_{\omega_{13}}] \rangle$	$L_{0,1,1,0} = \langle [R_{\omega_6}, R_{\omega_7}] \rangle$
$L_{-1,-2,-1,2} = \langle [R_{\omega_8}, R_{\omega_9}] \rangle$	$L_{1,0,0,-1} = \langle [R_{\omega_2}, R_{\omega_{21}}] \rangle$	$L_{0,1,1,-1} = \langle [R_{\omega_1}, R_{\omega_{19}}] \rangle$
$L_{-1,-1,0,2} = \langle [R_{\omega_7}, R_{\omega_8}] \rangle$	$L_{-1,-1,0,1} = \langle [R_{\omega_1}, R_{\omega_{17}}] \rangle$	$L_{1,1,1,-1} = \langle [R_{\omega_2}, R_{\omega_{27}}] \rangle$
$L_{0,1,1,-2} = \langle [R_{\omega_5}, R_{\omega_{11}}] \rangle$	$L_{0,-1,0,1} = \langle [R_{\omega_2}, R_{\omega_{25}}] \rangle$	$L_{1,1,1,-2} = \langle [R_{\omega_1}, R_{\omega_{11}}] \rangle$
$L_{-1,-1,0,0} = \langle [R_{\omega_5}, R_{\omega_9}] \rangle$	$L_{0,-1,0,0} = \langle [R_{\omega_1}, R_{\omega_9}] \rangle$	$L_{2,1,1,-2} = \langle [R_{\omega_4}, R_{\omega_{11}}] \rangle$
$L_{-1,0,1,0} = \langle [R_{\omega_5}, R_{\omega_7}] \rangle$	$L_{0,0,1,0} = \langle [R_{\omega_1}, R_{\omega_7}] \rangle$	$L_{1,-1,0,0} = \langle [R_{\omega_4}, R_{\omega_9}] \rangle$
$L_{0,0,1,-1} = \langle [R_{\omega_2}, R_{\omega_{23}}] \rangle$	$L_{1,0,1,0} = \langle [R_{\omega_4}, R_{\omega_7}] \rangle$	$L_{1,0,1,-1} = \langle [R_{\omega_1}, R_{\omega_{15}}] \rangle$
$L_{1,0,1,-2} = \langle [R_{\omega_9}, R_{\omega_{11}}] \rangle$	$L_{1,1,2,-2} = \langle [R_{\omega_7}, R_{\omega_{11}}] \rangle$	$L_{0,-1,1,0} = \langle [R_{\omega_7}, R_{\omega_9}] \rangle$
$L_{-1,0,0,0} = \langle [R_{\omega_1}, R_{\omega_5}] \rangle$	$L_{0,0,0,-1} = \langle [R_{\omega_1}, R_{\omega_{12}}] \rangle$	$L_{0,-1,-1,0} = \langle [R_{\omega_9}, R_{\omega_{10}}] \rangle$
$L_{1,2,1,-2} = \langle [R_{\omega_6}, R_{\omega_{11}}] \rangle$	$L_{-1,0,0,1} = \langle [R_{\omega_2}, R_{\omega_{20}}] \rangle$	$L_{0,-1,-1,1} = \langle [R_{\omega_1}, R_{\omega_{16}}] \rangle$
$L_{1,1,0,-2} = \langle [R_{\omega_{10}}, R_{\omega_{11}}] \rangle$	$L_{1,1,0,-1} = \langle [R_{\omega_1}, R_{\omega_{14}}] \rangle$	$L_{-1,-1,-1,1} = \langle [R_{\omega_2}, R_{\omega_{24}}] \rangle$
$L_{0,-1,-1,2} = \langle [R_{\omega_4}, R_{\omega_8}] \rangle$	$L_{0,1,0,-1} = \langle [R_{\omega_2}, R_{\omega_{22}}] \rangle$	$L_{-1,-1,-1,2} = \langle [R_{\omega_1}, R_{\omega_8}] \rangle$
$L_{1,1,0,0} = \langle [R_{\omega_4}, R_{\omega_6}] \rangle$	$L_{0,1,0,0} = \langle [R_{\omega_1}, R_{\omega_6}] \rangle$	$L_{-2,-1,-1,2} = \langle [R_{\omega_5}, R_{\omega_8}] \rangle$
$L_{1,0,-1,0} = \langle [R_{\omega_4}, R_{\omega_{10}}] \rangle$	$L_{0,0,-1,0} = \langle [R_{\omega_1}, R_{\omega_{10}}] \rangle$	$L_{-1,1,0,0} = \langle [R_{\omega_5}, R_{\omega_6}] \rangle$
$L_{0,0,1,-1} = \langle [R_{\omega_2}, R_{\omega_{26}}] \rangle$	$L_{-1,0,-1,0} = \langle [R_{\omega_5}, R_{\omega_{10}}] \rangle$	$L_{-1,0,-1,1} = \langle [R_{\omega_1}, R_{\omega_{18}}] \rangle$
$L_{-1,0,-1,2} = \langle [R_{\omega_6}, R_{\omega_8}] \rangle$	$L_{-1,-1,-2,2} = \langle [R_{\omega_8}, R_{\omega_{10}}] \rangle$	$L_{0,1,-1,0} = \langle [R_{\omega_6}, R_{\omega_{10}}] \rangle$.

This is of course the root decomposition relative to the Cartan subalgebra $\mathfrak{h} = L_{0,0,0,0}$. If Φ is the root system relative to $\mathfrak{h}$, and we take $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ the set of roots related to $[R_{\omega_8}, R_{\omega_9}]$, $[R_{\omega_6}, R_{\omega_7}]$, $[R_{\omega_1}, R_{\omega_{13}}]$ and $[R_{\omega_1}, R_{\omega_4}]$ respectively, it is straightforward to check that Δ is a basis of Φ such that the root space $L_{n_1\alpha_1+n_2\alpha_2+n_3\alpha_3+n_4\alpha_4}$ and the homogeneous component $L_{n_4-n_1,n_2-n_1,n_2-n_1,n_3+2n_1}$ coincide.

b) The $\mathbb{Z}_2^5$-nontoral grading on $\mathfrak{f}_4$.

We got the $\mathbb{Z}_2^5$-grading on J by looking at J as $H \oplus K \otimes C_0$. But $\mathfrak{f}_4$ is its algebra of derivations, hence there should exist some model of $\mathfrak{f}_4$ in terms of H, K and C. In fact we can see $\mathfrak{f}_4$ as

$$L = \mathrm{Der}(C) \oplus K \oplus H_0 \otimes C_0$$

identifying $\mathrm{Der}(H_3(F))$ in a natural way with K in the known Tits unified construction for the Lie exceptional algebras (for instance, see [9, p. 122]).

Consider a G_1-grading on the Jordan algebra $H = \oplus_{g\in G_1} H_g$. This grading will come from a grading on $M_3(F)$ so that the Lie algebra K has also an induced grading. Take now the $\mathbb{Z}_2^3$-grading on the Cayley algebra $C = \oplus_{g\in G_2=\mathbb{Z}_2^3} C_g$ and the induced grading $\mathrm{Der}(C) = \oplus_{g\in G_2} N_g$. All this material induces a $G_1 \times G_2$-grading on L by means of

$$L_{g_1,e} = K_{g_1}, \quad L_{e,g_2} = N_{g_2} \oplus (H_0)_e \otimes (C_0)_{g_2}, \quad L_{g_1,g_2} = (H_0)_{g_1} \otimes (C_0)_{g_2}, \tag{5}$$

which is just the grading induced by the $G_1 \times G_2$-grading on J described by (3).

In the case of the $\mathbb{Z}_2^5$-grading recall that $G_1 = \mathbb{Z}_2^2$,

$$\begin{array}{llll} H_e = \langle E_1, E_2, E_3 \rangle & H_{0,1} = \langle e_{12} + e_{21} \rangle & H_{1,1} = \langle e_{23} + e_{32} \rangle & H_{1,0} = \langle e_{13} + e_{31} \rangle \\ K_e = 0 & K_{0,1} = \langle e_{12} - e_{21} \rangle & K_{1,1} = \langle e_{23} - e_{32} \rangle & K_{1,0} = \langle e_{13} - e_{31} \rangle \end{array}$$

and $\dim(C_0)_g = 1$, $\dim N_g = 2$ for all $g \in \mathbb{Z}_2^3 \setminus \{(0,0,0)\}$. Therefore

$$\begin{aligned}
&\dim L_{e,e} = 0,\\
&\dim L_{e,g_2} = \dim N_{g_2} + \dim(H_0)_e \otimes (C_0)_{g_2} = 4,\\
&\dim L_{g_1,e} = \dim K_{g_1} = 1,\\
&\dim L_{g_1,g_2} = \dim(H_0)_{g_1} \otimes (C_0)_{g_2} = 1,
\end{aligned}$$

and so the grading is of type $(24,0,0,7)$, with all the homogeneous elements semisimple.

c) The $\mathbb{Z}_2^3 \times \mathbb{Z}$-nontoral grading on $\mathfrak{f}_4$.

We obtain the grading by the method just explained, with the $G_1 = \mathbb{Z}$-grading on H and K described in (4), and by crossing it with the $\mathbb{Z}_2^3$-grading on C. In such a way we get a $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading of type $(31,0,7)$, since

$$\begin{aligned}
&\dim L_{2,e} = 0,\\
&\dim L_{1,e} = \dim K_1 = 1,\\
&\dim L_{e,e} = \dim K_e = 1,\\
&\dim L_{2,g} = \dim H_2 \otimes (C_0)_g = 1,\\
&\dim L_{1,g} = \dim H_1 \otimes (C_0)_g = 1,\\
&\dim L_{e,g} = \dim N_g + \dim(H_0)_e \otimes (C_0)_g = 3,
\end{aligned}$$

and L_g is dual to L_{-g}, so they have the same dimensions.

The detailed description of the components of the last two gradings can be made by using (5), but it is not worth to be developed here.

d) The $\mathbb{Z}_3^3$-nontoral grading on $\mathfrak{f}_4$.

The easiest way to visualize this grading intrinsically, that is, with no reference to a particular basis or computer methods, is probably looking at the automorphisms inducing the grading. Adams gave a construction of the Lie algebra $\mathfrak{e}_6$ from three copies of $\mathfrak{a}_2$ ([1, p. 85]). Once the automorphisms have been given in $\mathfrak{e}_6$ we will restrict them to $\mathfrak{f}_4$.

Given a 3-dimensional F-vector space X in which a nonzero alternate trilinear map $\det\colon X \times X \times X \to F$ has been fixed, we can identify the exterior product with the dual space by $X \wedge X \overset{\approx}{\to} X^*$ such that $x \wedge y \mapsto \det(x,y,-) \in \hom(X,F)$. And in a dual way we can identify $X^* \wedge X^*$ with X through $\det^*$, the dual map of det. Consider three 3-dimensional vector spaces X_i $(i = 1,2,3)$, and define:

$$\mathcal{L} = \mathrm{sl}(X_1) \oplus \mathrm{sl}(X_2) \oplus \mathrm{sl}(X_3) \oplus \ X_1 \otimes X_2 \otimes X_3 \ \oplus \ X_1^* \otimes X_2^* \otimes X_3^*,$$

endowed with a Lie algebra structure with the product

$$\begin{aligned}
[\otimes f_i, \otimes x_i] \ &= \textstyle\sum_{\substack{k=1,2,3\\ i\neq j\neq k}} f_i(x_i) f_j(x_j)\big(f_k(-)x_k - \frac{1}{3} f_k(x_k)\mathrm{id}_{X_k}\big)\\
[\otimes x_i, \otimes y_i] \ &= \otimes(x_i \wedge y_i)\\
[\otimes f_i, \otimes g_i] \ &= \otimes(f_i \wedge g_i)
\end{aligned}$$

for any $x_i, y_i \in X_i$, $f_i, g_i \in X_i^*$, with the wedge products as above, and where the actions of the Lie subalgebra $\sum \mathrm{sl}(X_i)$ on $X_1 \otimes X_2 \otimes X_3$ and $X_1^* \otimes X_2^* \otimes X_3^*$ are the natural ones (the i-th simple ideal acts on the i-th slot). The Lie algebra $\mathcal{L}$ is isomorphic to $\mathfrak{e}_6$. The decomposition $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}} \oplus \mathcal{L}_{\bar{2}}$ is a $\mathbb{Z}_3$-grading for $\mathcal{L}_{\bar{0}} = \mathrm{sl}(X_1) \oplus \mathrm{sl}(X_2) \oplus \mathrm{sl}(X_3)$, $\mathcal{L}_{\bar{1}} = X_1 \otimes X_2 \otimes X_3$ and $\mathcal{L}_{\bar{2}} = X_1^* \otimes X_2^* \otimes X_3^*$. Take ϕ_1 the automorphism which induces the grading, that is, $\phi_1|_{\mathcal{L}_{\bar{i}}} = \omega^i \mathrm{id}_{L_{\bar{i}}}$ for ω a primitive cubic root of the unit. We are giving two automorphisms commuting with ϕ_1.

A family of automorphisms of the Lie algebra commuting with ϕ_1 is the following. If $\rho_i \colon X_i \to X_i$, $i = 1, 2, 3$, are linear maps preserving $\det \colon X_i^3 \to F$, the linear map $\rho_1 \otimes \rho_2 \otimes \rho_3 \colon \mathcal{L}_{\bar{1}} \to \mathcal{L}_{\bar{1}}$ can be uniquely extended to an automorphism of $\mathcal{L}$ such that its restriction to $\mathrm{sl}(V_i) \subset \mathcal{L}_{\bar{0}}$ is the conjugation map $g \mapsto \rho_i g \rho_i^{-1}$.

Fix now basis $\{u_0, u_1, u_2\}$ of X_1, $\{v_0, v_1, v_2\}$ of X_2, and $\{w_0, w_1, w_2\}$ of X_3 with $\det(u_0, u_1, u_2) = \det(v_0, v_1, v_2) = \det(w_0, w_1, w_2) = 1$. Consider ϕ_2 the unique automorphism of $\mathfrak{e}_6$ extending the map

$$u_i \otimes v_j \otimes w_k \mapsto u_{i+1} \otimes v_{j+1} \otimes w_{k+1}$$

(indices module 3). Finally let ϕ_3 be the unique automorphism of $\mathfrak{e}_6$ extending the map

$$u_i \otimes v_j \otimes w_k \mapsto \omega^i u_i \otimes \omega^j v_j \otimes \omega^k w_k = \omega^{i+j+k} u_i \otimes v_j \otimes w_k.$$

The set $\{\phi_i\}_{i=1}^3$ is a commutative set of semisimple automorphisms, and it induces a $\mathbb{Z}_3^3$-grading on $\mathfrak{e}_6$. The grading is nontoral since its zero homogeneous component is null. Some computations prove that the rest of the homogeneous components are all of them 3-dimensional.

The nice 3-symmetry described in $\mathfrak{e}_6$ is inherited by $\mathfrak{f}_4$. Indeed graphically speaking, $\mathfrak{f}_4$ arises by folding $\mathfrak{e}_6$. More precisely, taking $X_2 = X_3$ we can consider on $\mathfrak{e}_6$ the unique automorphism $\tau \colon \mathfrak{e}_6 \to \mathfrak{e}_6$ extension of $u \otimes v \otimes w \mapsto u \otimes w \otimes v$. This is an order two automorphism commuting with the previous ϕ_i for $i = 1, 2, 3$. The subalgebra of elements fixed by τ is

$$\mathrm{sl}(X_1) \oplus \mathrm{sl}(X_2) \oplus \; X_1 \otimes \mathrm{Sym}^2(X_2) \; \oplus \; X_1^* \otimes \mathrm{Sym}^2(X_2^*),$$

where $\mathrm{Sym}^n X_i$ denotes the symmetric powers. This is a simple Lie algebra of dimension 52, hence $\mathfrak{f}_4$. Furthermore, denoting also by $\phi_i \colon \mathfrak{f}_4 \to \mathfrak{f}_4$ the restriction of the corresponding automorphisms of $\mathfrak{e}_6$, the set $\{\phi_i\}_{i=1}^3$ is a set of commuting semisimple order three automorphisms of $\mathfrak{f}_4$ with no fixed points other than 0. So it induces a nontoral $\mathbb{Z}_3^3$-grading on $\mathfrak{f}_4$ of type $(0, 26)$, with all the homogeneous elements semisimple.

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Notes on a general compactification of symmetric spaces

Vadim Kaimanovich* Pedro J. Freitas†

Abstract

In this paper we outline the results regarding a construction of a general K-equivariant compactification of the symmetric space G/K, from a compactification of the Weyl chamber. A more detailed paper on this subject is expected soon. [1]

Symmetric spaces are a classical object of the Riemannian geometry and serve as a testing ground for numerous concepts and notions. The simplest symmetric space is the hyperbolic plane which can be naturally compactfied by the circle at infinity, which is essentailly the only reasonable compactifictaion of the hyperbolic plane. However, for higher rank symmetric spaces the situation is more complicated, and there one can define several different comapctifications—the visibility, the Furstenberg, the Martin, the Karpelevich—to name just the most popular ones. The present report is a part of an ongoing project aimed at understanding the nature and structure of general compactifications of symmetric spaces.

1 General Concepts

We start by defining the notation (either well known or taken from [GJT], with minor adjustments) and the concepts necessary. The results that follow can be found in [GJT] and [He].

We take G, a semisimple connected Lie group with finite center, and let K be a maximal compact subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{p}$ being the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, with respect to the Killing form B. The space $\mathfrak{p}$ can be identified with the tangent space to G/K at the coset K, which we'll denote by o. The restriction of the Killing form to this space is positive definite, and thus provides an inner product in $\mathfrak{p}$.

*IRMAR, Université Rennes-1, Campus de Beaulieu, 35042 Rennes Cedex, France

†Center for Linear and Combinatorial Structures, University of Lisbon, Av. Prof. Gama Pinto 2, 1649-003 Lisbon, Portugal

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We take $\mathfrak{a}$ to be a fixed Cartan subalgebra of $\mathfrak{p}$, $\mathfrak{a}^+$ a fixed Weyl chamber, Σ the set of all the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ (the so-called restricted roots), Σ^+ the set of positive roots, Δ the set of the simple roots. We denote by d be the rank of G (the dimension of $\mathfrak{a}$).

The action of G on G/K is by left multiplication. Every element of G can be written as $k.\exp(X)$ with $k \in K$ and $X \in \mathfrak{p}$—this is an easy consequence of the Cartan decomposition, which states that every element of G can be expressed as $k_1 \exp(X) k_2$, $k_1, k_2 \in K$, $X \in \overline{\mathfrak{a}^+}$ — as the closed Weyl chamber will be a very important object, we'll denote $\mathfrak{w} := \overline{\mathfrak{a}^+}$. From this we can easily conclude that every point in G/K can be presented as $k \exp(X).o$, $k \in K$, $X \in \mathfrak{w}$, which means that the K-orbit of $\exp(\mathfrak{w})$, is whole symmetric space. Moreover, the element $X \in \mathfrak{w}$ is uniquely defined, and is called the generalized radius. The element k is unique modulo the stabilizer of X for the adjoint action of K over $\mathfrak{w}$.

Given a topological group H, we'll say that a topological space A is an *H-space* if there is an action of H on A (which we'll denote by a dot) and the map

$$\begin{array}{rcl} H \times A & \rightarrow & A \\ (h, a) & \mapsto & h.a \end{array}$$

is continuous. If B is another H-space, and $\phi : A \rightarrow B$ is a continuous map, we say that ϕ is *H-equivariant* if, for any $h \in H, a \in A$, $\phi(h.a) = h.\phi(a)$.

If B is compact, ϕ is an embedding, and $\phi(A)$ is dense in B, we'll say that (ϕ, B) (or simply B if there is no confusion about the map involved) is a *compactification* of A. If ϕ is H-equivariant, we'll say that B is an *H-compactification*.

2 Building the compactification

We are now concerned with the definition of a compactification of the space G/K via compactifications of the closed Weyl chamber. There are a few compactifications of G/K that can be presented this way, as we will see later.

Now suppose we have a compactification of $\mathfrak{w}$, $\tilde{\mathfrak{w}}$, that is Hausdoff and satisfies the following condition:

($\star$) *For sequences $x_n, x'_n \in \mathfrak{w}$, If $x_n \rightarrow x \in \partial\tilde{\mathfrak{w}}$ and $d(x_n, x'_n) \rightarrow 0$, then $x'_n \rightarrow x$.*

Notice that if $x \in \mathfrak{w}$, we always have this property, since the topology in $\mathfrak{w}$ is given by the metric d.

Now, we are looking for a K-invariant compactification of G/K that restricted to $\mathfrak{w}$ will be $\tilde{\mathfrak{w}}$. We present a process of doing this.

Consider the space $K \times \mathfrak{w}$, and the map $\pi_1 : K \times \mathfrak{w} \to G/K$ defined naturally by $\pi_1(k, x) := k.\exp(x).o$. Consider also the compact space $K \times \tilde{\mathfrak{w}}$ and its quotient by the equivalence relation $\sim$ defined by the following rules:

(i) for $x, y \in \mathfrak{w}$, $(k, x) \sim (r, y) \Leftrightarrow x = y$ and $r \exp(x).o = s \exp(y).o$;

(ii) for $x, y \in \partial\tilde{\mathfrak{w}}$, $(k, x) \sim (r, y) \Leftrightarrow$ there exist convergent sequences (k, x_n) and (r, y_n) in $K \times \mathfrak{w}$ with $\lim x_n = x$, $\lim y_n = y$, such that $d(k \exp(x_n), r \exp(y_n)) \to 0$.

The relation $\sim$ is clearly an equivalence relation. Denote by $\mathcal{K}$ the quotient space.

Notation. Condition (i) assures that there is a bijection between G/K and $(K \times \mathfrak{w})/\sim$, name it ι. We will therefore use the notation $kE(x)$ to denote $(k, x)/\sim$, for $k \in K$ and $x \in \mathfrak{w}$. Thus, $\iota(k \exp(x).o) = kE(x)$ in $\mathcal{K}$.

Now take the inclusion and projection maps

$$\iota_1 : K \times \mathfrak{w} \to K \times \tilde{\mathfrak{w}} \qquad \pi_2 : K \times \tilde{\mathfrak{w}} \to \mathcal{K}.$$

We have that following diagram commutes.

$$\begin{array}{ccc} K \times \mathfrak{w} & \xrightarrow{\pi_1} & G/K \\ \downarrow \iota_1 & & \downarrow \iota \\ K \times \tilde{\mathfrak{w}} & \xrightarrow{\pi_2} & \mathcal{K} \end{array}$$

It can be proved that the image of $K \times \mathfrak{w}$ is dense in $\mathcal{K}$, and that ι is an embedding. This makes $\mathcal{K}$ into a compactification of G/K, since $\mathcal{K}$ is clearly compact. The following result is a parallel to the polar decomposition on G/K.

Proposition 2.1. *For $x, y \in \tilde{\mathfrak{w}}$, and $k, r \in K$, we have that $kE(x) = rE(y)$ if and only if $x = y$ and $k^{-1}r \in \mathrm{Stab}_K(E(x))$. In particular, the "generalized radius" x is well defined, even when $kE(x) \in \partial\mathcal{K}$.*

This furthers the analogy with elements in G/K. The compactification has the following properties.

Proposition 2.2. *1. The space $\mathcal{K}$ is Hausdorff.*

2. The projection $\pi_2 : K \times \tilde{\mathfrak{w}} \to \mathcal{K}$ is a closed map.

3. The space $\mathcal{K}$ is a K-space, and ι is K-equivariant.

Moreover, one can prove, using certain classes of converging sequences (just as it is done in [GJT]) that this compactification has some uniqueness properties.

Theorem 2.3. *The compactification $\mathcal{K}$ has the following properties.*

1. *It is a K-compactification.*

2. *When restricted to $\mathfrak{w}$ is $\tilde{\mathfrak{w}}$.*

3. *It respects intersections of Weyl chambers.*

4. *It is metrizable if $\tilde{\mathfrak{w}}$ is metrizable.*

5. *Being metrizable, it dominates any other compactification satisfying conditions 1–3.*

Examples. There are a few known compactifications that are particular cases of our compactification $\mathcal{K}$, originating from different compactifications of $\mathfrak{w}$. Among these are the compactifications of Furstenberg, Martin, Satake and Karpelevich.

3 An action of G and independence of the base point.

In the case $G = \mathrm{SL}(n, \mathbb{R})$ and $K = \mathrm{SO}(n)$, it is possible to define an action of G on $\mathcal{K}$. So, in this section, we take $G = \mathrm{SL}(n, \mathbb{R})$, plus the following assumptions on the compactification $\tilde{\mathfrak{w}}$.

1. *We'll assume that, in the compactification of the Cartan subalgebra $\mathfrak{a}$, if a sequence a_n converges, then for any $a \in \mathfrak{a}$, $a + a_n$ also converges.*

2. *The compactification $\tilde{\mathfrak{w}}$ is a refinement of the Furstenberg compactification.*

For $g \in G$, denote by $\mathcal{K}_g$ the compactification obtained by apllying the above process to G/K, but using $g.o$ as a reference point, instead of o.

Under these conditions, it is possible to identify a point of $\mathcal{K}$ with a point of $\mathcal{K}_g$, comparing the behaviour of sequences in both compactifications, and the rules of convergence for both—and prove they are the same. The proof available so far is quite technical, and the authors are working on a better version. We thus can prove the following result.

Proposition 3.1. *With the notation above, $\mathcal{K}$ and $\mathcal{K}_g$ are the same compactification.*

This allows us to define an action of G on $\mathcal{K}$, as follows: given a point $x \in G/K$, and $g \in G$, we take $g.x$ to be the point in $\mathcal{K}$ corresponding to the point $g.x \in \mathcal{K}_g$.

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Tridiagonal pairs, the q-tetrahedron algebra, $U_q(\mathfrak{sl}_2)$, and $U_q(\widehat{sl}_2)$

Darren Funk-Neubauer*

Abstract

Let V denote a finite dimensional vector space over an algebraically closed field. A *tridiagonal pair* is an ordered pair A, A^* of diagonalizable linear transformatons on V such that (i) the eigenspaces of A can be ordered as $\{V_i\}_{i=0}^d$ with $A^*V_i \subseteq V_{i-1}+V_i+V_{i+1}$ for $0 \leq i \leq d$; (ii) the eigenspaces of A^* can be ordered as $\{V_i^*\}_{i=0}^d$ with $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq d$; (iii) there are no nonzero proper subspaces of V which are invariant under both A and A^*. Tridiagonal pairs arise in the representation theory of various Lie algebras, associative algebras, and quantum groups. We recall the definition of one such algebra called the *q-tetrahedron algebra* and discuss its relation to the quantum groups $U_q(sl_2)$ and $U_q(\widehat{sl}_2)$. We discuss the role the q-tetrahedron algebra plays in the attempt to classify tridiagonal pairs. In particular, we state a theorem which connects the actions of a certain type of tridiagonal pair A, A^* on V to an irreducible action of the q-tetrahedron algebra on V.

1 Tridiagonal Pairs

In this paper we discuss the connection between tridiagonal pairs and representation theory. However, tridiagonal pairs originally arose in algebraic combinatorics through the study of a combinatorial object called a P- and Q-polynomial association scheme [4]. In addition, tridiagonal pairs are related to many other areas of mathematics. For example, they appear in the study of orthogonal polynomials and special functions [12], the theory of partially ordered sets [11], and statistical mechanics [13]. We now define a tridiagonal pair.

Definition 1.1. [4] Let V denote a finite dimensional vector space over an algebraically closed field $\mathbb{K}$. A *tridiagonal pair on V* is an ordered pair A, A^* where $A : V \to V$ and $A^* : V \to V$ are linear transformatons that satisfy the following conditions:

(i) Each of A, A^* is diagonalizable.

*Department of Mathematics, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706-1388, USA. E-mail: `neubauer@math.wisc.edu`

(ii) The eigenspaces of A can be ordered as $\{V_i\}_{i=0}^d$ with $A^*V_i \subseteq V_{i-1}+V_i+V_{i+1}$ $(0 \le i \le d)$, where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) The eigenspaces of A^* can be ordered as $\{V_i^*\}_{i=0}^\delta$ with $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ $(0 \le i \le \delta)$, where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

According to a common notational convention A^* denotes the conjugate-transpose of A. We am not using this convention; the linear transformations A, A^* are arbitrary subject to (i)–(iv) above.

Referring to Definition 1.1, it turns out $d = \delta$ [4]; we call this common value the *diameter* of the tridiagonal pair. We call an ordering of the eigenspaces of A (resp. A^*) *standard* whenever it satisfies (ii) (resp. (iii)) above. We call an ordering of the eigenvalues of A (resp. A^*) *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

The tridiagonal pairs for which the V_i, V_i^* all have dimension 1 are called *Leonard pairs*. The Leonard pairs are classified and correspond to a family of orthogonal polynomials consisting of the q-Racah polynomials and related polynomials in the Askey scheme [12]. Currently there is no classification of tridiagonal pairs. We will discuss the connection between tridiagonal pairs, the q-tetrahedron algebra, and the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{sl}_2)$. The hope is that this connection will eventually lead to a classification of tridiagonal pairs.

2 The eigenvalues of a tridiagonal pair

In this section we describe how the standard orderings of the eigenvalues of a tridiagonal pair satisfy a certain three term recurrance relation.

Let A, A^* denote a tridiagonal pair on V. Let $\{\theta_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A. Let $\{\theta_i^*\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A^*.

Theorem 2.1. [4, Theorem 11.1] *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \le i \le d-1$.

We now describe the solutions to the recurrance from Theorem 2.1.

Theorem 2.2. [4, Theorem 11.2] *Solving the recurrence in Theorem 2.1 we have the following. For some scalars* $q, a, b, c, a^*, b^*, c^* \in \mathbb{K}$ *the sequences* $\{\theta_i\}_{i=0}^d$ *and* $\{\theta_i^*\}_{i=0}^d$ *have one of the following forms:*
Case I: For $0 \leq i \leq d$

$$\theta_i = a + bq^i + cq^{-i},$$
$$\theta_i^* = a^* + b^*q^i + c^*q^{-i}.$$

Case II: For $0 \leq i \leq d$

$$\theta_i = a + bi + ci(i-1)/2,$$
$$\theta_i^* = a^* + b^*i + c^*i(i-1)/2.$$

Case III: The characteristic of $\mathbb{K}$ *is not* 2*, and for* $0 \leq i \leq d$

$$\theta_i = a + b(-1)^i + ci(-1)^i,$$
$$\theta_i^* = a^* + b^*(-1)^i + c^*i(-1)^i.$$

For the remainder of this paper we will be concerned with the tridiagonal pairs whose eigenvalues are as in Case I from Theorem 2.2. Such tridiagonal pairs are closely connected to representations of the quantum groups $U_q(sl_2)$ and $U_q(\widehat{sl}_2)$. The study of this connection inspired the definition of the q-tetrahedron algebra.

3 The q-tetrahedron algebra, $U_q(\mathfrak{sl}_2)$, and $U_q(\widehat{sl}_2)$

In this section we define the q-tetrahedron algebra and discuss its connection to the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{sl}_2)$.

For the remainder of the paper we will assume $q \in \mathbb{K}$ is nonzero and not a root of unity. We will use the following notation. For an integer $i \geq 0$ we define

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}} \qquad \text{and} \qquad [i]! = [i][i-1]\cdots[2][1].$$

We interpret $[0]! = 1$.

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 3.1. [7, Definition 6.1] Let $\boxtimes_q$ denote the unital associative $\mathbb{K}$-algebra that has generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4,\ j - i = 1 \text{ or } j - i = 2\}$$

and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij}x_{ji} = 1.$$

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i-h, j-i)$ is one of $(1,1), (1,2), (2,1)$,

$$\frac{qx_{hi}x_{ij} - q^{-1}x_{ij}x_{hi}}{q - q^{-1}} = 1.$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0. \tag{1}$$

We call $\boxtimes_q$ the *q-tetrahedron algebra.*

We now recall the definition of $U_q(\mathfrak{sl}_2)$.

Definition 3.2. [9, p. 9] Let $U_q(\mathfrak{sl}_2)$ denote the unital associative $\mathbb{K}$-algebra with generators $K^{\pm 1}$, $e^{\pm}$ and the following relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ Ke^{\pm}K^{-1} &= q^{\pm 2}e^{\pm}, \\ e^+e^- - e^-e^+ &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

We now recall an alternate presentation for $U_q(\mathfrak{sl}_2)$.

Lemma 3.3. [8, Theorem 2.1] *The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x^{\pm 1}$, y, z and the following relations:*

$$\begin{aligned} xx^{-1} = x^{-1}x &= 1, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, \\ \frac{qyz - q^{-1}zy}{q - q^{-1}} &= 1, \\ \frac{qzx - q^{-1}xz}{q - q^{-1}} &= 1. \end{aligned}$$

We now present a lemma which relates $U_q(\mathfrak{sl}_2)$ and $\boxtimes_q$.

Lemma 3.4. [7, Proposition 7.4] *For $i \in \mathbb{Z}_4$ there exists a $\mathbb{K}$-algebra homomorphism from $U_q(\mathfrak{sl}_2)$ to $\boxtimes_q$ that sends*

$$x \to x_{i,i+2}, \quad x^{-1} \to x_{i+2,i}, \quad y \to x_{i+2,i+3}, \quad z \to x_{i+3,i}.$$

We now recall the definition of $U_q(\widehat{sl}_2)$.

Definition 3.5. [1, Definition 2.2] Let $U_q(\widehat{sl}_2)$ denote the unital associative $\mathbb{K}$-algebra generated by $K_i^{\pm 1}$, $e_i^{\pm}$, $i \in \{0,1\}$ subject to the relations

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, && (2)\\
K_0 K_1 &= K_1 K_0, && (3)\\
K_i e_i^{\pm} K_i^{-1} &= q^{\pm 2} e_i^{\pm}, && (4)\\
K_i e_j^{\pm} K_i^{-1} &= q^{\mp 2} e_j^{\pm}, \qquad i \neq j, && (5)\\
e_i^{+} e_i^{-} - e_i^{-} e_i^{+} &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, && (6)\\
e_0^{\pm} e_1^{\mp} &= e_1^{\mp} e_0^{\pm}, && (7)
\end{aligned}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3](e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3] e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \qquad i \neq j. \tag{8}$$

We now recall an alternate presentation for $U_q(\widehat{sl}_2)$.

Theorem 3.6. [5, Theorem 2.1], [10] *The algebra $U_q(\widehat{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators x_i, y_i, z_i, $i \in \{0,1\}$ and the following relations:*

$$\begin{aligned}
x_0 x_1 = x_1 x_0 &= 1,\\
\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} &= 1,\\
\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} &= 1,\\
\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} &= 1,\\
\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} &= 1, \qquad i \neq j,
\end{aligned}$$

$$\begin{aligned}
y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \qquad i \neq j,\\
z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0, \qquad i \neq j.
\end{aligned}$$

We now present a lemma which relates $U_q(\widehat{sl}_2)$ and $\boxtimes_q$.

Lemma 3.7. [7, Proposition 8.3] *For $i \in \mathbb{Z}_4$ there exists a $\mathbb{K}$-algebra homomorphism from $U_q(\widehat{sl}_2)$ to $\boxtimes_q$ that sends*

$$\begin{aligned}
x_1 &\to x_{i,i+2}, \quad & y_1 &\to x_{i+2,i+3}, \quad & z_1 &\to x_{i+3,i},\\
x_0 &\to x_{i+2,i}, \quad & y_0 &\to x_{i,i+1}, \quad & z_0 &\to x_{i+1,i+2}.
\end{aligned}$$

4 The connection between tridiagonal pairs and $\boxtimes_q$

We now recall a theorem which connects $\boxtimes_q$ to the tridiagonal pairs whose eigenvalues are as in Case I of Theorem 2.2 (with $a = a^* = c = b^* = 0$ and $b, c^* \neq 0$).

Theorem 4.1. [6, Theorem 2.7], [7, Theorem 10.4] *Let A, A^* denote a tridiagonal pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Assume there exist nonzero scalars $b, c^* \in \mathbb{K}$ such that $\theta_i = bq^{2i-d}$ and $\theta_i^* = c^*q^{d-2i}$ for $0 \leq i \leq d$. Then there exists a unique irreducible representation of $\boxtimes_q$ on V such that bx_0 acts as A and c^*x_2 acts as A^*.*

Since the finite dimensional irreducible representations of $\boxtimes_q$ are completely understood [7] Theorem 4.1 classifies tridiagonal pairs where the eigenvalues of A (resp. A^*) are $\theta_i = bq^{2i-d}$ (resp. $\theta_i^* = c^*q^{d-2i}$).

Given Theorem 4.1 it is natural to ask the following question. If the eigenvalues of A and A^* are more general can we still construct a representation of $\boxtimes_q$ on V? More specifically, if the eigenvalues of A are $\theta_i = bq^{2i-d}$ for $0 \leq i \leq d$ and the eigenvalues of A^* are $\theta_i^* = b^*q^{2i-d} + c^*q^{d-2i}$ for $0 \leq i \leq d$ can we construct a representation of $\boxtimes_q$ on V that generalizes the construction in Theorem 4.1? We answer this question in the next section.

5 A generalization of Theorem 4.1

In this section we present a theorem which connects $\boxtimes_q$ to the tridiagonal pairs whose eigenvalues are as in Case I of Theorem 2.2 (with $a = a^* = c = 0$ and $b, b^*, c^* \neq 0$).

Before we state this theorem we have a number of prelimanary definitions.

Let A, A^* denote a tridiagonal pair on V and let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspace of A (resp. A^*). For $0 \leq i \leq d$ define $U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$. It turns out each of $\{U_i\}_{i=0}^d$ is nonzero and V is their direct sum [4]. The sequence $\{U_i\}_{i=0}^d$ is called the *split decomposition of* A, A^*. There exist linear transformations $R : V \to V$ and $L : V \to V$ such that (i) $U_0, \ldots, U_d$ are the common eigenspaces for $A - R, A^* - L$ and (ii) $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$ [4]. R (resp. L) is called the *raising* (resp. *lowering*) *map associated to* A, A^*.

Definition 5.1. [3] Let V denote a finite dimensional vector space over $\mathbb{K}$. Let A, A^* denote a tridiagonal pair on V and let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Assume for nonzero $b \in \mathbb{K}$ that $\theta_i = bq^{2i-d}$ for $0 \leq i \leq d$. Furthermore, assume for nonzero $b^*, c^* \in \mathbb{K}$ that $\theta_i^* = b^*q^{2i-d} + c^*q^{d-2i}$ for $0 \leq i \leq d$. Let $\{U_i\}_{i=0}^d$ denote the split decomposition of A, A^*. Let R (resp. L) denote the raising (resp.

lowering) map associated to A, A^*. It is known that $\dim(U_0) = 1$. Thus for $0 \le i \le d$ the space U_0 is an eigenspace of $L^i R^i$; let σ_i denote the coresponding eigenvalue.

The following polynomial will be used to state our theorem. It is a slight modification of the Drinfeld polynomial which is well known in representation theory [1, 2].

Definition 5.2. [3] With reference to Definition 5.1 define the polynomial $P \in \mathbb{K}[\lambda]$ by

$$P(\lambda) = \sum_{i=0}^{d} \frac{q^{i(1-i)}\,\sigma_i\,\lambda^i}{[i]!^{\,2}},$$

The following theorem uses P to explain the connection between finite dimensional irreducible representations of $\boxtimes_q$ and the tridiagonal pairs whose eigenvalues are as in Case I of Theorem 2.2 (with $a = a^* = c = 0$ and $b, b^*, c^* \neq 0$).

Theorem 5.3. [3] *With reference to Definition 5.1 and Definition 5.2, the following are equivalent:*

(i) *There exists a representation of $\boxtimes_q$ on V such that bx_{01} acts as A and $b^*x_{30} + c^*x_{23}$ acts as A^*.*

(ii) $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$.

Suppose (i),(ii) hold. Then the $\boxtimes_q$ representation on V is unique and irreducible.

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Quantum Lie algebras via modified reflection equation algebra

Dimitri Gurevich* Pavel Saponov†

1 Introduction

A Lie super-algebra was historically the first generalization of the notion of a Lie algebra. Lie super-algebras were introduced by physicists in studying dynamical models with fermions. In contrast with the usual Lie algebras defined via the classical flip P interchanging any two elements $P(X \otimes Y) = Y \otimes X$, the definition of a Lie super-algebra is essentially based on a super-analog of the permutation P. This super-analog is defined on a Z_2-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ where $\overline{0}, \overline{1} \in \mathrm{Z}_2$ is a "parity". On homogeneous elements (i.e. those belonging to either $V_{\overline{0}}$ or $V_{\overline{1}}$) its action is $P(X \otimes Y) = (-1)^{\overline{X}\,\overline{Y}} Y \otimes X$, where $\overline{X}$ stands for the parity of a homogeneous element $X \in V$.

Then a Lie super-algebra is the following data

$$(\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}},\ P : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g},\ [\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}),$$

where $\mathfrak{g}$ is a super-space, P is a super-flip, and $[\ ,\ ]$ is a Lie super-bracket, i.e. a linear operator which is subject to three axioms:

1. $[X, Y] = -(-1)^{\overline{X}\,\overline{Y}}[Y, X]$;
2. $[X, [Y, Z]] + (-1)^{\overline{X}(\overline{Y}+\overline{Z})}[Y, [Z, X]] + (-1)^{\overline{Z}(\overline{X}+\overline{Y})}[Z, [X, Y]] = 0$;
3. $\overline{[X, Y]} = \overline{X} + \overline{Y}$.

Here X, Y, Z are assumed to be arbitrary homogenous elements of $\mathfrak{g}$. Note that all axioms can be rewritten via the corresponding super-flip. For instance the axiom 3 takes the form

$$P(X \otimes [Y, Z]) = [\ ,\ ]_{12} P_{23} P_{12}(X \otimes Y \otimes Z).$$

(As usual, the indices indicate the space(s) where a given operator is applied.)

In this paper we discuss the problem what is a possible generalization of the notion of a Lie super-algebra related to "flips" of more general type.

*USTV, Université de Valenciennes, 59304 Valenciennes, France. E-mail: gurevich@univ-valenciennes.fr
†Division of Theoretical Physics, IHEP, 142284 Protvino, Russia. E-mail: Pavel.Saponov@ihep.ru

The first generalization of the notion of a Lie super-algebra was related to gradings different from Z_2. The corresponding Lie type algebras were called Γ-graded ones (cf. [Sh]).

The next step was done in [G1] where there was introduced a new generalization of the Lie algebra notion related to an involutive symmetry defined as follows. Let V be a vector space over a ground field $\mathbb{K}$ (usually $\mathbb{C}$ or $\mathbb{R}$) and $R : V^{\otimes 2} \to V^{\otimes 2}$ be a linear operator. It is called a *braiding* if it satisfies the quantum Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}\,,$$

where $R_{12} = R \otimes I,\ R_{23} = I \otimes R$ are operators in the space $V^{\otimes 3}$. If such a braiding satisfies the condition $R^2 = I$ (resp., $(R - q\,I)(R + q^{-1}\,I) = 0$, $q \in \mathbb{K}$) we call it an *involutive symmetry* (resp., a *Hecke symmetry*). In the latter case q is assumed to be generic[1].

Two basic examples of *generalized* Lie algebras are analogs of the Lie algebras $gl(n)$ and $sl(n)$ (or of their super-analogs $gl(m|n)$ and $sl(m|n)$). They can be associated to any "skew-invertible" (see Section 2) involutive symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$. We denote them $gl(V_R)$ and $sl(V_R)$ respectively. The generalized Lie algebras $gl(V_R)$ and $sl(V_R)$ are defined in the space $\mathrm{End}\,(V)$ of endomorphisms of the space V. Their enveloping algebras $U(gl(V_R))$ and $U(sl(V_R))$ (which can be defined in a natural way) are equipped with a braided Hopf structure such that the coproduct coming in its definition acts on the generators $X \in gl(V_R)$ or $sl(V_R)$ in the classical manner: $\Delta : X \to X \otimes 1 + 1 \otimes X$.

Moreover, if an involutive symmetry R is a deformation of the usual flip (or super-flip) the enveloping algebras $U(gl(V_R))$ and $U(sl(V_R))$ are deformations of their classical (or super-) counterparts.

There are known numerous attempts to define a quantum (braided) Lie algebra similar to generalized ones but without assuming R to be involutive. Let us mention some of them: [W], [LS], [DGG], [GM]. In this note we compare the objects defined there with gl type Lie algebras-like objects introduced recently in [GPS]. Note that the latter objects can be associated with any skew-invertible Hecke symmetry, in particular, that related to Quantum Groups (QG) of A_n series. Their enveloping algebras are treated in terms of the modified reflection equation algebra (mREA) defined bellow. These enveloping algebras have good deformation properties and the categories of their finite dimensional *equivariant* representations look like those of the Lie algebras $gl(m|n)$. Moreover, these algebras can be equipped with a structure of braided bi-algebras. Though the corresponding coproduct acts on the generators of the algebras in a non-classical way it is in a sense intrinsic (it has nothing in common with the coproduct in the QGs). Moreover, it allows to define braided analogs of (co)adjoint vectors fields.

[1]Note that there exists a big family of Hecke and involutive symmetries which are not deformations of the usual flip (cf. [G2]). Even the Poincaré-Hilbert (PH) series corresponding to the "symmetric" $\mathrm{Sym}\,(V) = T(V)/\langle \mathrm{Im}\,(qI - R)\rangle$ and "skew-symmetric" $\bigwedge(V) = T(V)/\langle \mathrm{Im}\,(q^{-1}I + R)\rangle$ algebras can drastically differ from the classical ones, whereas the PH series are stable under a deformation.

We think that apart from generalized Lie algebras related to involutive symmetries (described in Section 2) there is no general definition of a quantum (braided) Lie algebra. Moreover, reasonable quantum Lie algebras exist only for the A_n series (or more generally, for any skew-invertible Hecke symmetry). As for the quantum Lie algebras of the B_n, C_n, D_n series introduced in [DGG], their enveloping algebras are not deformations of their classical counterparts and for this reason they are somewhat pointless objects.

2 Generalized Lie algebras

Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be an involutive symmetry. Then the data

$$(V, R, [\ ,\ ] : V^{\otimes 2} \to V)$$

is called a *generalized Lie algebra* if the following holds

1. $[\ ,\ ]\, R(X \otimes Y) = -[X, Y]$;
2. $[\ ,\ ][\ ,\ ]_{12}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0$;
3. $R[\ ,\ ]_{12}(X \otimes Y \otimes Z) = [\ ,\ ]_{12}R_{23}R_{12}(X \otimes Y \otimes Z)$.

Such a generalized Lie algebra is denoted $\mathfrak{g}$.

Note that the generalized Jacobi identity (the axiom 2) can be rewritten in one of the following equivalent forms

$$[\ ,\ ][\ ,\ ]_{23}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0;$$

$$[\ ,\ ][\ ,\ ]_{12}(X(Y \otimes Z - R(Y \otimes Z))) = [X, [Y, Z]];$$

$$[\ ,\ ][\ ,\ ]_{23}((X \otimes Y - R(X \otimes Y))Z) = [[X, Y], Z].$$

Example 1. If R is the the usual flip then the third axiom is fulfilled automatically and we get a usual Lie algebra. If R is a super-flip then we get a Lie super-algebra. In the both cases R is involutive.

The enveloping algebras of the generalized Lie algebra $\mathfrak{g}$ can be defined in a natural way:

$$U(\mathfrak{g}) = T(V)/\langle X \otimes Y - R(X \otimes Y) - [X, Y]\rangle\,.$$

(Hereafter $\langle I\rangle$ stands for the ideal generated by a set I.) Let us introduce the symmetric algebra $\mathrm{Sym}\,(\mathfrak{g})$ of the generalized Lie algebra $\mathfrak{g}$ by the same formula but with 0 instead of the bracket in the denominator of the above formula.

For this algebra there exists a version of the Poincaré-Birhoff-Witt theorem.

Theorem 2. *The algebra $U(\mathfrak{g})$ is canonically isomorphic to $Sym(\mathfrak{g})$.*

A proof can be obtained via the Koszul property established in [G2] and the results of [PP]. Also, note that similarly to the classical case this isomorphism can be realized via a symmetric (w.r.t. the symmetry R) basis.

Definition 3. *We say that a given braiding $R: V^{\otimes 2}\to V^{\otimes 2}$ is skew-invertible if there exists a morphism $\Psi: V^{\otimes 2}\to V^{\otimes 2}$ such that*

$$\mathrm{Tr}_2\Psi_{12}R_{23} = P_{13} = \mathrm{Tr}_2\Psi_{23}R_{12}$$

where P is the usual flip.

If R is a skew-invertible braiding, a "categorical significance" can be given to the dual space of V. Let V^* be the vector space dual to V. This means that there exist a non-degenerated pairing $\langle\ ,\ \rangle: V^*\otimes V\to \mathbb{K}$ and an extension of the symmetry R to the space $(V^*\oplus V)^{\otimes 2}\to (V^*\oplus V)^{\otimes 2}$ (we keep the same notation for the extended braiding) such that the above pairing is R-invariant. This means that on the space $V^*\otimes V\otimes W$ (resp., $W\otimes V^*\otimes V$) where either $W=V$ or $W=V^*$ the following relations hold

$$R\langle\ ,\ \rangle_{12} = \langle\ ,\ \rangle_{23}\, R_{12}\, R_{23} \qquad (\text{resp.,}\quad R\langle\ ,\ \rangle_{23} = \langle\ ,\ \rangle_{12}\, R_{23}\, R_{12})\,.$$

(Here as usual, we identify $X\in W$ with $X\otimes 1$ and $1\otimes X$.)

Note that if such an extension exists it is unique. By fixing bases $x_i\in V$ and $x_i\otimes x_j\in V^{\otimes 2}$ we can identify the operators R and Ψ with matrices $\|R_{ij}^{kl}\|$ and $\|\Psi_{ij}^{kl}\|$ respectively. For example,

$$R(x_i\otimes x_j) = R_{ij}^{kl}\, x_k\otimes x_l$$

(from now on we assume the summation over the repeated indices).

Then the above definition can be presented in the following matrix form

$$R_{ij}^{kl}\,\Psi_{lm}^{jn} = \delta_m^k\delta_i^n.$$

If ${}^i x$ is the *left dual* basis of the space V^*, i.e. such that $\langle {}^j x, x_i\rangle = \delta_i^j$ then we put

$$\langle x_i,\, {}^j x\rangle = \langle\ ,\ \rangle\,\Psi_{ik}^{jl}\ {}^k x\otimes x_l = C_i^j, \quad \text{where}\quad C_i^j = \Psi_{ik}^{jk}.$$

(Note that the operator Ψ is a part of the braiding R extended to the space $(V^*\oplus V)^{\otimes 2}$.) By doing so, we ensure R-invariance of the pairing $V\otimes V^*\to\mathbb{K}$.

As shown in [GPS] for any skew-invertible Hecke symmetry R the following holds

$$C_i^j\, B_j^k = q^{-2a}\delta_i^k, \quad \text{where}\quad B_i^j = \Psi_{ki}^{kj}$$

with an integer a depending on the the HP series of the algebra $\mathrm{Sym}\,(V)$ (see footnote 1). So, if $q\neq 0$ the operators C and B (represented by the matrices $\|C_i^j\|$ and $\|B_j^k\|$ respectively) are invertible. Therefore, we get a non-trivial pairing

$$\langle\ ,\ \rangle : (V\oplus V^*)^{\otimes 2}\to\mathbb{K}$$

which is R-invariant.

Note that these operators B and C can be introduced without fixing any basis in the space V as follows

$$B_2 = Tr_{(1)}(\Psi_{12}), \quad C_1 = Tr_{(2)}(\Psi_{12}). \tag{1}$$

Let us exhibit an evident but very important property of these operators

$$Tr_{(1)}(B_1 \, R_{12}) = I, \quad Tr_{(2)}(C_2 \, R_{12}) = I. \tag{2}$$

By fixing the basis $h_i^j = x_i \otimes {}^j x$ in the space $\mathrm{End}\,(V) \cong V \otimes V^*$ equipped with the usual product

$$\circ : \mathrm{End}\,(V)^{\otimes 2} \to \mathrm{End}\,(V)$$

we get the following multiplication table $h_i^j \circ h_k^l = \delta_k^j \, h_i^l$.

Below we use another basis in this algebra, namely that $l_i^j = x_i \otimes x^j$ where x^j is the *right dual* basis in the space V^*, i.e. such that $\langle x_i, x^j \rangle = \delta_i^j$. Note that the multiplication table for the the product $\circ$ in this basis is $l_i^j \circ l_k^m = B_k^j \, l_i^m$ (also see formula (6)).

Let R be the above extension of a skew-invertible braiding to the space $(V^* \oplus V)^{\otimes 2}$. Then a braiding $R_{\mathrm{End}\,(V)} : \mathrm{End}\,(V)^{\otimes 2} \to \mathrm{End}\,(V)^{\otimes 2}$ can be defined in a natural way:

$$R_{\mathrm{End}\,(V)} = R_{23} \, R_{34} R_{12} \, R_{23} \,,$$

where we used the isomorphism $\mathrm{End}\,(V) \cong V \otimes V^*$.

Observe that the product $\circ$ in the space $\mathrm{End}\,(V)$ is R-invariant and therefore $R_{\mathrm{End}\,(V)}$-invariant. Namely, we have

$$R_{\mathrm{End}\,(V)}(X \circ Y, Z) = \circ_{23}(R_{\mathrm{End}\,(V)})_{12}(R_{\mathrm{End}\,(V)})_{23}(X \otimes Y \otimes Z)\,,$$

$$R_{\mathrm{End}\,(V)}(X, Y \circ Z) = \circ_{12}(R_{\mathrm{End}\,(V)})_{23}(R_{\mathrm{End}\,(V)})_{12}(X \otimes Y \otimes Z)\,.$$

Example 4. Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a skew-invertible involutive symmetry. Define a generalized Lie bracket by the rule

$$[X, Y] = X \circ Y - \circ R_{\mathrm{End}\,(V)}(X \otimes Y)\,.$$

Then the data $(\mathrm{End}\,(V), R_{\mathrm{End}\,(V)}, [\ ,\ ])$ is a generalized Lie algebra (denoted $gl(V_R)$).

Besides, define the R-trace $\mathrm{Tr}_R : \mathrm{End}\,(V) \to \mathbb{K}$ as follows

$$\mathrm{Tr}_R(h_i^j) = B_j^i \, h_i^j, \qquad X \in \mathrm{End}\,(V)\,.$$

The R-trace possesses the following properties :

- The pairing

  $$\mathrm{End}\,(V) \otimes \mathrm{End}\,(V) \to \mathbb{K} : \; X \otimes Y \mapsto \langle X, Y \rangle = Tr_R(X \circ Y)$$

  is non-degenerated;

- It is $R_{\mathrm{End}\,(V)}$-invariant in the following sense

$$R_{\mathrm{End}\,(V)}((\mathrm{Tr}_R X)\otimes Y) = (I\otimes \mathrm{Tr}_R)R_{\mathrm{End}\,(V)}(X\otimes Y)\,,$$

$$R_{\mathrm{End}\,(V)}(X\otimes (\mathrm{Tr}_R Y)) = (\mathrm{Tr}_R\otimes I)R_{\mathrm{End}\,(V)}(X\otimes Y)\,;$$

- $\mathrm{Tr}_R\,[\ ,\ ] = 0$.

Therefore the set $\{X\in gl(V_R)|Tr_R\,X = 0\}$ is closed w.r.t. the above bracket. Moreover, this subspace squared is invariant w.r.t the symmetry $R_{\mathrm{End}\,(V)}$. Therefore this subspace (denoted $sl(V_R)$) is a generalized Lie subalgebra.

Observe that the enveloping algebra of any generalized Lie algebra possesses a braided Hopf algebra structure such that the coproduct Δ and antipode S are defined on the generators in the classical way

$$\Delta(X) = X\otimes 1 + 1\otimes X, \qquad S(X) = -X\,.$$

For details the reader is referred to [G2].

Also, observe that while R is a super-flip the generalized Lie algebra $gl(V_R)$ (resp., $sl(V_R)$) is nothing but the Lie super-algebras $gl(m|n)$ (resp., $sl(m|n)$).

3 Quantum Lie algebras for B_n, C_n, D_n series

In this Section we restrict ourselves to the braidings coming from the QG $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra of one of the series B_n, C_n, D_n. By the Jacobi identity, the usual Lie bracket

$$[\ ,\ ]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$$

is a $\mathfrak{g}$-morphism.

Let us equip the space $\mathfrak{g}$ with a $U_q(\mathfrak{g})$ action which is a deformation of the usual adjoint one. The space $\mathfrak{g}$ equipped with such an action is denoted $\mathfrak{g}_q$. Our immediate goal is to define an operator

$$[\ ,\ ]_q:\mathfrak{g}_q\otimes\mathfrak{g}_q\to\mathfrak{g}_q$$

which would be a $U_q(\mathfrak{g})$-covariant deformation of the initial Lie bracket. This means that the q-bracket satisfies the relation

$$[\ ,\ ]_q(a_1(X)\otimes a_2(Y)) = a([X\otimes Y]_q)\,,$$

where a is an arbitrary element of the QG $U_q(\mathfrak{g})$, $a_1\otimes a_2 = \Delta(a)$ is the Sweedler notation for the QG coproduct Δ, and $a(X)$ stands for the result of applying the element $a\in U_q(\mathfrak{g})$ to an element $X\in\mathfrak{g}_q$.

Let us show that $U_q(\mathfrak{g})$-covariance of the bracket entails its R-invariance where $R = P\,\pi_{\mathfrak{g}\otimes\mathfrak{g}}(\mathcal{R})$ is the image of the universal quantum R-matrix $\mathcal{R}$ composed with the flip P. Indeed, due to the relation

$$\Delta_{12}(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}\,,$$

we have (by omitting $\pi_{\mathfrak{g}\otimes\mathfrak{g}}$)

$$\begin{aligned}
&R[\ ,\ ]_1(X\otimes Y\otimes Z) = P\mathcal{R}([X,Y]\otimes Z) = P[\ ,\ ]_{12}\Delta_{12}\mathcal{R}(X\otimes Y\otimes Z) = \\
&P[\ ,\ ]_{12}\mathcal{R}_{13}\mathcal{R}_{23}(X\otimes Y\otimes Z) = P[\ ,\ ]_{12}P_{13}R_{13}P_{23}R_{23}(X\otimes Y\otimes Z) = \\
&P[\ ,\ ]_{12}P_{13}P_{23}R_{12}R_{23}(X\otimes Y\otimes Z) = [\ ,\ ]_{23}R_{12}R_{23}(X\otimes Y\otimes Z)\,.
\end{aligned}$$

Finally, we have

$$R[\ ,\ ]_{12} = [\ ,\ ]_{23}R_{12}R_{23},\ \ R[\ ,\ ]_{23} = [\ ,\ ]_{12}R_{23}R_{12}$$

(the second relation can be obtained in a similar way).

Thus, the $U_q(\mathfrak{g})$-covariance of the bracket $[\ ,\ ]_q$ can be considered as an analog of the axiom 3 from the above list. In fact, if $\mathfrak{g}$ belongs to one of the series B_n, C_n or D_n, this property suffices for unique (up to a factor) definition of the bracket $[\ ,\ ]_q$. Indeed, in this case it is known that if one extends the adjoint action of $\mathfrak{g}$ to the space $\mathfrak{g}\otimes\mathfrak{g}$ (via the coproduct in the enveloping algebra), then the latter space is multiplicity free with respect to this action. This means that there is no isomorphic irreducible $\mathfrak{g}$-modules in the space $\mathfrak{g}\otimes\mathfrak{g}$. In particular, the component isomorphic to $\mathfrak{g}$ itself appears only in the skew-symmetric subspace of $\mathfrak{g}\otimes\mathfrak{g}$. A similar property is valid for decomposition of the space $\mathfrak{g}_q\otimes\mathfrak{g}_q$ into a direct sum of irreducible $U_q(\mathfrak{g})$-modules (recall that q is assumed to be generic).

Thus, the map $[\ ,\ ]_q$, being a $U_q(\mathfrak{g})$-morphism, must kill all components in the decomposition of $\mathfrak{g}_q\otimes\mathfrak{g}_q$ into a direct sum of irreducible $\mathfrak{g}_q$-submodules except for the component isomorphic to $\mathfrak{g}_q$. Being restricted to this component, the map $[\ ,\ ]_q$ is an isomorphism. This property uniquely defines the map $[\ ,\ ]_q$ (up to a non-zero factor). For an explicit computation of the structure constants of the q-bracket $[\ ,\ ]_q$ the reader is referred to the paper [DGG]. Note that the authors of that paper embedded the space $\mathfrak{g}_q$ in the QG $U_q(\mathfrak{g})$. Nevertheless, it is possible to do all the calculations without such an embedding but using the QG just as a substitute of the corresponding symmetry group.

Now, we want to define the enveloping algebra of a quantum Lie algebra $\mathfrak{g}_q$. Since the space $\mathfrak{g}_q\otimes\mathfrak{g}_q$ is multiplicity free, we conclude that there exists a unique $U_q(\mathfrak{g})$-morphism $P_q : \mathfrak{g}_q\otimes\mathfrak{g}_q \to \mathfrak{g}_q\otimes\mathfrak{g}_q$ which is a deformation of the usual flip and such that $P_q^2 = I$. Indeed, in order to introduce such an operator it suffices to define q-analogs of symmetric and skew-symmetric components in $\mathfrak{g}_q\otimes\mathfrak{g}_q$. Each of them can be defined as a direct sum of irreducible $U_q(\mathfrak{g})$-submodules of $\mathfrak{g}_q\otimes\mathfrak{g}_q$ which are q-counterparts of the $U_q(\mathfrak{g})$-modules entering the usual symmetric and skew-symmetric subspaces respectively.

Now, the enveloping algebra can be defined as a quotient

$$U(\mathfrak{g}_q) = T(\mathfrak{g}_q)/\langle X \otimes Y - P_q(X \otimes Y) - [\ ,\ ]_q \rangle .$$

Thus, we have defined the quantum Lie algebra $\mathfrak{g}_q$ and its enveloping algebra related to the QG of B_n, C_n, D_n series. However, the question what properties of these quantum Lie algebras are similar to those of generalized Lie algebras is somewhat pointless since the algebra $U(\mathfrak{g}_q)$ is not a deformation of its classical counterpart. Moreover, its "q-commutative" analog (which is defined similarly to the above quotient but without the q-bracket $[\ ,\ ]_q$ in the denominator) is not a deformation of the algebra $\mathrm{Sym}\,(\mathfrak{g})$. For the proof, it suffices to verify that the corresponding semiclassical term is not a Poisson bracket. (However, it becomes Poisson bracket upon restriction to the corresponding algebraic group.)

Remark 5. A similar construction of a quantum Lie algebra is valid for any skew-invertible braiding of the Birman-Murakami-Wenzl type. But for the same reason it is out of our interest.

Also, note that the Lie algebra $sl(2)$ possesses a property similar to that above: the space $sl(2) \otimes sl(2)$ being equipped with the extended adjoint action is a multiplicity free $sl(2)$-module. So, the corresponding quantum Lie algebra and its enveloping algebra can be constructed via the same scheme. However, the latter algebra is a deformation of its classical counterpart. This case is consider in the next Sections as a part of our general construction related to Hecke symmetries.

4 Modified reflection equation algebra and its representation theory

In this section we shortly describe the modified reflection equation algebra (mREA) and the quasitensor Schur-Weyl category of its finite dimensional *equivariant* representations. Our presentation is based on the work [GPS], where these objects were considered in full detail.

The starting point of all constructions is a Hecke symmetry R. As was mentioned in Introduction, the Hecke symmetry is a linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$, satisfying the quantum Yang-Baxter equation and the additional Hecke condition

$$(R - q\,I)(R + q^{-1}I) = 0\,,$$

where a nonzero $q \in \mathbb{K}$ is generic, in particular, is not a primitive root of unity. Besides, we assume R to be skew-invertible (see Definition 3).

Fixing bases $x_i \in V$ and $x_i \otimes x_j \in V^{\otimes 2}$, $1 \le i, j \le N = \dim V$, we identify R with a $N^2 \times N^2$ matrix $\|R_{ij}^{kl}\|$. Namely, we have

$$R(x_i \otimes x_j) = R_{ij}^{kl}\, x_k \otimes x_l\,, \tag{3}$$

where the lower indices label the rows of the matrix, the upper ones — the columns.

As is known, the Hecke symmetry R allows to define a representations ρ_R of the A_{k-1} series Hecke algebras $H_k(q)$, $k \geq 2$, in tensor powers $V^{\otimes k}$:

$$\rho_R : H_k(q) \to \mathrm{End}(V^{\otimes k}) \quad \rho_R(\sigma_i) = R_i := I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(k-i-1)} ,$$

where elements σ_i, $1 \leq i \leq k-1$ form the set of the standard generators of $H_k(q)$.

The Hecke algebra $H_k(q)$ possesses the primitive idempotents $e_a^\lambda \in H_k(q)$, which are in one-to-one correspondence with the set of all standard Young tableaux (λ, a), corresponding to all possible partitions $\lambda \vdash k$. The index a labels the tableaux of a given partition λ in accordance with some ordering.

Under the representation ρ_R, the primitive idempotents e_a^λ are mapped into the projection operators

$$E_a^\lambda(R) = \rho_R(e_a^\lambda) \in \mathrm{End}\,(V^{\otimes k}) , \tag{4}$$

these projectors being some polynomials in R_i, $1 \leq i \leq k-1$.

Under the action of these projectors the spaces $V^{\otimes k}$, $k \geq 2$, are expanded into the direct sum

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} \bigoplus_{a=1}^{d_\lambda} V_{(\lambda,a)}, \qquad V_{(\lambda,a)} = \mathrm{Im}(E_a^\lambda) , \tag{5}$$

where the number d_λ stands for the total number of the standard Young tableaux, which can be constructed for a given partition λ.

Since the projectors E_a^λ with different a are connected by invertible transformations, all spaces $V_{(\lambda,a)}$ with fixed λ and different a are isomorphic. Note, that the isomorphic spaces $V_{(\lambda,a)}$ (at a fixed λ) in decomposition (5) are treated as particular embeddings of the space V_λ into the tensor product $V^{\otimes k}$. Hereafter we use the notation V_λ for the class of the spaces $V_{(\lambda,a)}$ equipped with one or another embedding in $V^{\otimes k}$.

In a similar way we define classes V_μ^*. First, note that the Hecke symmetry being extended to the space $(V^*)^{\otimes 2}$ is given in the basis $x^i \otimes x^j$ as follows

$$R(x^i \otimes x^j) = R_{lk}^{ji}\; x^k \otimes x^l$$

(and similarly in the basis ${}^i x \otimes {}^j x$). It is not difficult to see that the operator R so defined in the space $(V^*)^{\otimes 2}$ is a Hecke symmetry. Thus, by using the above method we can introduce spaces $V^*_{(\mu,a)}$ looking like those from (5) and define the classes V_μ^*.

Now, let us define a rigid quasitensor Schur-Weyl category $\mathrm{SW}(V)$ whose objects are spaces V_λ and V_μ^* labelled by partitions of nonnegative integers, as well as their tensor products $V_\lambda \otimes V_\mu^*$ and all finite sums of these spaces.

Among the morphisms of the category $\mathrm{SW}(V)$ are the above left and right pairings and the set of braidings $R_{U,W} : U \otimes W \to W \otimes U$ for any pair of objects U and W. These braidings can be defined in a natural way. In order to define them on a couple of objects of the form

$V_\lambda \otimes V_\mu^*$ we embed them into appropriate products $V^{\otimes k} \otimes (V^*)^{\otimes l}$ and define the braiding $R_{U,W}$ as an appropriate restriction. Note, that all these braidings are R-invariant maps (cf. [GPS] for detail). Note that the category $\mathrm{SW}(V)$ is monoidal quasitensor rigid according to the standard terminology (cf. [CP]).

Now we are aiming at introducing modified reflection equation algebra and equipping objects of the category $\mathrm{SW}(V)$ with a structure of its modules.

Again, consider the space $\mathrm{End}\,(V)$ equipped with the basis l_i^j (see Section 2). Note that the element l_i^j acts on the elements of the space V as follows

$$l_i^j(x_k) := x_i\,\langle x^j, x_k\rangle = x_i B_k^j\,. \tag{6}$$

Introduce the $N \times N$ matrix $L = \|l_i^j\|$. Also, define its "copies" by the iterative rule

$$L_{\overline{1}} := L_1 := L \otimes I, \qquad L_{\overline{k+1}} := R_k L_{\overline{k}} R_k^{-1}\,. \tag{7}$$

Observe that the isolated spaces $L_{\overline{k}}$ have no meaning (except for that $L_{\overline{1}}$). They can be only correctly understood in the products $L_{\overline{1}}L_{\overline{2}}$, $L_{\overline{1}}L_{\overline{2}}L_{\overline{3}}$ and so on, but this notation is useful in what follows.

Definition 6. The associative algebra generated by the unit element $e_{\mathcal{L}}$ and the indeterminates l_i^j $1 \le i, j \le N$ subject to the following matrix relation

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} - \hbar\,(R_{12}\,L_1 - L_1\,R_{12}) = 0\,, \tag{8}$$

is called the modified reflection equation algebra (mREA) and denoted $\mathcal{L}(R_q, \hbar)$.

Note, that at $\hbar = 0$ the above algebra is known as the reflection equation algebra $\mathcal{L}(R_q)$. Actually, at $q \neq \pm 1$ one has $\mathcal{L}(R_q, \hbar) \cong \mathcal{L}(R_q)$. Since at $\hbar \neq 0$ it is always possible to renormalize generators $L \mapsto \hbar\,L$. So, below we consider the case $\hbar = 1$.

Thus, the mREA is the quotient algebra of the free tensor algebra $T(\mathrm{End}\,(V))$ over the two-sided ideal, generated by the matrix elements of the left hand side of (8). It can be shown, that the relations (8) are R-invariant, that is the above two-sided ideal is invariant when commuting with any object U under the action of the braidings $R_{U,\mathrm{End}\,(V)}$ or $R_{\mathrm{End}\,(V),U}$ of the category SW(V).

Taking into account (2) one can easily prove, that the action (6) gives a basic (vector) representation of the mREA $\mathcal{L}(R_q, 1)$ in the space V

$$\rho_1(l_i^j) \triangleright x_k = x_i B_k^j\,, \tag{9}$$

where the symbol $\triangleright$ stands for the (left) action of a linear operator onto an element. Since B is non-degenerated, the representation is irreducible.

Another basic (covector) representation $\rho_1^* : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(V^*)$ is given by

$$\rho_1^*(l_i^j) \triangleright x^k = -x^r\,R_{ri}^{kj}\,. \tag{10}$$

one can prove, that the maps $\mathrm{End}\,(V) \to \mathrm{End}\,(V)$ and $\mathrm{End}\,(V) \to \mathrm{End}\,(V^*)$ generated by

$$l_i^j \mapsto \rho_1(l_i^j) \quad \text{and} \quad l_i^j \mapsto \rho_1^*(l_i^j)$$

are the morphisms of the category $\mathrm{SW}(V)$.

Definition 7. *A representation $\rho : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(U)$ where U is an object of the category $\mathrm{SW}(V)$ is called equivariant if its restriction to $\mathrm{End}\,(V)$ is a categorical morphism.*

Thus, the above representations ρ_1 and ρ_1^* are equivariant.

Note that there are known representations of the mREA which are nor equivariant. However, the class of equivariant representations of the mREA is very important. In particular, because the tensor product of two equivariant $\mathcal{L}(R_q, 1)$-modules can be also equipped with a structure of an equivariant $\mathcal{L}(R_q, 1)$-module via a "braided bialgebra structure" of the mREA.

Let us briefly describe this structure. It consists of two maps: the braided coproduct Δ and counit ε.

The coproduct Δ is an algebra homomorphism of $\mathcal{L}(R_q, 1)$ into the associative algebra $\mathbf{L}(R_q)$ which is defined as follows.

- As a vector space over the field $\mathbb{K}$ the algebra $\mathbf{L}(R_q)$ is isomorphic to the tensor product of two copies of mREA
$$\mathbf{L}(R_q) = \mathcal{L}(R_q, 1) \otimes \mathcal{L}(R_q, 1)\,.$$
- The product $\star : (\mathbf{L}(R_q))^{\otimes 2} \to \mathbf{L}(R_q)$ is defined by the rule
$$(a_1 \otimes b_1) \star (a_2 \otimes b_2) := a_1 a_2' \otimes b_1' b_2\,, \qquad a_i \otimes b_i \in \mathbf{L}(R_q)\,, \tag{11}$$
where $a_1 a_2'$ and $b_1 b_2'$ are the usual product of mREA elements, while a_1' and b_1' result from the action of the braiding $R_{\mathrm{End}(V)}$ (see Section 2) on the tensor product $b_1 \otimes a_2$
$$a_2' \otimes b_1' := R_{\mathrm{End}(V)}(b_1 \otimes a_2)\,. \tag{12}$$

The braided coproduct Δ is now defined as a linear map $\Delta : \mathcal{L}(R_q, 1) \to \mathbf{L}(R_q)$ with the following properties:

$$\begin{aligned}
&\Delta(e_{\mathcal{L}}) := e_{\mathcal{L}} \otimes e_{\mathcal{L}} \\
&\Delta(l_i^j) := l_i^j \otimes e_{\mathcal{L}} + e_{\mathcal{L}} \otimes l_i^j - (q - q^{-1}) \textstyle\sum_k l_i^k \otimes l_k^j \\
&\Delta(ab) := \Delta(a) \star \Delta(b) \qquad \forall\, a, b \in \mathcal{L}(R_q, 1)\,.
\end{aligned} \tag{13}$$

In addition to (13), we introduce a linear map $\varepsilon : \mathcal{L}(R_q, 1) \to \mathbb{K}$

$$\begin{aligned}
&\varepsilon(e_{\mathcal{L}}) := 1 \\
&\varepsilon(l_i^j) := 0 \\
&\varepsilon(ab) := \varepsilon(a)\varepsilon(b) \qquad \forall\, a, b \in \mathcal{L}(R_q, 1)\,.
\end{aligned} \tag{14}$$

One can show (cf. [GPS]) that the maps Δ and ε are indeed algebra homomorphisms and that they satisfy the relation

$$(\mathrm{id} \otimes \varepsilon)\,\Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\,\Delta\,.$$

Let now U and W be two equivariant mREA-modules with representations $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(U)$ and $\rho_W : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(W)$ respectively. Consider the map $\rho_{U\otimes W} : \mathbf{L}(R_q) \to \mathrm{End}\,(U \otimes W)$ defined as follows

$$\rho_{U\otimes W}(a \otimes b) \triangleright (u \otimes w) = (\rho_U(a) \triangleright u') \otimes (\rho_W(b') \triangleright w)\,, \qquad a \otimes b \in \mathbf{L}(R_q)\,, \tag{15}$$

where

$$u' \otimes b' := R_{\mathrm{End}\,(V),U}(b \otimes u)\,.$$

Definition (15) is self-consistent since the map $b \mapsto \rho_W(b')$ is also a representation of the mREA $\mathcal{L}(R_q, 1)$.

The following proposition holds true.

Proposition 8. ([GPS]) *The action* (15) *defines a representation of the algebra* $\mathbf{L}(R_q)$.

Note again, that the equivariance of the representations in question plays a decisive role in the proof of the above proposition.

As an immediate corollary of the proposition 8 we get the rule of tensor multiplication of equivariant $\mathcal{L}(R_q, 1)$-modules.

Corollary 9. *Let U and W be two $\mathcal{L}(R_q, 1)$-modules with equivariant representations ρ_U and ρ_W. Then the map $\mathcal{L}(R_q, 1) \to \mathrm{End}\,(U \otimes W)$ given by the rule*

$$a \mapsto \rho_{U\otimes W}(\Delta(a))\,, \qquad \forall\, a \in \mathcal{L}(R_q, 1) \tag{16}$$

is an equivariant representation. Here the coproduct Δ and the map $\rho_{U\otimes W}$ are given respectively by formulae (13) *and* (15).

Thus, by using (16) we can extend the basic representations ρ_1 and ρ_1^* to the representations ρ_k and ρ_l^* in tensor products $V^{\otimes k}$ and $(V^{*\otimes l})$ respectively. These representations are reducible, and their restrictions on the representations $\rho_{\lambda,a}$ in the invariant subspaces $V_{(\lambda,a)}$ (see (5)) are given by the projections

$$\rho_{\lambda,a} = E_a^\lambda \circ \rho_k \tag{17}$$

and similarly for the subspaces $V^*_{(\mu,a)}$. By using (16) once more we can equip each object of the category $\mathrm{SW}(V)$ with the structure of an equivariant $\mathcal{L}(R_q, 1)$-module.

5 Quantum Lie algebras related to Hecke symmetries

In this section we consider the question to which extent one can use the scheme of section 2 in the case of non-involutive Hecke symmetry R for definition of the corresponding Lie algebra-like object. For such an object related to a Hecke symmetry R we use the term *quantum* or *braided* Lie algebra. Besides, we require the mREA, connected with the same symmetry R, to be an analog of the enveloping algebra of the quantum Lie algebra. Finally, we compare the properties of the above generalized Lie algebras and quantum ones.

Let us recall the interrelation of a usual Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$. As is known, the universal enveloping algebra for a Lie algebra $\mathfrak{g}$ is a unital associative algebra $U(\mathfrak{g})$ possessing the following properties:

- There exists a linear map $\tau : \mathfrak{g} \to U(\mathfrak{g})$ such that 1 and $\mathrm{Im}\,\tau$ generate the whole $U(\mathfrak{g})$.
- The Lie bracket $[x, y]$ of any two elements of $\mathfrak{g}$ has the image

$$\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x).$$

Let us rewrite these formulae in an equivalent form. Note that the tensor square $\mathfrak{g} \otimes \mathfrak{g}$ splits into the direct sum of symmetric and skew symmetric components

$$\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a, \qquad \mathfrak{g}_s = \mathrm{Im}\,\mathcal{S}, \quad \mathfrak{g}_a = \mathrm{Im}\,\mathcal{A}\,,$$

where $\mathcal{S}$ and $\mathcal{A}$ are the standard (skew)symmetrizing operators

$$\mathcal{S}(x \otimes y) = x \otimes y + y \otimes x, \quad \mathcal{A}(x \otimes y) = x \otimes y - y \otimes x\,,$$

where we neglect the usual normalizing factor 1/2. Then the skew-symmetry property of the classical Lie bracket is equivalent to the requirement

$$[\ ,\ ]\,\mathcal{S}(x \otimes y) = 0\,. \tag{18}$$

The image of the bracket in $U(\mathfrak{g})$ is presented as follows

$$\tau([x, y]) = \circ\mathcal{A}\,(\tau(x) \otimes \tau(y))\,, \tag{19}$$

where $\circ$ stands for the product in the associative algebra $U(\mathfrak{g})$.

- The Jacobi identity for the Lie bracket $[\ ,\ ]$ translates into the requirement that the correspondence $x \mapsto [x,\ ]$ generate the (adjoint) representation of $U(\mathfrak{g})$ in the linear subspace $\tau(\mathfrak{g}) \subset U(\mathfrak{g})$.

So, we define a braided Lie algebra as a linear subspace $\mathcal{L}_1 = \mathrm{End}\,(V)$ of the mREA $\mathcal{L}(R_q, 1)$, which generates the whole algebra and is equipped with the quantum Lie bracket. We want the bracket to satisfy some skew-symmetry condition, generalizing (18), and define a representation of the mREA in the same linear subspace $\mathcal{L}_1$ via an analog of the Jacobi identity.

As $\mathcal{L}_1$, let us take the linear span of mREA generators

$$\mathcal{L}_1 = \mathrm{End}\,(V) \cong V \otimes V^* .$$

Together with the unit element this subspace generate the whole $\mathcal{L}(R_q, 1)$ by definition.

In order to find the quantum Lie bracket, consider a particular representation of $\mathcal{L}(R_q, 1)$ in the space $\mathrm{End}\,(V)$. In this case the general formula (16) reads

$$l_i^j \mapsto \rho_{V\otimes V^*}(\Delta(l_i^j))\,,$$

where we should take the basic representations (9) and (10) as $\rho_V(l_i^j)$ and $\rho_{V^*}(l_i^j)$ respectively. Omitting straightforward calculations, we write the final result in the compact matrix form

$$\rho_{V\otimes V^*}(L_{\overline{1}}) \triangleright L_{\overline{2}} = L_1 R_{12} - R_{12} L_1\,, \tag{20}$$

where the matrix $L_{\overline{k}}$ is defined in (7).

Let us define

$$[L_{\overline{1}}, L_{\overline{2}}] = L_1 R_{12} - R_{12} L_1\,. \tag{21}$$

The generalized skew-symmetry (the axiom 1 from Section 2) of this bracket is now modified as follows. In the space $\mathcal{L}_1 \otimes \mathcal{L}_1$ one can construct two projection operators $\mathcal{S}_q$ and $\mathcal{A}_q$ which are interpreted as q-symmetrizer and q-skew-symmetrizer respectively (cf. [GPS]). Then straightforward calculations show that the above bracket satisfies the relation

$$[\ ,\ ]\,\mathcal{S}_q(L_{\overline{1}} \otimes L_{\overline{2}}) = 0\,, \tag{22}$$

which is the generalized skew-symmetry condition, analogous to (18).

Moreover, if we rewrite the defining commutation relations of the mREA (8) in the equivalent form

$$L_{\overline{1}} L_{\overline{2}} - R_{12}^{-1} L_{\overline{1}} L_{\overline{2}} R_{12} = L_1 R_{12} - R_{12} L_1\,, \tag{23}$$

we come to a generalization of the formula (19).

By introducing an operator

$$Q : \mathcal{L}_1^{\otimes 2} \to \mathcal{L}_1^{\otimes 2}, \quad Q(L_{\overline{1}} L_{\overline{2}}) = R_{12}^{-1} L_{\overline{1}} L_{\overline{2}} R_{12}$$

we can present the relation (23) as follows

$$L_{\overline{1}} L_{\overline{2}} - Q(L_{\overline{1}} L_{\overline{2}}) = [L_{\overline{1}}, L_{\overline{2}}]. \tag{24}$$

It looks like the defining relation of the enveloping algebra of a generalized Lie algebra. (Though we prefer to use the notations $L_{\overline{k}}$ it is possible to exhibit the maps Q and $[\ ,\ ]$ in the basis $l_i^j \otimes l_k^m$.) Observe that the map Q is a braiding. Also, note that the operators $\mathcal{S}_q$ and $\mathcal{A}_q$ can be expressed in terms of Q and its inverse (cf. [GPS]).

We call the data $(\mathfrak{g} = \mathcal{L}_1, Q, [\ ,\ ])$ the gl type quantum (braided) Lie algebra. Note that if $q = 1$ (i.e. the symmetry R is involutive) then $Q = R_{\mathrm{End}\,(V)}$ and this quantum Lie algebra is nothing but the generalized Lie algebra $gl(V_R)$ and the corresponding mREA becomes isomorphic to its enveloping algebra.

Let us list the properties of the the quantum Lie algebra in question.

- The bracket $[\ ,\ ]$ is skew-symmetric in the sense of (22).
- The q-Jacobi identity is valid in the following form

$$[\ ,\ ][\ ,\ ]_{12} = [\ ,\ ][\ ,\ ]_{23}(I - Q_{12})\,. \tag{25}$$

- The bracket $[\ ,\ ]$ is R-invariant. Essentially, this means that the following relations hold

$$\begin{aligned} R_{\mathrm{End}\,(V)}[\ ,\ ]_{23} &= [\ ,\ ]_{12}(R_{\mathrm{End}\,(V)})_{23}(R_{\mathrm{End}\,(V)})_{12}, \\ R_{\mathrm{End}\,(V)}[\ ,\ ]_{12} &= [\ ,\ ]_{23}(R_{\mathrm{End}\,(V)})_{12}(R_{\mathrm{End}\,(V)})_{23}\,. \end{aligned} \tag{26}$$

So, the adjoint action

$$L_{\overline{1}} \triangleright L_{\overline{2}} = [L_{\overline{1}}, L_{\overline{2}}]$$

is indeed a representation. By chance (!) the representation $\rho_{V\otimes V^*}$ coincides with this adjoint action.

Turn now to the question of the "sl-reduction", that is, the passing from the mREA $\mathcal{L}(R_q, 1)$ to the quotient algebra

$$\mathcal{SL}(R_q) := \mathcal{L}(R_q, 1)/\langle \mathrm{Tr}_R L\rangle\,, \qquad \mathrm{Tr}_R L := \mathrm{Tr}(CL)\,, \tag{27}$$

(see Section 2 for the operator C). The element $\ell := \mathrm{Tr}_R L$ is central in the mREA, which can be easily proved by calculating the R-trace in the second space of the matrix relation (8).

To describe the quotient algebra $\mathcal{SL}(R_q)$ explicitly, we pass to a new set of generators $\{f_i^j, \ell\}$, connected with the initial one by a linear transformation:

$$l_i^j = f_i^j + (\mathrm{Tr}(C))^{-1}\delta_i^j\,\ell \quad \text{or} \quad L = F + (\mathrm{Tr}(C))^{-1} I\,\ell\,, \tag{28}$$

where $F = \|f_i^j\|$. Hereafter we assume that $\mathrm{Tr}\,C = \ell_i^i \neq 0$. (So, the Lie super-algebras $gl(m|m)$ and their q-deformations are forbidden.) Obviously, $\mathrm{Tr}_R F = 0$, i.e. the generators f_i^j are dependent.

In terms of the new generators, the commutation relations of the mREA read

$$\begin{cases} R_{12}F_1R_{12}F_1 - F_1R_{12}F_1R_{12} = (e_{\mathcal{L}} - \dfrac{\omega}{Tr(C)}\,\ell)(R_{12}F_1 - F_1R_{12}) \\ \ell\, F = F\,\ell\,, \end{cases}$$

where $\omega = q - q^{-1}$. Now, it is easy to describe the quotient (27) . The matrix $F = \|f_i^j\|$ of $\mathcal{SL}(R_q)$ generators satisfy the same commutation relations (8) as the matrix L

$$R_{12}F_1R_{12}F_1 - F_1R_{12}F_1R_{12} = R_{12}F_1 - F_1R_{12}\,, \tag{29}$$

but the generators f_i^j are linearly dependent.

Rewriting this relation in the form similar to (24) we can introduce an sl-type bracket. However for such a bracket the q-Jacobi identity fails. This is due to the fact the element ℓ comes in the relations for f_i^j (at $q = 1$ this effect disappears). Nevertheless, we can construct a representation

$$\rho_{V\otimes V^*} : \mathcal{SL}(R_q) \to \mathrm{End}\,(V \otimes V^*)$$

which is an analog of the adjoint representation. In order to do so, we rewrite the representation (20) in terms of the generators f_i^j and ℓ. Taking relation (28) into account, we find, after a short calculation

$$\begin{aligned}
&\rho_{V\otimes V^*}(\ell) \triangleright \ell = 0, \qquad \rho_{V\otimes V^*}(F_1) \triangleright \ell = 0\,,\\
&\rho_{V\otimes V^*}(\ell) \triangleright F_1 = -\omega\,\mathrm{Tr}(C)\,F_1\\
&\rho_{V\otimes V^*}(F_{\overline{1}}) \triangleright F_{\overline{2}} = F_1R_{12} - R_{12}F_1 + \omega R_{12}F_1R_{12}^{-1}\,.
\end{aligned} \tag{30}$$

Namely, the last formula from this list defines the representation $\rho_{V\otimes V^*}$. However, in contrast with the mREA $\mathcal{L}(R_q, 1)$, this map is different from that defined by the bracket $[\ ,\ ]$ reduced to the space $\mathrm{span}\,(f_i^j)$. This is reason why the "q-adjoint" representation cannot be presented in the form (25). (Also, note that though ℓ is central it acts in a non-trivial way on the elements f_i^j.)

Moreover, any object U of the category SW(V) above such that

$$\rho_U(\ell) = \chi\, I_U\,, \qquad \chi \in \mathbb{K}$$

is a scalar operator, can be equipped with an $\mathcal{SL}(R_q)$-module structure. First, let us observe that for any representation $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(U)$ and for any $z \in \mathbb{K}$ the map

$$\rho_U^z : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(U), \qquad \rho_U^z(l_i^j) = z\rho_U(l_i^j) + \delta_i^j(1 - z)(q - q^{-1})^{-1}\, I_U$$

is a representation of this algebra as well.

By using this freedom we can convert a given representation $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}\,(U)$ with the above property into that ρ_U^z such that $\rho_U^z(\ell) = 0$. Thus we get a representation of the algebra $\mathcal{SL}(R_q)$. Explicitly, this representation is given by the formula

$$\tilde{\rho}(f_i^j) = \frac{1}{\xi}\left(\rho(l_i^j) - (\mathrm{Tr}(C))^{-1}\rho(\ell)\,\delta_i^j\right), \qquad \xi = 1 - (q - q^{-1})(\mathrm{Tr}(C))^{-1}\chi\,. \tag{31}$$

The data $(\mathrm{span}\,(f_i^j), Q, [\ ,\ ])$ where the bracket stands for the l.h.s. of (24) restricted to $\mathrm{span}\,(f_i^j)$ is called the sl-type quantum (braided) Lie algebra.

Note that in the particular case related to the QG $U_q(sl(n))$ this quantum algebra can be treated in terms of [LS] where an axiomatic approach to the corresponding Lie algebra-like object is given. However, we think that any general axiomatic definition of such objects is somewhat useless (unless the corresponding symmetry is involutive). Our viewpoint is motivated by the fact that for B_n, C_n, D_n series there do not exist "quantum Lie algebras" such that their enveloping algebras have good deformation properties. As for the A_n series (or more generally, for any skew-invertible Hecke symmetry) such objects exist and can be explicitly exhibited via the mREA. Their properties differ from those listed in [W, GM] in the framework of an axiomatic approach to Lie algebra-like objects.

Completing the paper, we want to emphasize that the above coproduct can be useful for definition of a "braided (co)adjoint vector field". In the $\mathcal{L}(R_q, 1)$ case these fields are naturally introduced through the above adjoint action extended to the symmetric algebra of the space $\mathcal{L}_1$ by means of this coproduct. The symmetric algebra can be defined via the above operators $\mathcal{S}_q$ and $\mathcal{A}_q$. In the $\mathcal{SL}(R_q)$ case a similar treatment is possible if $\mathrm{Tr}\, C \neq 0$.

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On the automorphism groups of q-enveloping algebras of nilpotent Lie algebras

Stéphane Launois*

Abstract

We investigate the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of certain simple complex Lie algebras $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Studying its action on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ we compute this group in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ confirming a Conjecture of Andruskiewitsch and Dumas regarding the automorphism group of $U_q^+(\mathfrak{g})$. In the case where $\mathfrak{g} = \mathfrak{sl}_3$, we retrieve the description of the automorphism group of the quantum Heisenberg algebra that was obtained independently by Alev and Dumas, and Caldero. In the case where $\mathfrak{g} = \mathfrak{so}_5$, the automorphism group of $U_q^+(\mathfrak{g})$ was computed in [16] by using previous results of Andruskiewitsch and Dumas. In this paper, we give a new (simpler) proof of the Conjecture of Andruskiewitsch and Dumas in the case where $\mathfrak{g} = \mathfrak{so}_5$ based both on the original proof and on graded arguments developed in [17] and [18].

Introduction

In the classical situation, there are few results about the automorphism group of the enveloping algebra $U(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ over $\mathbb{C}$; except when $\dim \mathcal{L} \leq 2$, these groups are known to possess "wild" automorphisms and are far from being understood. For instance, this is the case when $\mathcal{L}$ is the three-dimensional abelian Lie algebra [22], when $\mathcal{L} = \mathfrak{sl}_2$ [14] and when $\mathcal{L}$ is the three-dimensional Heisenberg Lie algebra [1].

In this paper we study the quantum situation. More precisely, we study the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of a finite dimensional simple complex Lie algebra $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Although it is a common belief that quantum algebras are "rigid" and so should possess few symmetries, little is known about the automorphism group of $U_q^+(\mathfrak{g})$.

*School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland. E-mail: stephane.launois@ed.ac.uk. Current address (From 01 August 2007): Institute of Mathematics, Statistics and Actuarial Science, University of Kent at Canterbury, CT2 7NF, UK. This research was supported by a Marie Curie Intra-European Fellowship within the 6$^{\text{th}}$ European Community Framework Programme.

Indeed, until recently, this group was known only in the case where $\mathfrak{g} = \mathfrak{sl}_3$ whereas the structure of the automorphism group of the augmented form $\check{U}_q(\mathfrak{b}^+)$, where $\mathfrak{b}^+$ is the positive Borel subalgebra of $\mathfrak{g}$, has been described in [9] in the general case.

The automorphism group of $U_q^+(\mathfrak{s}l_3)$ was computed independently by Alev-Dumas, [2], and Caldero, [8], who showed that

$$\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) \simeq (\mathbb{C}^*)^2 \rtimes S_2.$$

Recently, Andruskiewitsch and Dumas, [4] have obtained partial results on the automorphism group of $U_q^+(\mathfrak{so}_5)$. In view of their results and the description of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$, they have proposed the following conjecture.

Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):

$$\mathrm{Aut}(U_q^+(\mathfrak{g})) \simeq (\mathbb{C}^*)^{\mathrm{rk}(\mathfrak{g})} \rtimes \mathrm{autdiagr}(\mathfrak{g}),$$

where $\mathrm{autdiagr}(\mathfrak{g})$ denotes the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.

Recently we proved this conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$, [16], and, in collaboration with Samuel Lopes, in the case where $\mathfrak{g} = \mathfrak{sl}_4$, [18]. The techniques in these two cases are very different. Our aim in this paper is to show how one can prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ by first studying the action of $\mathrm{Aut}(U_q^+(\mathfrak{g}))$ on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ - this was the main idea in [16] -, and then using graded arguments as developed in [17] and [18]. This strategy leads us to a new (simpler) proof of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and q is a nonzero complex number that is not a root of unity.

1 Preliminaries

In this section, we present the $\mathcal{H}$-stratification theory of Goodearl and Letzter for the positive part $U_q^+(\mathfrak{g})$ of the quantised enveloping algebra of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$. In particular, we present a criterion (due to Goodearl and Letzter) that characterises the primitive ideals of $U_q^+(\mathfrak{g})$ among its prime ideals. In the next section, we will use this criterion in order to describe the primitive spectrum of $U_q^+(\mathfrak{g})$ in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

1.1 Quantised enveloping algebras and their positive parts.

Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra of rank n. We denote by $\pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that π is a basis of an euclidean vector space E over $\mathbb{R}$, whose inner product is denoted by $(\ ,\ )$ (E is usually denoted by $\mathfrak{h}^*_{\mathbb{R}}$ in Bourbaki). We denote by W the Weyl group of $\mathfrak{g}$, that is, the subgroup of the orthogonal group of E generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \dots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, $i \in \{1, \dots, n\}$. The length of $w \in W$ is denoted by $l(w)$. Further, we denote by w_0 the longest element of W. We denote by R^+ the set of positive roots and by R the set of roots. Set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$ and $Q := \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data. As $\mathfrak{g}$ is simple, $a_{ij} \in \{0, -1, -2, -3\}$ for all $i \neq j$.

Recall that the scalar product of two roots (α, β) is always an integer. As in [5], we assume that the short roots have length $\sqrt{2}$.

For all $i \in \{1, \dots, n\}$, set $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and

$$\left[\begin{array}{c} m \\ k \end{array} \right]_i := \frac{(q_i - q_i^{-1}) \dots (q_i^{m-1} - q_i^{1-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \dots (q_i^k - q_i^{-k})(q_i - q_i^{-1}) \dots (q_i^{m-k} - q_i^{k-m})}$$

for all integers $0 \leq k \leq m$. By convention,

$$\left[\begin{array}{c} m \\ 0 \end{array} \right]_i := 1.$$

The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by the indeterminates $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \text{ and } K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_i E_i^{1-a_{ij}-k} E_j E_i^k = 0 \ (i \neq j) \tag{1}$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_i F_i^{1-a_{ij}-k} F_j F_i^k = 0 \ (i \neq j).$$

We refer the reader to [5, 13, 15] for more details on this (Hopf) algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \dots, E_n$ (resp. $F_1, \dots, F_n$) and by U^0 the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$. Moreover, for all $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n \in Q$, we set

$$K_\alpha := K_1^{a_1} \cdots K_n^{a_n}.$$

As in the classical case, there is a triangular decomposition as vector spaces:

$$U_q^-(\mathfrak{g}) \otimes U^0 \otimes U_q^+(\mathfrak{g}) \simeq U_q(\mathfrak{g}).$$

In this paper we are concerned with the algebra $U_q^+(\mathfrak{g})$ that admits the following presentation, see [13, Theorem 4.21]. The algebra $U_q^+(\mathfrak{g})$ is (isomorphic to) the $\mathbb{C}$-algebra generated by n indeterminates $E_1, \dots, E_n$ subject to the quantum Serre relations (1).

1.2 PBW-basis of $U_q^+(\mathfrak{g})$.

To each reduced decomposition of the longest element w_0 of the Weyl group W of $\mathfrak{g}$, Lusztig has associated a PBW basis of $U_q^+(\mathfrak{g})$, see for instance [19, Chapter 37], [13, Chapter 8] or [5, I.6.7]. The construction relates to a braid group action by automorphisms on $U_q^+(\mathfrak{g})$. Let us first recall this action. For all $s \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, we set

$$[s]_i := \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \quad \text{and} \quad [s]_i! := [1]_i \dots [s-1]_i [s]_i.$$

As in [5, I.6.7], we denote by $\mathrm{T_i}$, for $1 \leq i \leq n$, the automorphism of $U_q^+(\mathfrak{g})$ defined by:

$$\begin{aligned}
T_i(E_i) &= -F_i K_i, \\
T_i(E_j) &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)}, \quad i \neq j \\
T_i(F_i) &= -K_i^{-1} E_i, \\
T_i(F_j) &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)}, \quad i \neq j \\
T_i(K_\alpha) &= K_{s_i(\alpha)}, \quad \alpha \in Q,
\end{aligned}$$

where $E_i^{(s)} := \frac{E_i^s}{[s]_i!}$ and $F_i^{(s)} := \frac{F_i^s}{[s]_i!}$ for all $s \in \mathbb{N}$. It was proved by Lusztig that the automorphisms T_i satisfy the braid relations, that is, if $s_i s_j$ has order m in W, then

$$T_i T_j T_i \cdots = T_j T_i T_j \dots,$$

where there are exactly m factors on each side of this equality.

The automorphisms T_i can be used in order to describe PBW bases of $U_q^+(\mathfrak{g})$ as follows. It is well-known that the length of w_0 is equal to the number N of positive roots of $\mathfrak{g}$. Let $s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of w_0. For $k \in \{1, \dots, N\}$, we set $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then $\{\beta_1, \dots, \beta_N\}$ is exactly the set of positive roots of $\mathfrak{g}$. Similarly, we define elements E_{β_k} of $U_q(\mathfrak{g})$ by

$$E_{\beta_k} := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}).$$

Note that the elements E_{β_k} depend on the reduced decomposition of w_0. The following well-known results were proved by Lusztig and Levendorskii-Soibelman.

Theorem 1.1 (Lusztig and Levendorskii-Soibelman)**.**

1. *For all $k \in \{1, \ldots, N\}$, the element E_{β_k} belongs to $U_q^+(\mathfrak{g})$.*

2. *If $\beta_k = \alpha_i$, then $E_{\beta_k} = E_i$.*

3. *The monomials $E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$, with $k_1, \ldots, k_N \in \mathbb{N}$, form a linear basis of $U_q^+(\mathfrak{g})$.*

4. *For all $1 \leq i < j \leq N$, we have*

$$E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum a_{k_{i+1}, \ldots, k_{j-1}} E_{\beta_{i+1}}^{k_{i+1}} \cdots E_{\beta_{j-1}}^{k_{j-1}},$$

   where each $a_{k_{i+1}, \ldots, k_{j-1}}$ belongs to $\mathbb{C}$.

As a consequence of this result, $U_q^+(\mathfrak{g})$ can be presented as a skew-polynomial algebra:

$$U_q^+(\mathfrak{g}) = \mathbb{C}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \cdots [E_{\beta_N}; \sigma_N, \delta_N],$$

where each σ_i is a linear automorphism and each δ_i is a σ_i-derivation of the appropriate subalgebra. In particular, $U_q^+(\mathfrak{g})$ is a noetherian domain and its group of invertible elements is reduced to nonzero complex numbers.

1.3 Prime and primitive spectra of $U_q^+(\mathfrak{g})$.

We denote by $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ the set of prime ideals of $U_q^+(\mathfrak{g})$. First, as q is not a root of unity, it was proved by Ringel [21] (see also [10, Theorem 2.3]) that, as in the classical situation, every prime ideal of $U_q^+(\mathfrak{g})$ is completely prime.

In order to study the prime and primitive spectra of $U_q^+(\mathfrak{g})$, we will use the stratification theory developed by Goodearl and Letzter. This theory allows the construction of a partition of these two sets by using the action of a suitable torus on $U_q^+(\mathfrak{g})$. More precisely, the torus $\mathcal{H} := (\mathbb{C}^*)^n$ acts naturally by automorphisms on $U_q^+(\mathfrak{g})$ via:

$$(h_1, \ldots, h_n).E_i = h_i E_i \text{ for all } i \in \{1, \ldots, n\}.$$

(It is easy to check that the quantum Serre relations are preserved by the group $\mathcal{H}$.) Recall (see [4, 3.4.1]) that this action is rational. (We refer the reader to [5, II.2.] for the defintion of a rational action.) A non-zero element x of $U_q^+(\mathfrak{g})$ is an *$\mathcal{H}$-eigenvector of* $U_q^+(\mathfrak{g})$ if $h.x \in \mathbb{C}^* x$ for all $h \in \mathcal{H}$. An ideal I of $U_q^+(\mathfrak{g})$ is *$\mathcal{H}$-invariant* if $h.I = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}$-$\mathrm{Spec}(U_q^+(\mathfrak{g}))$ the set of all $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{g})$. It turns out that this is a finite set by a theorem of Goodearl and Letzter about iterated Ore extensions, see [11, Proposition 4.2]. In fact, one can be even more precise in our situation. Indeed, in [12], Gorelik has also constructed a stratification of the prime spectrum of $U_q^+(\mathfrak{g})$ using tools coming from representation theory. It turns out that her stratification coincides with the $\mathcal{H}$-stratification, so that we deduce from [12, Corollary 7.1.2] that

Proposition 1.2 (Gorelik)**.** *$U_q^+(\mathfrak{g})$ has exactly $|W|$ $\mathcal{H}$-invariant prime ideals.*

The action of $\mathcal{H}$ on $U_q^+(\mathfrak{g})$ allows via the $\mathcal{H}$-stratification theory of Goodearl and Letzter (see [5, II.2]) the construction of a partition of $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ as follows. If J is an $\mathcal{H}$-invariant prime ideal of $U_q^+(\mathfrak{g})$, we denote by $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ the *$\mathcal{H}$-stratum of* $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ *associated to* J. Recall that $\mathrm{Spec}_J(U_q^+(\mathfrak{g})) := \{P \in \mathrm{Spec}(U_q^+(\mathfrak{g})) \mid \bigcap_{h\in\mathcal{H}} h.P = J\}$. Then the $\mathcal{H}$-strata $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ ($J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$) form a partition of $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ (see [5, II.2]):

$$\mathrm{Spec}(U_q^+(\mathfrak{g})) = \bigsqcup_{J\in\mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))} \mathrm{Spec}_J(U_q^+(\mathfrak{g})).$$

Naturally, this partition induces a partition of the set $\mathrm{Prim}(U_q^+(\mathfrak{g}))$ of all (left) primitive ideals of $U_q^+(\mathfrak{g})$ as follows. For all $J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$, we set $\mathrm{Prim}_J(U_q^+(\mathfrak{g})) := \mathrm{Spec}_J(U_q^+(\mathfrak{g})) \cap \mathrm{Prim}(U_q^+(\mathfrak{g}))$. Then it is obvious that the $\mathcal{H}$-strata $\mathrm{Prim}_J(U_q^+(\mathfrak{g}))$ ($J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$) form a partition of $\mathrm{Prim}(U_q^+(\mathfrak{g}))$:

$$\mathrm{Prim}(U_q^+(\mathfrak{g})) = \bigsqcup_{J\in\mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))} \mathrm{Prim}_J(U_q^+(\mathfrak{g})).$$

More interestingly, because of the finiteness of the set of $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{g})$, the $\mathcal{H}$-stratification theory provides a useful tool to recognise primitive ideals without having to find all its irreductible representations! Indeed, following previous works of Hodges-Levasseur, Joseph, and Brown-Goodearl, Goodearl and Letzter have characterised the primitive ideals of $U_q^+(\mathfrak{g})$ as follows, see [11, Corollary 2.7] or [5, Theorem II.8.4].

Theorem 1.3 (Goodearl-Letzter)**.** *$\mathrm{Prim}_J(U_q^+(\mathfrak{g}))$ ($J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$) coincides with those primes in $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ that are maximal in $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$.*

2 Automorphism group of $U_q^+(\mathfrak{g})$

In this section, we investigate the automorphism group of $U_q^+(\mathfrak{g})$ viewed as the algebra generated by n indeterminates $E_1, \dots, E_n$ subject to the quantum Serre relations. This algebra has some well-identified automorphisms. First, there are the so-called torus automorphisms; let $\mathcal{H} = (\mathbb{C}^*)^n$, where n still denotes the rank of $\mathfrak{g}$. As $U_q^+(\mathfrak{g})$ is the $\mathbb{C}$-algebra generated by n indeterminates subject to the quantum Serre relations, it is easy to check that each $\bar{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathcal{H}$ determines an algebra automorphism $\phi_{\bar{\lambda}}$ of $U_q^+(\mathfrak{g})$ with $\phi_{\bar{\lambda}}(E_i) = \lambda_i E_i$ for $i \in \{1, \dots, n\}$, with inverse $\phi_{\bar{\lambda}}^{-1} = \phi_{\bar{\lambda}^{-1}}$. Next, there are the so-called diagram automorphisms coming from the symmetries of the Dynkin diagram of $\mathfrak{g}$. Namely, let w be an automorphism of the Dynkin diagram of $\mathfrak{g}$, that is, w is an element of the symmetric group S_n such that $(\alpha_i, \alpha_j) = (\alpha_{w(i)}, \alpha_{w(j)})$ for all $i, j \in \{1, \dots, n\}$. Then one defines an automorphism, also denoted w, of $U_q^+(\mathfrak{g})$ by: $w(E_i) = E_{w(i)}$. Observe that

$$\phi_{\bar{\lambda}} \circ w = w \circ \phi_{(\lambda_{w(1)},\dots,\lambda_{w(n)})}.$$

We denote by G the subgroup of $\mathrm{Aut}(U_q^+(\mathfrak{g}))$ generated by the torus automorphisms and the diagram automorphisms. Observe that

$$G \simeq \mathcal{H} \rtimes \mathrm{autdiagr}(\mathfrak{g}),$$

where $\mathrm{autdiagr}(\mathfrak{g})$ denotes the set of diagram automorphisms of $\mathfrak{g}$.

The group $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ was computed independently by Alev and Dumas, see [2, Proposition 2.3] , and Caldero, see [8, Proposition 4.4]; their results show that, in the case where $\mathfrak{g} = \mathfrak{sl}_3$, we have

$$\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) = G.$$

About ten years later, Andruskiewitsch and Dumas investigated the case where $\mathfrak{g} = \mathfrak{so}_5$, see [4]. In this case, they obtained partial results that lead them to the following conjecture.

Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):

$$\mathrm{Aut}(U_q^+(\mathfrak{g})) = G.$$

This conjecture was recently confirmed in two new cases: $\mathfrak{g} = \mathfrak{so}_5$, [16], and $\mathfrak{g} = \mathfrak{sl}_4$, [18]. Our aim in this section is to show how one can use the action of the automorphism group of $U_q^+(\mathfrak{g})$ on the primitive spectrum of this algebra in order to prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

2.1 Normal elements of $U_q^+(\mathfrak{g})$

Recall that an element a of $U_q^+(\mathfrak{g})$ is normal provided the left and right ideals generated by a in $U_q^+(\mathfrak{g})$ coincide, that is, if

$$aU_q^+(\mathfrak{g}) = U_q^+(\mathfrak{g})a.$$

In the sequel, we will use several times the following well-known result concerning normal elements of $U_q^+(\mathfrak{g})$.

Lemma 2.1. *Let u and v be two nonzero normal elements of $U_q^+(\mathfrak{g})$ such that $\langle u \rangle = \langle v \rangle$. Then there exist $\lambda, \mu \in \mathbb{C}^*$ such that $u = \lambda v$ and $v = \mu u$.*

Proof. It is obvious that units λ, μ exist with these properties. However, the set of units of $U_q^+(\mathfrak{g})$ is precisely $\mathbb{C}^*$. $\square$

2.2 $\mathbb{N}$-grading on $U_q^+(\mathfrak{g})$ and automorphisms

As the quantum Serre relations are homogeneous in the given generators, there is an $\mathbb{N}$-grading on $U_q^+(\mathfrak{g})$ obtained by assigning to E_i degree 1. Let

$$U_q^+(\mathfrak{g}) = \bigoplus_{i \in \mathbb{N}} U_q^+(\mathfrak{g})_i \tag{2}$$

be the corresponding decomposition, with $U_q^+(\mathfrak{g})_i$ the subspace of homogeneous elements of degree i. In particular, $U_q^+(\mathfrak{g})_0 = \mathbb{C}$ and $U_q^+(\mathfrak{g})_1$ is the n-dimensional space spanned by the generators $E_1, \dots, E_n$. For $t \in \mathbb{N}$ set $U_q^+(\mathfrak{g})_{\geq t} = \bigoplus_{i \geq t} U_q^+(\mathfrak{g})_i$ and define $U_q^+(\mathfrak{g})_{\leq t}$ similarly.

We say that the nonzero element $u \in U_q^+(\mathfrak{g})$ has degree t, and write $\deg(u) = t$, if $u \in U_q^+(\mathfrak{g})_{\leq t} \setminus U_q^+(\mathfrak{g})_{\leq t-1}$ (using the convention that $U_q^+(\mathfrak{g})_{\leq -1} = \{0\}$). As $U_q^+(\mathfrak{g})$ is a domain, $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$.

Definition 2.2. *Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ which is generated as an algebra by $A_1 = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. If for each $i \in \{1, \dots, n\}$ there exist $0 \neq a \in A$ and a scalar $q_{i,a} \neq 1$ such that $x_i a = q_{i,a} a x_i$, then we say that A is an $\mathbb{N}$-graded algebra with enough q-commutation relations.*

The algebra $U_q^+(\mathfrak{g})$, endowed with the grading just defined, is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations. Indeed, if $i \in \{1, \dots, n\}$, then there exists $u \in U_q^+(\mathfrak{g})$ such that $E_i u = q^{\bullet} u E_i$ where $\bullet$ is a nonzero integer. This can be proved as follows. As $\mathfrak{g}$ is simple, there exists an index $j \in \{1, \dots, n\}$ such that $j \neq i$ and $a_{ij} \neq 0$, that is, $a_{ij} \in \{-1, -2, -3\}$. Then $s_i s_j$ is a reduced expression in W, so that one can find a reduced expression of w_0 starting with $s_i s_j$, that is, one can write

$$w_0 = s_i s_j s_{i_3} \dots s_{i_N}.$$

With respect to this reduced expression of w_0, we have with the notation of Section 1.2:

$$\beta_1 = \alpha_i \quad \text{and} \quad \beta_2 = s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

Then it follows from Theorem 1.1 that $E_{\beta_1} = E_i$, $E_{\beta_2} = E_{\alpha_j - a_{ij}\alpha_i}$ and

$$E_i E_{\beta_2} = q^{(\alpha_i, \alpha_j - a_{ij}\alpha_i)} E_{\beta_2} E_i,$$

that is,

$$E_i E_{\beta_2} = q^{-(\alpha_i, \alpha_j)} E_{\beta_2} E_i.$$

As $a_{ij} \neq 0$, we have $(\alpha_i, \alpha_j) \neq 0$ and so $q^{-(\alpha_i, \alpha_j)} \neq 1$ since q is not a root of unity. So we have just proved:

Proposition 2.3. *$U_q^+(\mathfrak{g})$ is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations.*

One of the advantages of $\mathbb{N}$-graded algebras with enough q-commutation relations is that any automorphism of such an algebra must conserve the valuation associated to the $\mathbb{N}$-graduation. More precisely, as $U_q^+(\mathfrak{g})$ is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations, we deduce from [18] (see also [17, Proposition 3.2]) the following result.

Corollary 2.4. *Let $\sigma \in \mathrm{Aut}(U_q^+(\mathfrak{g}))$ and $x \in U_q^+(\mathfrak{g})_d \setminus \{0\}$. Then $\sigma(x) = y_d + y_{>d}$, for some $y_d \in U_q^+(\mathfrak{g})_d \setminus \{0\}$ and $y_{>d} \in U_q^+(\mathfrak{g})_{\geq d+1}$.*

2.3 The case where $\mathfrak{g} = \mathfrak{sl}_3$

In this section, we investigate the automorphism group of $U_q^+(\mathfrak{g})$ in the case where $\mathfrak{g} = \mathfrak{sl}_3$. In this case the Cartan matrix is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so that $U_q^+(\mathfrak{s}l_3)$ is the $\mathbb{C}$-algebra generated by two indeterminates E_1 and E_2 subject to the following relations:

$$E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2 = 0 \tag{3}$$

$$E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2 = 0 \tag{4}$$

We often refer to this algebra as the quantum Heisenberg algebra, and sometimes we denote it by $\mathbb{H}$, as in the classical situation the enveloping algebra of $\mathfrak{sl}_3^+$ is the so-called Heisenberg algebra.

We now make explicit a PBW basis of $\mathbb{H}$. The Weyl group of $\mathfrak{sl}_3$ is isomorphic to the symmetric group S_3, where s_1 is identified with the transposition $(1\ 2)$ and s_2 is identified with $(2\ 3)$. Its longest element is then $w_0 = (13)$; it has two reduced decompositions: $w_0 = s_1s_2s_1 = s_2s_1s_2$. Let us choose the reduced decomposition $s_1s_2s_1$ of w_0 in order to construct a PBW basis of $U_q^+(\mathfrak{s}l_3)$. According to Section 1.2, this reduced decomposition leads to the following root vectors:

$$E_{\alpha_1} = E_1,\ E_{\alpha_1+\alpha_2} = T_1(E_2) = -E_1E_2 + q^{-1}E_2E_1 \text{ and } E_{\alpha_2} = T_1T_2(E_1) = E_2.$$

In order to simplify the notation, we set $E_3 := -E_1E_2 + q^{-1}E_2E_1$. Then, it follows from Theorem 1.1 that

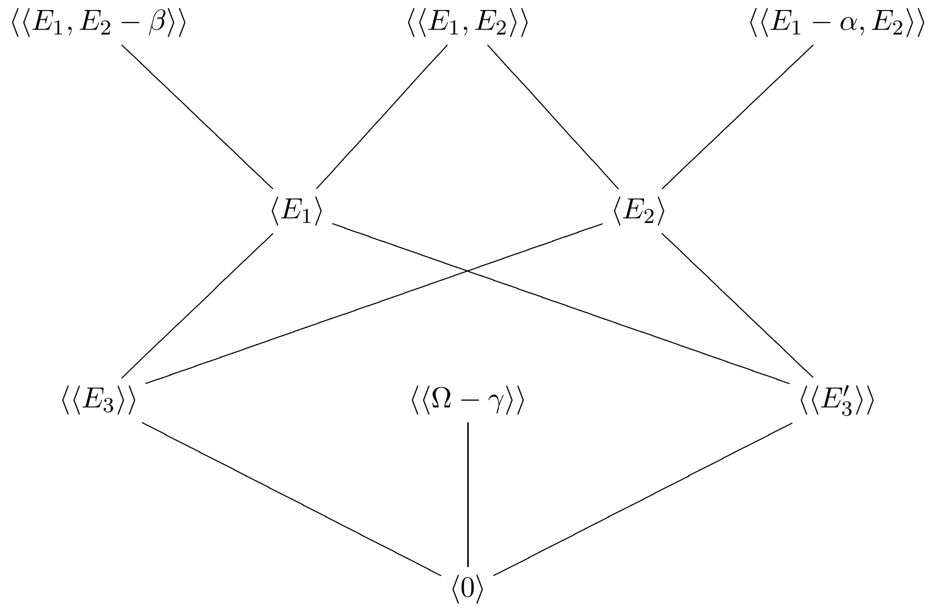
- The monomials $E_1^{k_1}E_3^{k_3}E_2^{k_2}$, with k_1, k_2, k_3 nonnegative integers, form a PBW-basis of $U_q^+(\mathfrak{s}l_3)$.
- $\mathbb{H}$ is the iterated Ore extension over $\mathbb{C}$ generated by the indeterminates E_1, E_3, E_2 subject to the following relations:

  $$E_3E_1 = q^{-1}E_1E_3, \quad E_2E_3 = q^{-1}E_3E_2, \quad E_2E_1 = qE_1E_2 + qE_3.$$

  In particular, $\mathbb{H}$ is a Noetherian domain, and its group of invertible elements is reduced to $\mathbb{C}^*$.
- It follows from the previous commutation relations between the root vectors that E_3 is a normal element in $\mathbb{H}$, that is, $E_3\mathbb{H} = \mathbb{H}E_3$.

In order to describe the prime and primitive spectra of $\mathbb{H}$, we need to introduce two other elements. The first one is the root vector $E_3' := T_2(E_1) = -E_2E_1 + q^{-1}E_1E_2$. This root vector would have appeared if we have choosen the reduced decomposition $s_2s_1s_2$ of w_0 in order to construct a PBW basis of $\mathbb{H}$. It follows from Theorem 1.1 that E_3' q-commutes with E_1 and E_2, so that E_3' is also a normal element of $\mathbb{H}$. Moreover, one can describe the centre of $\mathbb{H}$ using the two normal elements E_3 and E_3'. Indeed, in [3, Corollaire 2.16], Alev and Dumas have described the centre of $U_q^+(\mathfrak{s}l_n)$; independently Caldero has described the centre of $U_q^+(\mathfrak{g})$ for arbitrary $\mathfrak{g}$, see [7]. In our particular situation, their results show that the centre $Z(\mathbb{H})$ of $\mathbb{H}$ is a polynomial ring in one variable $Z(\mathbb{H}) = \mathbb{C}[\Omega]$, where $\Omega = E_3E_3'$.

We are now in position to describe the prime and primitive spectra of $\mathbb{H} = U_q^+(sl(3))$; this was first achieved by Malliavin who obtained the following picture for the poset of prime ideals of $\mathbb{H}$, see [20, Théorème 2.4]:



where $\alpha, \beta, \gamma \in \mathbb{C}^*$.

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts on $U_q^+(\mathfrak{s}l_3)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(\mathfrak{s}l_3)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{s}l_3)$. Using this description of $\mathrm{Spec}(U_q^+(\mathfrak{s}l_3))$, it is easy to identify the $6 = |W|$ $\mathcal{H}$-invariant prime ideals of $\mathbb{H}$ and their corresponding $\mathcal{H}$-strata. As E_1, E_2, E_3 and E_3' are $\mathcal{H}$-eigenvectors, the 6 $\mathcal{H}$-invariant primes are:

$$\langle 0\rangle,\ \langle E_3\rangle,\ \langle E_3'\rangle,\ \langle E_1\rangle,\ \langle E_2\rangle \text{ and } \langle E_1, E_2\rangle.$$

Moreover the corresponding $\mathcal{H}$-strata are:
$\mathrm{Spec}_{\langle 0\rangle}(\mathbb{H}) = \{\langle 0\rangle\} \cup \{\langle \Omega - \gamma\rangle \mid \gamma \in \mathbb{C}^*\}$,
$\mathrm{Spec}_{\langle E_3\rangle}(\mathbb{H}) = \{\langle E_3\rangle\}$,
$\mathrm{Spec}_{\langle E_3'\rangle}(\mathbb{H}) = \{\langle E_3'\rangle\}$,
$\mathrm{Spec}_{\langle E_1\rangle}(\mathbb{H}) = \{\langle E_1\rangle\} \cup \{\langle E_1, E_2 - \beta\rangle \mid \beta \in \mathbb{C}^*\}$,
$\mathrm{Spec}_{\langle E_2\rangle}(\mathbb{H}) = \{\langle E_2\rangle\} \cup \{\langle E_1 - \alpha, E_2\rangle \mid \alpha \in \mathbb{C}^*\}$
and $\mathrm{Spec}_{\langle E_1, E_2\rangle}(\mathbb{H}) = \{\langle E_1, E_2\rangle\}$.

We deduce from this description of the $\mathcal{H}$-strata and the the fact that primitive ideals are exactly those primes that are maximal within their $\mathcal{H}$-strata, see Theorem 1.3, that the primitive ideals of $U_q^+(\mathfrak{sl}_3)$ are exactly those primes that appear in double brackets in the previous picture.

We now investigate the group of automorphisms of $\mathbb{H} = U_q^+(\mathfrak{sl}_3)$. In that case, the torus acting naturally on $U_q^+(\mathfrak{sl}_3)$ is $\mathcal{H} = (\mathbb{C}^*)^2$, there is only one non-trivial diagram automorphism w that exchanges E_1 and E_2, and so the subgroup G of $\mathrm{Aut}(U_q^+(\mathfrak{sl}_3))$ generated by the torus and diagram automorphisms is isomorphic to the semi-direct product $(\mathbb{C}^*)^2 \rtimes S_2$. We want to prove that $\mathrm{Aut}(U_q^+(\mathfrak{sl}_3)) = G$.

In order to do this, we study the action of $\mathrm{Aut}(U_q^+(\mathfrak{sl}_3))$ on the set of primitive ideals that are not maximal. As there are only two of them, $\langle E_3 \rangle$ and $\langle E_3' \rangle$, an automorphism of $\mathbb{H}$ will either fix them or permute them.

Let σ be an automorphism of $U_q^+(\mathfrak{sl}_3)$. It follows from the previous observation that

$$\text{either } \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } \sigma(\langle E_3' \rangle) = \langle E_3' \rangle,$$

$$\text{or } \sigma(\langle E_3 \rangle) = \langle E_3' \rangle \text{ and } \sigma(\langle E_3' \rangle) = \langle E_3 \rangle.$$

As it is clear that the diagram automorphism w permutes the ideals $\langle E_3 \rangle$ and $\langle E_3' \rangle$, we get that there exists an automorphism $g \in G$ such that

$$g \circ \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } g \circ \sigma(\langle E_3' \rangle) = \langle E_3' \rangle.$$

Then, as E_3 and E_3' are normal, we deduce from Lemma 2.1 that there exist $\lambda, \lambda' \in \mathbb{C}^*$ such that

$$g \circ \sigma(E_3) = \lambda E_3 \text{ and } g \circ \sigma(E_3') = \lambda' E_3'.$$

In order to prove that $g \circ \sigma$ is an element of G, we now use the $\mathbb{N}$-graduation of $U_q^+(\mathfrak{sl}_3)$ introduced in Section 2.2. With respect to this graduation, E_1 and E_2 are homogeneous of degree 1, and so E_3 and E_3' are homogeneous of degree 2. Moreover, as $(q^{-2} - 1)E_1E_2 = E_3 + q^{-1}E_3'$, we deduce from the above discussion that

$$g \circ \sigma(E_1E_2) = \frac{1}{q^{-2} - 1} \left(\lambda E_3 + q^{-1}\lambda' E_3'\right)$$

has degree two. On the other hand, as $U_q^+(\mathfrak{sl}_3)$ is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations by Proposition 2.3, it follows from Corollary 2.4 that $\sigma(E_1) = a_1E_1 + a_2E_2 + u$ and $\sigma(E_2) = b_1E_1 + b_2E_2 + v$, where $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, and $u, v \in U_q^+(\mathfrak{sl}_3)$ are linear combinations of homogeneous elements of degree greater than one. As $g \circ \sigma(E_1).g \circ \sigma(E_2)$ has degree two, it is clear that $u = v = 0$. To conclude that $g \circ \sigma \in G$, it just remains to prove that $a_2 = 0 = b_1$. This can be easily shown by using the fact that $g \circ \sigma(-E_1E_2 + q^{-1}E_2E_1) = g \circ \sigma(E_3) = \lambda E_3$; replacing $g \circ \sigma(E_1)$ and $g \circ \sigma(E_2)$ by $a_1E_1 + a_2E_2$ and $b_1E_1 + b_2E_2$ respectively, and then identifying the coefficients in the PBW basis, leads to

$a_2 = 0 = b_1$, as required. Hence we have just proved that $g \circ \sigma \in G$, so that σ itself belongs to G the subgroup of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ generated by the torus and diagram automorphisms. Hence one can state the following result that confirms the Andruskiewitsch-Dumas Conjecture.

Proposition 2.5. $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) \simeq (\mathbb{C}^*)^2 \rtimes \mathrm{autdiagr}(\mathfrak{sl}_3)$

This result was first obtained independently by Alev and Dumas, [2, Proposition 2.3], and Caldero, [8, Proposition 4.4], but using somehow different methods; they studied this automorphism group by looking at its action on the set of normal elements of $U_q^+(\mathfrak{s}l_3)$.

2.4 The case where $\mathfrak{g} = \mathfrak{so}_5$

In this section we investigate the automorphism group of $U_q^+(\mathfrak{g})$ in the case where $\mathfrak{g} = \mathfrak{so}_5$. In this case there are no diagram automorphisms, so that the Andruskiewitsch-Dumas Conjecture asks whether every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism. In [16] we have proved their conjecture when $\mathfrak{g} = \mathfrak{so}_5$. The aim of this section is to present a slightly different proof based both on the original proof and on the recent proof by S.A. Lopes and the author of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g}$ is of type A_3.

In the case where $\mathfrak{g} = \mathfrak{so}_5$, the Cartan matrix is $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, so that $U_q^+(\mathfrak{so}_5)$ is the $\mathbb{C}$-algebra generated by two indeterminates E_1 and E_2 subject to the following relations:

$$E_1^3E_2 - (q^2+1+q^{-2})E_1^2E_2E_1 + (q^2+1+q^{-2})E_1E_2E_1^2 + E_2E_1^3 = 0 \tag{5}$$
$$E_2^2E_1 - (q^2+q^{-2})E_2E_1E_2 + E_1E_2^2 = 0 \tag{6}$$

We now make explicit a PBW basis of $U_q^+(\mathfrak{so}_5)$. The Weyl group of $\mathfrak{so}_5$ is isomorphic to the dihedral group $D(4)$. Its longest element is $w_0 = -id$; it has two reduced decompositions: $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$. Let us choose the reduced decomposition $s_1s_2s_1s_2$ of w_0 in order to construct a PBW basis of $U_q^+(\mathfrak{so}_5)$. According to Section 1.2, this reduced decomposition leads to the following root vectors:

$$E_{\alpha_1} = E_1,\ E_{2\alpha_1+\alpha_2} = T_1(E_2) = \frac{1}{(q+q^{-1})}\left(E_1^2E_2 - q^{-1}(q+q^{-1})E_1E_2E_1 + q^{-2}E_2E_1^2\right),$$

$$E_{\alpha_1+\alpha_2} = T_1T_2(E_1) = -E_1E_2 + q^{-2}E_2E_1 \text{ and } E_{\alpha_2} = T_1T_2T_1(E_2) = E_2.$$

In order to simplify the notation, we set $E_3 := -E_{\alpha_1+\alpha_2}$ and $E_4 := E_{2\alpha_1+\alpha_2}$. Then, it follows from Theorem 1.1 that

- The monomials $E_1^{k_1}E_4^{k_4}E_3^{k_3}E_2^{k_2}$, with k_1, k_2, k_3, k_4 nonnegative integers, form a PBW-basis of $U_q^+(\mathfrak{so}_5)$.

- $U_q^+(\mathfrak{so}_5)$ is the iterated Ore extension over $\mathbb{C}$ generated by the indeterminates E_1, E_4, E_3, E_2 subject to the following relations:

$$\begin{array}{lll} E_4E_1 = q^{-2}E_1E_4 & & \\ E_3E_1 = E_1E_3 - (q+q^{-1})E_4, & E_3E_4 = q^{-2}E_4E_3, & \\ E_2E_1 = q^2E_1E_2 - q^2E_3, & E_2E_4 = E_4E_2 - \frac{q^2-1}{q+q^{-1}}E_3^2, & E_2E_3 = q^{-2}E_3E_2. \end{array}$$

  In particular, $U_q^+(\mathfrak{so}_5)$ is a Noetherian domain, and its group of invertible elements is reduced to $\mathbb{C}^*$.

Before describing the automorphism group of $U_q^+(\mathfrak{so}_5)$, we first describe the centre and the primitive ideals of $U_q^+(\mathfrak{so}_5)$. The centre of $U_q^+(\mathfrak{g})$ has been described in general by Caldero, [7]. In the case where $\mathfrak{g} = \mathfrak{so}_5$, his result shows that $Z(U_q^+(\mathfrak{so}_5))$ is a polynomial algebra in two indeterminates

$$Z(U_q^+(\mathfrak{so}_5)) = \mathbb{C}[z, z'],$$

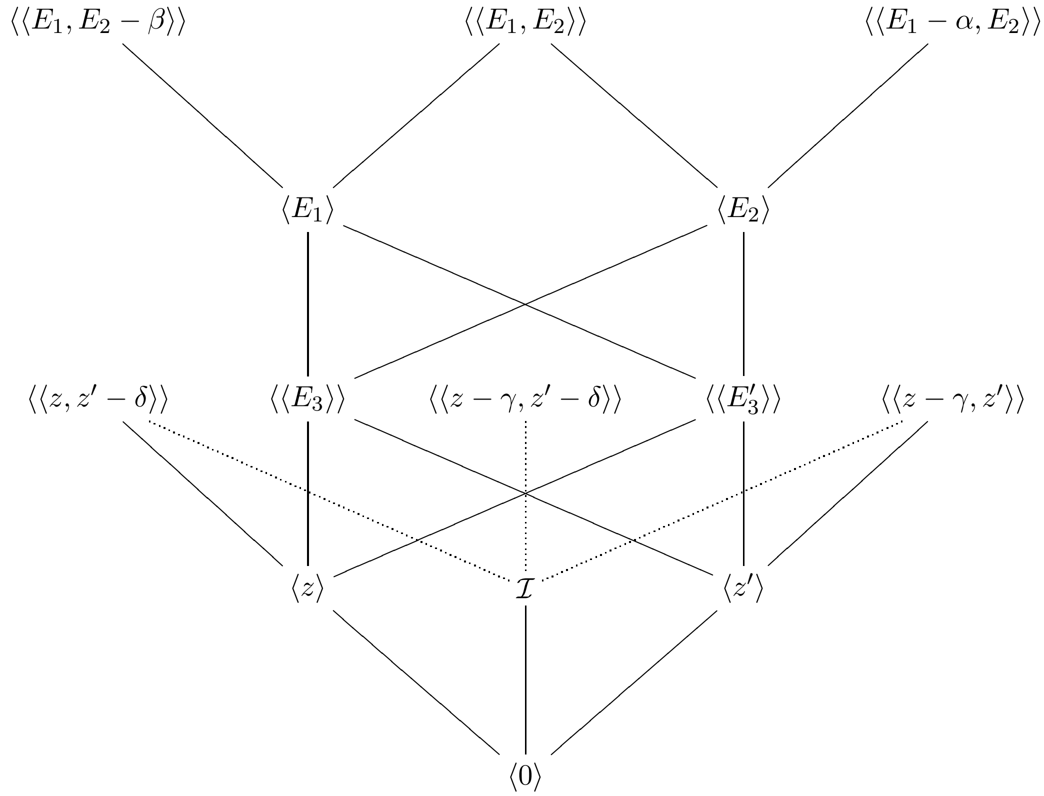
where

$$z = (1-q^2)E_1E_3 + q^2(q+q^{-1})E_4$$

and

$$z' = -(q^2-q^{-2})(q+q^{-1})E_4E_2 + q^2(q^2-1)E_3^2.$$

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts on $U_q^+(\mathfrak{so}_5)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(\mathfrak{so}_5)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $8 = |W|$ $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{so}_5)$. In [16], we have described these eight $\mathcal{H}$-strata. More precisely, we have obtained the following picture for the poset $\mathrm{Spec}(U_q^+(\mathfrak{so}_5))$,



where $\alpha, \beta, \gamma, \delta \in \mathbb{C}^*$, $E_3' := E_1E_2 - q^2E_2E_1$ and

$$\mathcal{I} = \{\langle P(z, z')\rangle \mid P \text{ is a unitary irreductible polynomial of } \mathbb{C}[z, z'],\ P \neq z, z'\}.$$

As the primitive ideals are those primes that are maximal in their $\mathcal{H}$-strata, see Theorem 1.3, we deduced from this description of the prime spectrum that the primitive ideals of $U_q^+(\mathfrak{so}_5)$ are the following:

- $\langle z - \alpha, z' - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
- $\langle E_3\rangle$ and $\langle E_3'\rangle$.
- $\langle E_1 - \alpha, E_2 - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\beta = 0$.

(They correspond to the "double brackets" prime ideals in the above picture.)

Among them, two only are not maximal, $\langle E_3\rangle$ and $\langle E_3'\rangle$. Unfortunately, as E_3 and E_3' are not normal in $U_q^+(\mathfrak{so}_5)$, one cannot easily obtain information using the fact that any automorphism of $U_q^+(\mathfrak{so}_5)$ will either preserve or exchange these two prime ideals. Rather than using this observation, we will use the action of $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ on the set of maximal ideals of height two. Because of the previous description of the primitive spectrum of $U_q^+(\mathfrak{so}_5)$, the height two maximal ideals in $U_q^+(\mathfrak{so}_5)$ are those $\langle z - \alpha, z' - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In [16, Proposition 3.6], we have proved that the group of units of the factor algebra $U_q^+(\mathfrak{so}_5)/\langle z - \alpha, z' - \beta\rangle$ is reduced to $\mathbb{C}^*$ if and only if both α and β are nonzero. Consequently, if σ is an automorphism of $U_q^+(\mathfrak{so}_5)$ and $\alpha \in \mathbb{C}^*$, we get that:

$$\sigma(\langle z-\alpha, z'\rangle) = \langle z-\alpha', z'\rangle \text{ or } \langle z, z'-\beta'\rangle,$$

where $\alpha', \beta' \in \mathbb{C}^*$. Similarly, if σ is an automorphism of $U_q^+(\mathfrak{so}_5)$ and $\beta \in \mathbb{C}^*$, we get that:

$$\sigma(\langle z, z'-\beta\rangle) = \langle z-\alpha', z'\rangle \text{ or } \langle z, z'-\beta'\rangle, \tag{7}$$

where $\alpha', \beta' \in \mathbb{C}^*$.

We now use this information to prove that the action of $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ on the centre of $U_q^+(\mathfrak{so}_5)$ is trivial. More precisely, we are now in position to prove the following result.

Proposition 2.6. *Let $\sigma \in \mathrm{Aut}(U_q^+(\mathfrak{so}_5))$. There exist $\lambda, \lambda' \in \mathbb{C}^*$ such that*

$$\sigma(z) = \lambda z \quad \textit{and} \quad \sigma(z') = \lambda' z'.$$

Proof. We only prove the result for z. First, using the fact that $U_q^+(\mathfrak{so}_5)$ is noetherian, it is easy to show that, for any family $\{\beta_i\}_{i\in\mathbb{N}}$ of pairwise distinct nonzero complex numbers, we have:

$$\langle z\rangle = \bigcap_{i\in\mathbb{N}} P_{0,\beta_i} \text{ and } \langle z'\rangle = \bigcap_{i\in\mathbb{N}} P_{\beta_i,0},$$

where $P_{\alpha,\beta} := \langle z-\alpha, z'-\beta\rangle$. Indeed, if the inclusion

$$\langle z\rangle \subseteq I := \bigcap_{i\in\mathbb{N}} P_{0,\beta_i}$$

is not an equality, then any P_{0,β_i} is a minimal prime over I for height reasons. As the P_{0,β_i} are pairwise distinct, I is a two-sided ideal of $U_q^+(\mathfrak{so}_5)$ with infinitely many prime ideals minimal over it. This contradicts the noetherianity of $U_q^+(\mathfrak{so}_5)$. Hence

$$\langle z\rangle = \bigcap_{i\in\mathbb{N}} P_{0,\beta_i} \text{ and } \langle z'\rangle = \bigcap_{i\in\mathbb{N}} P_{\beta_i,0},$$

and so

$$\sigma(\langle z\rangle) = \bigcap_{i\in\mathbb{N}} \sigma(P_{0,\beta_i}).$$

It follows from (7) that, for all $i \in \mathbb{N}$, there exists $(\gamma_i, \delta_i) \neq (0,0)$ with $\gamma_i = 0$ or $\delta_i = 0$ such that

$$\sigma(P_{0,\beta_i}) = P_{\gamma_i,\delta_i}.$$

Naturally, we can choose the family $\{\beta_i\}_{i\in\mathbb{N}}$ such that either $\gamma_i = 0$ for all $i \in \mathbb{N}$, or $\delta_i = 0$ for all $i \in \mathbb{N}$. Moreover, observe that, as the β_i are pairwise distinct, so are the γ_i or the δ_i.

Hence, either

$$\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} P_{\gamma_i,0},$$

or

$$\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} P_{0,\delta_i},$$

that is,

$$\text{either } \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z' \rangle \text{ or } \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z \rangle.$$

As z, $\sigma(z)$ and z' are all central, it follows from Lemma 2.1 that there exists $\lambda \in \mathbb{C}^*$ such that either $\sigma(z) = \lambda z$ or $\sigma(z) = \lambda z'$.

To conclude, it just remains to show that the second case cannot happen. In order to do this, we use a graded argument. Observe that, with respect to the $\mathbb{N}$-graduation of $U_q^+(\mathfrak{so}_5)$ defined in Section 2.2, z and z' are homogeneous of degree 3 and 4 respectively. Thus, if $\sigma(z) = \lambda z'$, then we would obtain a contradiction with the fact that every automorphism of $U_q^+(\mathfrak{so}_5)$ preserves the valuation, see Corollary 2.4. Hence $\sigma(z) = \lambda z$, as desired. The corresponding result for z' can be proved in a similar way, so we omit it. $\square$

Andruskiewitsch and Dumas, [4, Proposition 3.3], have proved that the subgroup of those automorphisms of $U_q^+(\mathfrak{so}_5)$ that stabilize $\langle z \rangle$ is isomorphic to $(\mathbb{C}^*)^2$. Thus, as we have just shown that every automorphism of $U_q^+(\mathfrak{so}_5)$ fixes $\langle z \rangle$, we get that $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ itself is isomorphic to $(\mathbb{C}^*)^2$. This is the route that we have followed in [16] in order to prove the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$. Recently, with Samuel Lopes, we proved this Conjecture in the case where $\mathfrak{g} = \mathfrak{sl}_4$ using different methods and in particular graded arguments. We are now using (similar) graded arguments to prove that every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism (witout using results of Andruskiewitsch and Dumas).

In the proof, we will need the following relation that is easily obtained by straightforward computations.

Lemma 2.7. $(q^2-1)E_3E_3' = (q^4-1)zE_2 + q^2z'$.

Proposition 2.8. *Let σ be an automorphism of $U_q^+(\mathfrak{so}_5)$. Then there exist $a_1, b_2 \in \mathbb{C}^*$ such that*

$$\sigma(E_1) = a_1E_1 \quad \textit{and} \quad \sigma(E_2) = b_2E_2.$$

Proof. For all $i \in \{1, \ldots, 4\}$, we set $d_i := \deg(\sigma(E_i))$. We also set $d_3' := \deg(\sigma(E_3'))$. It follows from Corollary 2.4 that $d_1, d_2 \geq 1$, $d_3, d_3' \geq 2$ and $d_4 \geq 3$. First we prove that $d_1 = d_2 = 1$.

Assume first that $d_1 + d_3 > 3$. As $z = (1-q^2)E_1E_3 + q^2(q+q^{-1})E_4$ and $\sigma(z) = \lambda z$ with $\lambda \in \mathbb{C}^*$ by Proposition 2.6, we get:

$$\lambda z = (1-q^2)\sigma(E_1)\sigma(E_3) + q^2(q+q^{-1})\sigma(E_4). \tag{8}$$

Recall that $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$, as $U_q^+(\mathfrak{g})$ is a domain. Thus, as $\deg(z) = 3 < \deg(\sigma(E_1)\sigma(E_3)) = d_1 + d_3$, we deduce from (8) that $d_1 + d_3 = d_4$. As $z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2-1)E_3^2$ and $\deg(z') = 4 < d_1 + d_3 + d_2 = d_4 + d_2 = \deg(\sigma(E_4)\sigma(E_2))$, we get in a similar manner that $d_2 + d_4 = 2d_3$. Thus $d_1 + d_2 = d_3$. As $d_1 + d_3 > 3$, this forces $d_3 > 2$ and so $d_3 + d_3' > 4$. Thus we deduce from Lemma 2.7 that $d_3 + d_3' = 3 + d_2$. Hence $d_1 + d_3' = 3$. As $d_1 \geq 1$ and $d_3' \geq 2$, this implies $d_1 = 1$ and $d_3' = 2$.

Thus we have just proved that $d_1 = \deg(\sigma(E_1)) = 1$ and either $d_3 = 2$ or $d_3' = 2$. To prove that $d_2 = 1$, we distinguish between these two cases.

If $d_3 = 2$, then as previously we deduce from the relation $z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2-1)E_3^2$ that $d_2 + d_4 = 4$, so that $d_2 = 1$, as desired.

If $d_3' = 2$, then one can use the definition of E_3' and the previous expression of z' in order to prove that $z' = q^{-2}(q^2-1)E_3'^2 + E_2u$, where u is a nonzero homogeneous element of $U_q^+(\mathfrak{so}_5)$ of degree 3. (u is nonzero since $\langle z' \rangle$ is a completely prime ideal and $E_3' \notin \langle z' \rangle$ for degree reasons.) As $d_3' = 2$ and $\deg(\sigma(z')) = 4$, we get as previously that $d_2 = 1$.

To summarise, we have just proved that $\deg(\sigma(E_1)) = 1 = \deg(\sigma(E_2))$, so that $\sigma(E_1) = a_1E_1 + a_2E_2$ and $\sigma(E_2) = b_1E_1 + b_2E_2$, where $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. To conclude that $a_2 = b_1 = 0$, one can for instance use the fact that $\sigma(E_1)$ and $\sigma(E_2)$ must satisfy the quantum Serre relations. $\square$

We have just confirmed the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Theorem 2.9. *Every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism, so that*

$$\mathrm{Aut}(U_q^+(\mathfrak{so}_5)) \simeq (\mathbb{C}^*)^2.$$

2.5 Beyond these two cases

To finish this overview paper, let us mention that recently the Andruskiewitsch-Dumas Conjecture was confirmed by Samuel Lopes and the author, [18], in the case where $\mathfrak{g} = \mathfrak{sl}_4$. The crucial step of the proof is to prove that, up to an element of G, every normal element of $U_q^+(\mathfrak{sl}_4)$ is fixed by every automorphism. This step was dealt with by first computing the Lie algebra of derivations of $U_q^+(\mathfrak{sl}_4)$, and this already requires a lot of computations!

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An Artinian theory for Lie algebras

Antonio Fernández López[*] Esther García[†] Miguel Gómez Lozano[‡]

Abstract

We summarize in this contribution the main results of our paper [5]. Only those definitions which are necessary to understand the statements are provided, but no proofs, which can be found in [5].

1 Lie algebras and Jordan pairs

1. Throughout this paper, and at least otherwise specified, we will be dealing with Lie algebras L [7], [11], and Jordan pairs $V = (V^+, V^-)$ [8], over a ring of scalars Φ containing $\frac{1}{30}$.

2. Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$, and V is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors.

3. Given a Jordan pair $V = (V^+, V^-)$, an *inner ideal* of V is any Φ-submodule B of V^σ such that $\{B, V^{-\sigma}, B\} \subset B$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[B, [B, L]] \subset B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

4. (a) Recall that the *socle* of a nondegenerate Jordan pair V is $\mathrm{Soc}\, V = (\mathrm{Soc}\, V^+, \mathrm{Soc}\, V^-)$ where $\mathrm{Soc}\, V^\sigma$ is the sum of all minimal inner ideals of V contained in V^σ [9]. The *socle* of a nondegenerate Lie algebra L is $\mathrm{Soc}\, L$, defined as the sum of all minimal inner ideals of L [3].

[*]Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain. Email: `emalfer@agt.cie.uma.es`. Partially supported by the MEC and Fondos FEDER, MTM2004-03845, and by Junta de Andalucía FQM264.

[†]Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles. Madrid, Spain. Email: `esther.garcia@urjc.es`. Partially supported by the MEC and Fondos FEDER, MTM2004-06580-C02-01, and by FICYT-IB05-017.

[‡]Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain. Email: `magomez@agt.cie.uma.es`. Partially supported by the MEC and Fondos FEDER, MTM2004-03845, by Junta de Andalucía FQM264, and by FICYT-IB05-017.

(b) By [9, Theorem 2] (for the Jordan pair case) and [3, Theorem 2.5] (for the Lie case), the socle of a nondegenerate Jordan pair or Lie algebra is the direct sum of its simple ideals. Moreover, each simple component of $\operatorname{Soc} L$ is either inner simple or contains an abelian minimal inner ideal [2, Theorem 1.12].

(c) A Lie algebra L or Jordan pair V is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals.

(d) By definition, a properly ascending chain $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n$ of inner ideals of a Lie algebra L has length n. The *length* of an inner ideal M is the supremum of the lengths of chains of inner ideals of L contained in M.

2 Complemented Lie algebras

The module-theoretic characterization of semiprime Artinian rings (R is unital and completely reducible as a left R-module) cannot be translated to Jordan systems by merely replacing left ideals by inner ideals: if we take, for instance, the Jordan algebra $M_2(F)^{(+)}$ of 2×2-matrices over a field F, any nontrivial inner ideal of $M_2(F)^{(+)}$ has dimension 1, so it cannot be complemented as a F-subspace by any other inner ideal. Nevertheless, O. Loos and E. Neher succeeded in getting the appropriate characterization by introducing the notion of kernel of an inner ideal [10]:

A Jordan pair $V = (V^+, V^-)$ (over an arbitrary ring of scalars) is a direct sum of simple Artinian nondegenerate Jordan pairs if and only if it is complemented in the following sense: for any inner ideal B of V^σ there exists an inner ideal C of $V^{-\sigma}$ such that each of them is complemented as a submodule by the kernel of the other. In particular, a simple Jordan pair is complemented if and only if is nondegenerate and Artinian. A similar characterization works for Lie algebras.

1.

Let M be an inner ideal of a Lie algebra L. The *kernel* of M

$$\operatorname{Ker} M = \{x \in L : [M, [M, x]] = 0\}$$

is a Φ-submodule of L. An *inner complement* of M is an inner ideal N of L such that

$$L = M \oplus \operatorname{Ker} N = N \oplus \operatorname{Ker} M.$$

A Lie algebra L will be called *complemented* if any inner ideal of L has an inner complement, and *abelian complemented* if any abelian inner ideal has an inner complement which is abelian. Our main result, which can be regarded as a Lie analogue of the module-theoretic characterization of semiprime Artinian rings, proves.

Theorem 2. *For a Lie algebra L, the following notions are equivalent:*

(i) *L is complemented.*

(ii) *L is a direct sum of ideals each of which is a simple nondegenerate Artinian Lie algebra.*

Moreover,

(iii) *Complemented Lie algebras are abelian complemented.*

A key tool used in the proof of (i) $\Rightarrow$ (ii) is the notion of subquotient of a Lie algebra with respect to an abelian inner ideal:

3.

For any abelian inner ideal M of L, the pair of Φ-modules

$$V = (M, L/\operatorname{Ker}_L M)$$

with the triple products given by

$$\begin{aligned} \{m, \overline{a}, n\} &:= [[m,a],n] \quad \text{for every} \quad m,n \in M \quad \text{and} \quad a \in L \\ \{\overline{a}, m, \overline{b}\} &:= \overline{[[a,m],b]} \quad \text{for every} \quad m \in M \quad \text{and} \quad a,b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\operatorname{Ker}_L M$, is a Jordan pair called the subquotient of L with respect to M.

Subquotients inherit, on the one hand, regularity conditions from the Lie algebra, and, on the other hand, keep the inner ideal structure of L within them. This fact turns out to be crucial for using results of Jordan theory. For instance, it is used to prove that any prime abelian complemented Lie algebra satisfies the ascending and descending chain conditions on abelian inner ideals. Moreover, as proved in [6], an abelian inner ideal B of finite length in a nondegenerate Lie algebra L is not just complemented by abelian inner ideals, but there exists a short grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$, and hence B is complemented by L_{-n}.

It must be noted that, while any nondegenerate Artinian Jordan pair is a direct sum of finitely many simple nondegenerate Artinian Jordan pairs, a nondegenerate Artinian Lie algebra does only have essential socle [3, Corollary 2.6]. In fact, there exist strongly prime finite dimensional Lie algebras (over a field of characteristic $p > 5$) with nontrivial ideals [12, p. 152]. Therefore, unlike the Jordan case, nondegenerate Artinian Lie algebras are not necessarily complemented: they are only abelian complemented.

3 Simple nondegenerate Artinian Lie algebras

Let L be a simple nondegenerate Lie algebra containing an abelian minimal inner ideal. Then L has a 5-grading. Hence, by [13, Theorem 1], L is one of the following: (i) a simple Lie algebra of type G_2, F_4, E_6, E_7 or E_8, (ii) $L = \overline{R}' = [R,R]/Z(R) \cap [R,R]$, where R is a simple associative algebra such that $[R,R]$ is not contained in $Z(R)$, or (iii) $L = \overline{K}' = [K,K]/Z(R) \cap [K,K]$, where $K = \mathrm{Skew}(R,*)$ and R is a simple associative algebra, $*$ is an involution of R, and either $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16. (Actually, the list of simple Lie algebras with gradings given in [13, Theorem 1], contains two additional algebras: the Tits-Kantor-Koecher algebra of a nondegenerate symmetric bilinear form and D_4. However, because of we are not interested in describing the gradings, both algebras can be included in case (iii): $\overline{K}' = K' = K = \mathrm{Skew}(R,*)$, where R is a simple algebra with orthogonal involution.)

By using the inner ideal structure of Lie algebras of traceless operators of finite rank which are continuous with respect to an infinite dimensional pair of dual vector spaces over a division algebra, and that of the Lie algebras of finite rank skew operators on an infinite dimensional self dual vector space (extending the work of G. Benkart [1] for the finite dimensional case, and a previous one of the authors [4] for finitary Lie algebras), we can refine the above list in the case of a simple nondegenerate Artinian Lie algebra.

4.

A Lie algebra L will be called a *division Lie algebra* if it is nonzero, nondegenerate and has no nontrivial inner ideals. Two examples are given below:

1. Let Δ be a division associative algebra such that $[[\Delta,\Delta],\Delta] \neq 0$. Then $[\Delta,\Delta]/[\Delta,\Delta] \cap Z(\Delta)$ is a division Lie algebra, [1, Corollary 3.15].

2. Let R be a simple associative algebra with involution $*$ and nonzero socle. Suppose that $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16, and set $K := \mathrm{Skew}(R,*)$. Then $L = [K,K]/[K,K] \cap Z(R)$ is a division Lie algebra if and only if $(R,*)$ has no nonzero *isotropic* right ideals, i.e., those right ideals I such that $I^*I = 0$. This is a direct consequence of the inner ideal structure of L: [1, Theorem 5.5] when R is Artinian, and [4] when R is not Artinian.

Theorem 5. *Let L be a simple Lie algebra over a field F of characteristic 0 or greater than 7. Then L is Artinian and nondegenerate if and only if it is one of the following:*

1. *A division Lie algebra.*

2. *A simple Lie algebra of type G_2, F_4, E_6, E_7 or E_8 containing abelian minimal inner ideals.*

3. *$[R,R]/[R,R] \cap Z(R)$, where R is a simple Artinian (but not division) associative algebra.*

4. *$[K,K]/[K,K] \cap Z(R)$, where $K = \mathrm{Skew}(R,*)$ and R is a simple associative algebra with involution $*$ which coincides with its socle, such that $Z(R) = 0$ or the dimension of*

R over $Z(R)$ is greater than 16, and $(R, *)$ satisfies the descending chain condition on isotropic right ideals.

Summarizing, we can say that, as conjectured by G. Benkart in the introduction of [2], inner ideals in Lie algebras play a role analogous to Jordan inner ideals in the development of an Artinian theory for Lie algebras.

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Quantum groupoids with projection

J.N. Alonso Álvarez* J.M. Fernández Vilaboa† R. González Rodríguez‡

Abstract

In this survey we explain in detail how Radford's ideas and results about Hopf algebras with projection can be generalized to quantum groupoids in a strict symmetric monoidal category with split idempotents.

Introduction

Let H be a Hopf algebra over a field K and let A be a K-algebra. A well-known result of Radford [23] gives equivalent conditions for an object $A \otimes H$ equipped with smash product algebra and coalgebra to be a Hopf algebra and characterizes such objects via bialgebra projections. Majid in [16] interpreted this result in the modern context of Yetter-Drinfeld modules and stated that there is a correspondence between Hopf algebras in this category, denoted by ${}_H^H\mathcal{YD}$, and Hopf algebras B with morphisms of Hopf algebras $f : H \rightarrow B$, $g : B \rightarrow H$ such that $g \circ f = id_H$. Later, Bespalov proved the same result for braided categories with split idempotents in [5]. The key point in Radford-Majid-Bespalov's theorem is to define an object B_H, called the algebra of coinvariants, as the equalizer of $(B \otimes g) \circ \delta_B$ and $B \otimes \eta_H$. This object is a Hopf algebra in the category ${}_H^H\mathcal{YD}$ and there exists a Hopf algebra isomorphism between B and $B_H \bowtie H$ (the smash (co)product of B_H and H). It is important to point out that in the construction of $B_H \bowtie H$ they use that B_H is the image of the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (f \circ \lambda_H \circ g)) \circ \delta_B$.

In [11], Bulacu and Nauwelaerts generalize Radford's theorem about Hopf algebras with projection to the quasi-Hopf algebra setting. Namely, if H and B are quasi-Hopf algebras with bijective antipode and with morphisms of quasi-Hopf algebras $f : H \rightarrow B$, $g : B \rightarrow H$ such that $g \circ f = id_H$, then they define a subalgebra B^i (the generalization of B_H to this setting) and with some additional structures B^i becomes, a Hopf algebra in the category of left-left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$ defined by Majid in [17]. Moreover, as the main

*Departamento de Matemáticas, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36280 Vigo, Spain. E-mail: jnalonso@uvigo.es

†Departamento de Álxebra, Universidad de Santiago de Compostela. E-15771 Santiago de Compostela, Spain. E-mail: alvila@usc.es

‡Departamento de Matemática Aplicada II, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36310 Vigo, Spain. E-mail: rgon@dma.uvigo.es

result in [11], Bulacu and Nauwelaerts state that $B^i \times H$ is isomorphic to B as quasi-Hopf algebras where the algebra structure of $B^i \times H$ is the smash product defined in [10] and the quasi-coalgebra structure is the one introduced in [11].

The basic motivation of this survey is to explain in detail how the above ideas and results can be generalized to quantum groupoids in a strict symmetric monoidal category with split idempotents. Quantum groupoids or weak Hopf algebras have been introduced by Böhm, Nill and Szlachányi [7] as a new generalization of Hopf algebras and groupoid algebras. Roughly speaking, a weak Hopf algebra H in a symmetric monoidal category is an object that has both algebra and coalgebra structures with some relations between them and that possesses an antipode λ_H which does not necessarily verify $\lambda_H \wedge id_H = id_H \wedge \lambda_H = \varepsilon_H \otimes \eta_H$ where ε_H, η_H are the counity and unity morphisms respectively and $\wedge$ denotes the usual convolution product. The main differences with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, are the following: weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity η_H or, equivalently, the counity is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras (see [20]).

The survey is organized as follows.

In Section 1 we give basis definitions and examples of quantum groupoids without finiteness conditions. Also we introduce the category of left-left Yetter-Drinfeld modules defined by Böhm for a quantum groupoid with invertible antipode. As in the case of Hopf algebras this category is braided monoidal but in this case is not strict.

The exposition of the theory of crossed products associated to projections of quantum groupoids in Section 2 follows [2] and is the good generalization of the classical theory developed by Blattner, Cohen and Montgomery in [6]. The main theorem in this section generalizes a well know result, due to Blattner, Cohen and Montgomery, which shows that if $B \xrightarrow{\pi} H \to 0$ is an exact sequence of Hopf algebras with coalgebra splitting then $B \approx A\sharp_\sigma H$, where A is the left Hopf kernel of π and σ is a suitable cocycle (see Theorem (4.14) of [6]). In this section we show that if $g : B \to H$ is a morphism of quantum groupoids and there exists a morphism of coalgebras $f : H \to B$ such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$, using the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B$ it is possible to construct an equalizer diagram and an algebra B_H, i.e, the algebra of coinvariants or the Hopf kernel of g, and morphisms $\varphi_{B_H} : H \otimes B_H \to B_H$ (the weak measuring), $\sigma_{B_H} : H \otimes H \to B_H$ (the weak cocycle) such that there exists an idempotent endomorphism of $B_H \otimes H$ which image, denoted by $B_H \times H$, is isomorphic with B as algebras being the algebra structure (crossed product algebra)

$$\eta_{B_H \times H} = r_B \circ (\eta_{B_H} \otimes \eta_H),$$

$$\mu_{B_H \times H} = r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_{H \otimes H}) \circ$$

$$(B_H \otimes H \otimes c_{H,B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),$$

where s_B is the inclusion of $B_H \times H$ in $B_H \otimes H$ and r_B the projection of $B_H \otimes H$ on $B_H \times H$. Of course, when H, B are Hopf algebras we recover the result of Blattner, Cohen and Montgomery. For this reason, we denote the algebra $B_H \times H$ by $B_H \sharp_{\sigma_{B_H}} H$. If moreover f is an algebra morphism, the cocycle is trivial in a weak sense and then we obtain that $\mu_{B_H \times H}$ is the weak version of the smash product used by Radford in the Hopf algebra setting. Also, we prove the dual results using similar arguments but passing to the opposite category, for a morphism of quantum groupoids $h : H \to B$ and an algebra morphism $t : B \to H$ such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$.

Finally, in Section 3, linking the information of section 2 with the results of [1], [2], [3] and [4], we obtain our version of Radford's Theorem for quantum groupoids with projection. In this section we prove that the algebra of coinvariants B_H associated to a quantum groupoid projection (i.e. a pair of morphisms of quantum groupoids $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$) can be obtained as an equalizer or, by duality, as a coequalizer (in this case the classical theory developed in Section 2 and the dual one provide the same object B_H with dual algebraic structures, algebra-coalgebra, module-comodule, etc...). Therefore, it is possible to find an algebra coalgebra structure for B_H and morphisms $\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B) : H \otimes B_H \to B_H$ and $\varrho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B : B_H \to H \otimes B_H$ such that (B_H, φ_{B_H}) is a left H-module and (B_H, ϱ_{B_H}) is a left H-comodule. We show that B_H is a Hopf algebra in the category of left-left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$ and, using the the the weak smash product and the weak smash coproduct of B_H and H we give a good weak Hopf algebra interpretation of the theorems proved by Radford [23] and Majid [16] in the Hopf algebra setting, obtaining an isomorphism of quantum groupoids between $B_H \times H$ and B.

1 Quantum groupoids in monoidal categories

In this section we give definitions and discuss basic properties of quantum groupoids in monoidal categories.

Let $\mathcal{C}$ be a category. We denote the class of objects of $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $X \in |\mathcal{C}|$, the identity morphism by $id_X : X \to X$.

A monoidal category $(\mathcal{C}, \otimes, K, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with an object K, called the unit of the monoidal category, with a natural isomorphism $a : \otimes(id \times \otimes) \to \otimes(\otimes \times id)$, called the associativity constrain, and with natural isomorphisms $l : \otimes(K \times id) \to id$, $r : \otimes(id \times K) \to id$, called left unit constraint and right unit constraint respectively, such that the Pentagon Axiom

$$(a_{U,V,W} \otimes id_X) \circ a_{U,V \otimes W,X} \circ (id_U \otimes a_{V,W,X}) = a_{U \otimes V,W,X} \circ a_{U,V,W \otimes X}$$

and the Triangle Axiom

$$id_V \otimes l_W = (r_V \otimes id_W) \circ a_{V,K,W}$$

are satisfied.

The monoidal category is said to be strict if the associativity and the unit constraints a, l, r are all identities of the category.

Let $\Psi : \mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$ be the flip functor defined by $\Psi(V, W) = (W, V)$ on any pair of objects of $\mathcal{C}$. A commutativity constrain is a natural isomorphism $c : \otimes \to \otimes\Psi$. If $(\mathcal{C}, \otimes, K, a, l, r)$ is a monoidal category, a braiding in $\mathcal{C}$ is a commutativity constraint satisfying the Hexagon Axiom

$$a_{W,U,V} \circ c_{U\otimes V,W} \circ a_{U,V,W} = (c_{U,W} \otimes id_V) \circ a_{U,W,V} \circ (id_U \otimes c_{V,W}),$$
$$a^{-1}_{V,W,U} \circ c_{U,V\otimes W} \circ a^{-1}_{U,V,W} = (id_V \otimes c_{U,W}) \circ a^{-1}_{V,U,W} \circ (c_{U,V} \otimes id_W).$$

A braided monoidal category is a monoidal category with a braiding c. These categories generalizes the classical notion of symmetric monoidal category introduced earlier by category theorists. A braided monoidal category is symmetric if the braiding satisfies $c_{W,V} \circ c_{V,W} = id_{V\otimes W}$ for all $V, W \in |\mathcal{C}|$.

From now on we assume that $\mathcal{C}$ is strict symmetric and admits split idempotents, i.e., for every morphism $\nabla_Y : Y \to Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$ there exist an object Z and morphisms $i_Y : Z \to Y$ and $p_Y : Y \to Z$ such that $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$. There is not loss of generality in assuming the strict character for $\mathcal{C}$ because it is well know that given a monoidal category we can construct a strict monoidal category $\mathcal{C}^{st}$ which is tensor equivalent to $\mathcal{C}$ (see [15] for the details). For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Definition 1.1. An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B is a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

By quantum groupoids or weak Hopf algebras we understand the objects introduced in [7], as a generalization of ordinary Hopf algebras. Here, for the convenience of the reader, we recall the definition of these objects and some relevant results from [7] without proof, thus making our exposition self-contained.

Definition 1.2. A quantum groupoid H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) verifiying:

(a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Note that, in this definition, the conditions (a2), (a3) weaken the conditions of multiplicativity of the counit, and comultiplicativity of the unit that we can find in the Hopf algebra definition. On the other hand, axioms (a4-1), (a4-2) and (a4-3) weaken the properties of the antipode in a Hopf algebra. Therefore, a quantum groupoid is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

1.3. If H is a quantum groupoid in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L (target morphism), Π_H^R (source morphism), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H)$,

$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

it is straightforward to show that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H, \quad \Pi_H^R = \lambda_H \wedge id_H.$$

Moreover, we have that

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R,$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R.$$

Also it is easy to show the formulas

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R,$$

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L.$$

If λ_H is an isomorphism (for example, when H is finite), we have the equalities:

$$\overline{\Pi}_H^L = \mu_H \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H, \quad \overline{\Pi}_H^R = \mu_H \circ (\lambda_H^{-1} \otimes H) \circ c_{H,H} \circ \delta_H.$$

If the antipode of H is an isomorphism, the opposite operator and the coopposite operator produce quantum groupoids from quantum groupoids. In the first one the product μ_H is replaced by the opposite product $\mu_{H^{op}} = \mu_H \circ c_{H,H}$ while in the second the coproduct δ_H is replaced by $\delta_{H^{coop}} = c_{H,H} \circ \delta_H$. In both cases the antipode λ_H is replaced by λ_H^{-1}.

A morphism between quantum groupoids H and B is a morphism $f : H \to B$ which is both algebra and coalgebra morphism. If $f : H \to B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see Proposition 1.4 of [1]).

Examples 1.4. (i) As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of quantum groupoids. Recall that a groupoid G is simply a category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of G will be denoted by G_0 and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by id_x and for a morphism $\sigma : x \to y$ in G_1, we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of σ.

Let G be a groupoid, and R a commutative ring. The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. The algebra RG is a cocommutative quantum groupoid, with coproduct δ_{RG}, counit ε_{RG} and antipode λ_{RG} given by the formulas:

$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

For the quantum groupoid RG the morphisms target and source are respectively,

$$\Pi^L_{RG}(\sigma) = id_{t(\sigma)}, \quad \Pi^R_{RG}(\sigma) = id_{s(\sigma)}$$

and $\lambda_{RG} \circ \lambda_{RG} = id_{RG}$, i.e. the antipode is involutive.

If G_1 is finite, then RG is free of a finite rank as a R-module, hence $GR = (RG)^* = Hom_R(RG, R)$ is a commutative quantum groupoid with involutory antipode. As R-module

$$GR = \bigoplus_{\sigma \in G_1} Rf_\sigma$$

with $\langle f_\sigma, \tau \rangle = \delta_{\sigma,\tau}$. The algebra structure is given by the formulas $f_\sigma f_\tau = \delta_{\sigma,\tau} f_\sigma$ and $1_{GR} = \sum_{\sigma \in G_1} f_\sigma$. The coalgebra structure is

$$\delta_{GR}(f_\sigma) = \sum_{\tau\rho=\sigma} f_\tau \otimes f_\rho = \sum_{\rho \in G_1} f_{\sigma\rho^{-1}} \otimes f_\rho, \quad \varepsilon_{GR}(f_\sigma) = \delta_{\sigma, id_{t(\sigma)}}.$$

The antipode is given by $\lambda_{GR}(f_\sigma) = f_{\sigma^{-1}}$.

(ii) It is known that any group action on a set gives rise to a groupoid (see [24]). In [20] Nikshych and Vainerman extend this construction associating a quantum groupoid with any action of a Hopf algebra on a separable algebra.

(iii) It was shown in [19] that any inclusion of type Π_1 factors with finite index and depth give rise to a quantum groupoid describing the symmetry of this inclusion. In [20] can be found an example of this construction applied to the case of Temperley-Lieb algebras (see [13]).

(iv) In [22] Nill proved that Hayashi's face algebras [14] are examples of quantum groupoids whose counital subalgebras, i.e., the images of Π^L_H and Π^R_H, are commutative. Also, in [22] we can find that Yamanouchi's generalized Kac algebras (see [25]) are exactly C^*-quantum groupoids with involutive antipode.

1.5. Let H be a quantum groupoid. We say that (M, φ_M) is a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_H \otimes M) = id_M$, $\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M)$. Given two left H-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-modules if $\varphi_N \circ (H \otimes f) = f \circ \varphi_M$. We denote the category of right H-modules by ${}_H\mathcal{C}$. In an analogous way we define the category of right H-modules and we denote it by $\mathcal{C}_H$.

If (M, φ_M) and (N, φ_N) are left H-modules we denote by $\varphi_{M\otimes N}$ the morphism $\varphi_{M\otimes N} : H \otimes M \otimes N \to M \otimes N$ defined by

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

We say that (M, ϱ_M) is a left H-comodule if M is an object in $\mathcal{C}$ and $\varrho_M : M \to H \otimes M$ is a morphism in $\mathcal{C}$ satisfying $(\varepsilon_H \otimes M) \circ \varrho_M = id_M$, $(H \otimes \varrho_M) \circ \varrho_M = (\delta_H \otimes M) \circ \varrho_M$. Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if $\varrho_N \circ f = (H \otimes f) \circ \varrho_M$. We denote the category of left H-comodules by ${}^H\mathcal{C}$. Analogously, $\mathcal{C}^H$ denotes the category of right H-comodules.

For two left H-comodules (M, ϱ_M) and (N, ϱ_N), we denote by $\varrho_{M\otimes N}$ the morphism $\varrho_{M\otimes N} : M \otimes N \to H \otimes M \otimes N$ defined by

$$\varrho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

Let (M, φ_M), (N, φ_N) be left H-modules. Then the morphism

$$\nabla_{M\otimes N} = \varphi_{M\otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$$

is idempotent. In this setting we denote by $M \times N$ the image of $\nabla_{M\otimes N}$ and by $p_{M,N} : M \otimes N \to M \times N$, $i_{M,N} : M \times N \to M \otimes N$ the morphisms such that $i_{M,N} \circ p_{M,N} = \nabla_{M\otimes N}$ and $p_{M,N} \circ i_{M,N} = id_{M\times N}$. Using the definition of $\times$ we obtain that the object $M \times N$ is a left H-module with action $\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}) : H \otimes (M \times N) \to M \times N$ (see [20]). Note that, if $f : M \to M'$ and $g : N \to N'$ are morphisms of left H-modules then $(f \otimes g) \circ \nabla_{M\otimes N} = \nabla_{M'\otimes N'} \circ (f \otimes g)$.

In a similar way, if (M, ϱ_M) and (N, ϱ_N) are left H-comodules the morphism

$$\nabla'_{M\otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \varrho_{M\otimes M} : M \otimes N \to M \otimes N$$

is idempotent. We denote by $M \odot N$ the image of $\nabla'_{M\otimes N}$ and by $p'_{M,N} : M \otimes N \to M \odot N$, $i'_{M,N} : M \odot N \to M \otimes N$ the morphisms such that $i'_{M,N} \circ p'_{M,N} = \nabla'_{M\otimes N}$ and $p'_{M,N} \circ i'_{M,N} = id_{M\odot N}$. Using the definition of $\odot$ we obtain that the object $M \odot N$ is a left H-comodule with coaction $\varrho_{M\odot N} = (H \otimes p'_{M,N}) \circ \varrho_{M\otimes N} \circ i'_{M,N} : M \odot N \to H \otimes (M \odot N)$. If $f : M \to M'$ and $g : N \to N'$ are morphisms of left H-comodules then $(f \otimes g) \circ \nabla'_{M\otimes N} = \nabla'_{M'\otimes N'} \circ (f \otimes g)$.

Let (M, φ_M), (N, φ_N), (P, φ_P) be left H-modules. Then the following equalities hold (Lemma 1.7 of [3]):

$$\varphi_{M\otimes N} \circ (H \otimes \nabla_{M\otimes N}) = \varphi_{M\otimes N},$$

$$\nabla_{M\otimes N} \circ \varphi_{M\otimes N} = \varphi_{M\otimes N} = \varphi_{M\otimes N} \circ \nabla_{M\otimes N},$$

$$(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) = (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}),$$

$$(M\otimes i_{N,P})\circ\nabla_{M\otimes(N\times P)}\circ(M\otimes p_{N,P}) = (\nabla_{M\otimes N}\otimes P)\circ(M\otimes\nabla_{N\otimes P}) = (M\otimes\nabla_{N\otimes P})\circ(\nabla_{M\otimes N}\otimes P).$$

Furthermore, by a similar calculus, if (M, ϱ_M), (N, ϱ_N), (P, ϱ_P) be left H-comodules we have

$$(H \otimes \nabla'_{M\otimes N}) \circ \varrho_{M\otimes N} = \varrho_{M\otimes N},$$

$$\varrho_{M\otimes N} \circ \nabla'_{M\otimes N} = \varrho_{M\otimes N} = \nabla'_{M\otimes N} \circ \varrho_{M\otimes N},$$

$$(i'_{M,N}\otimes P)\circ\nabla'_{(M\odot N)\otimes P}\circ(p'_{M,N}\otimes P)=(M\otimes i'_{N,P})\circ\nabla'_{M\otimes(N\odot P)}\circ(M\otimes p'_{N,P}),$$

$$(M\otimes i'_{N,P})\circ\nabla'_{M\otimes(N\odot P)}\circ(M\otimes p'_{N,P})=(\nabla'_{M\otimes N}\otimes P)\circ(M\otimes\nabla'_{N\otimes P})=(M\otimes\nabla'_{N\otimes P})\circ(\nabla'_{M\otimes N}\otimes P).$$

Yetter-Drinfeld modules over finite dimensional weak Hopf algebras over fields have been introduced by Böhm in [9]. It is shown in [9] that the category of finite dimensional Yetter-Drinfeld modules is monoidal and in [18] it is proved that this category is isomorphic to the category of finite dimensional modules over the Drinfeld double. In [12], the results of [18] are generalized, using duality results between entwining structures and smash product structures, and more properties are given.

Definition 1.6. Let H be a weak Hopf algebra. We shall denote by ${}^{H}_{H}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. That is, $M=(M,\varphi_M,\varrho_M)$ is an object in ${}^{H}_{H}\mathcal{YD}$ if (M,φ_M) is a left H-module, (M,ϱ_M) is a left H-comodule and

(b1) $\quad(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ((\varrho_M\circ\varphi_M)\otimes H)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)$

$$=(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M).$$

(b2) $\quad(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ((\delta_H\circ\eta_H)\otimes\varrho_M)=\varrho_M.$

Let M, N in ${}^{H}_{H}\mathcal{YD}$. The morphism $f:M\to N$ is a morphism of left-left Yetter-Drinfeld modules if $f\circ\varphi_M=\varphi_N\circ(H\otimes f)$ and $(H\otimes f)\circ\varrho_M=\varrho_N\circ f$.

Note that if (M,φ_M,ϱ_M) is a left-left Yetter-Drinfeld module then (b2) is equivalent to

(b3) $\quad((\varepsilon_H\circ\mu_H)\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M)=\varphi_M.$

and we have the identity $\varphi_M\circ(\Pi_H^L\otimes M)\circ\varrho_M=id_M$.

The conditions (b1) and (b2) of the last definition can also be restated (see Proposition 2.2 of [12]) in the following way: suppose that $(M,\varphi_M)\in|\,{}_H\mathcal{C}|$ and $(M,\varrho_M)\in|\,{}^H\mathcal{C}|$, then M is a left-left Yetter-Drinfeld module if and only if

$$\varrho_M\circ\varphi_M=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ$$

$$(((\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M))\otimes\lambda_H)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M).$$

Moreover, the following Proposition, proved in [4], guaranties the equality between the morphisms $\nabla_{M\otimes N}$ and $\nabla'_{M\otimes N}$ defined in 1.5 for all $M,N\in|\,{}^{H}_{H}\mathcal{YD}|$.

Proposition 1.7. *Let H be a weak Hopf algebra. Let (M,φ_M,ϱ_M) and (N,φ_N,ϱ_N) be left-left Yetter-Drinfeld modules over H. Then we have the following assertions.*

(i) $\nabla_{M\otimes N}=((\varphi_M\circ(\overline{\Pi}_H^L\otimes M)\circ c_{M,H})\otimes N)\circ(M\otimes\varrho_N).$

(ii) $\nabla'_{M\otimes N}=(M\otimes\varphi_N)\circ(((M\otimes\overline{\Pi}_H^R)\circ c_{H,M}\circ\varrho_M)\otimes N).$

(iii) $\nabla_{M\otimes N} = \nabla'_{M\otimes N}$.

(iv) $\nabla_{M\otimes H} = ((\varphi_M \circ (\overline{\Pi}_H^L \otimes M) \circ c_{M,H}) \otimes H) \circ (M \otimes \delta_H)$.

(v) $\nabla'_{M\otimes H} = (M \otimes \mu_H) \circ (((M \otimes \overline{\Pi}_H^R) \circ c_{H,M} \circ \varrho_M) \otimes H)$.

(vi) $\nabla_{M\otimes H} = \nabla'_{M\otimes H}$.

1.8. It is a well know fact that, if the antipode of a weak Hopf algebra H is invertible, ${}^H_H\mathcal{YD}$ is a non-strict braided monoidal category. In the following lines we give a brief resume of the braided monoidal structure that we can construct in the category ${}^H_H\mathcal{YD}$ (see Proposition 2.7 of [18] for modules over a field K or Theorem 2.6 of [12] for modules over a commutative ring).

For two left-left Yetter-Drinfeld modules $(M, \varphi_M, \varrho_M)$, $(N, \varphi_N, \varrho_N)$ the tensor product is defined as object as the image of $\nabla_{M\otimes N}$ (see 1.5). As a consequence, by (iii) of Proposition 1.7, $M \times N = M \odot N$ and this object is a left-left Yetter-Drinfeld module with the following action and coaction:

$$\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}), \quad \varrho_{M\times N} = (H \otimes p_{M,N}) \circ \varrho_{M\otimes N} \circ i_{M,N}.$$

The base object is $H_L = Im(\Pi_H^L)$ or, equivalently, the equalizer of δ_H and $\zeta_H^1 = (H \otimes \Pi_H^L) \circ \delta_H$ (see (9)) or the equalizer of δ_H and $\zeta_H^2 = (H \otimes \overline{\Pi}_H^R) \circ \delta_H$. The structure of left-left Yetter-Drinfeld module for H_L is the one derived of the following morphisms

$$\varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L), \quad \varrho_{H_L} = (H \otimes p_L) \circ \delta_H \circ i_L.$$

where $p_L : H \to H_L$ and $i_L : H_L \to H$ are the morphism such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$.

The unit constrains are:

$$l_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L,M} : H_L \times M \to M,$$

$$r_M = \varphi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_L)) \circ i_{M,H_L} : M \times H_L \to M.$$

These morphisms are isomorphisms with inverses:

$$l_M^{-1} = p_{H_L,M} \circ (p_L \otimes \varphi_M) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to H_L \times M,$$

$$r_M^{-1} = p_{M,H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to M \times H_L.$$

If M, N, P are objects in the category ${}^H_H\mathcal{YD}$, the associativity constrains are defined by

$$a_{M,N,P} = p_{(M\times N),P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,(N\times P)} : M \times (N \times P) \to (M \times N) \times P$$

where the inverse is the morphism

$$a_{M,N,P}^{-1} = a_{M,N,P} = p_{M,(N\times P)} \circ (M \otimes p_{N,P}) \circ (i_{M,N} \otimes P) \circ i_{(M\times N),P} : (M\times N)\times P \to M\times(N\times P).$$

If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M' \times N'} \circ (\gamma \otimes \phi) \circ i_{M,N} : M \times N \to M' \times N'$$

is a morphism in ${}_H^H\mathcal{YD}$ and $(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi)$, where $\gamma' : M' \to M''$ and $\phi' : N' \to N''$ are morphisms in ${}_H^H\mathcal{YD}$.

Finally, the braiding is

$$\tau_{M,N} = p_{N,M} \circ t_{M,N} \circ i_{M,N} : M \times N \to N \times M$$

where $t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \to N \otimes M$. The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau^{-1}_{M,N} = p_{M,N} \circ t'_{M,N} \circ i_{N,M} : N \times M \to M \times N$$

where $t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M)$.

2 Projections, quantum groupoids and crossed products

In this section we give basic properties of quantum groupoids with projection. The material presented here can be found in [1] and [2]. For example, in Theorem 2.2 we will show that if H, B are quantum groupoids in $\mathcal{C}$ and $g : B \to H$ is a quantum groupoid morphism such that there exist a coalgebra morphism $f : H \to B$ verifiying $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$ then, it is possible to find an object B_H, defined by an equalizer diagram an called the algebra of coinvariants, morphisms $\varphi_{B_H} : H \otimes B_H \to B_H$, $\sigma_{B_H} : H \otimes H \to B_H$ and an isomorphism of algebras and comodules $b_H : B \to B_H \times H$ being $B_H \times H$ a subobject of $B_H \otimes H$ with its algebra structure twisted by the morphism σ_{B_H}. Of course, the multiplication in $B_H \times H$ is a generalization of the crossed product and in the Hopf algebra case the Theorem 2.2 is the classical and well know result obtained by Blattner, Cohen and Montgomery in [6].

The following Proposition is a generalization to the quantum groupoid setting of classic result obtained by Radford in [23].

Proposition 2.1. *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$ be a morphism of quantum groupoids and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$. Then the following morphism is an idempotent in $\mathcal{C}$:*

$$q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B.$$

Proof. See Proposition 2.1 of [2].

As a consequence of this proposition, we obtain that there exist an epimorphism p_H^B, a monomorphism i_H^B and an object B_H such that the diagram

$$\begin{tikzcd} B \arrow[rr, "q_H^B"] \arrow[dr, "p_H^B"'] & & B \\ & B_H \arrow[ur, "i_H^B"'] & \end{tikzcd}$$

commutes and $p_H^B\circ i_H^B=id_{B_H}$. Moreover, we have that

$$\begin{tikzcd} B_H \arrow[r, "i_H^B"] & B \arrow[r, shift left, "(B\otimes g)\circ\delta_B"] \arrow[r, shift right, "(B\otimes(\Pi_H^L\circ g))\circ\delta_B"'] & B\otimes H \end{tikzcd}$$

is an equalizer diagram.

Now, let η_{B_H} and μ_{B_H} be the factorizations, through the equalizer i_H^B, of the morphisms η_B and $\mu_B\circ(i_H^B\otimes i_H^B)$. Then $(B_H,\eta_{B_H}=p_H^B\circ\eta_B,\mu_{B_H}=p_H^B\circ\mu_B\circ(i_H^B\otimes i_H^B))$ is an algebra in $\mathcal{C}$.

On the other hand, by Proposition 2.4 of [2] we have that there exists an unique morphism $\varphi_{B_H}:H\otimes B_H\rightarrow B_H$ such that $i_H^B\circ\varphi_{B_H}=y_B$ where $y_B:H\otimes B_H\rightarrow B$ is the morphism defined by $y_B=\mu_B\circ(B\otimes(\mu_B\circ c_{B,B}))\circ(f\otimes(\lambda_B\circ f)\otimes B)\circ(\delta_H\otimes i_H^B)$. The morphism φ_{B_H} satisfies:

$$\varphi_{B_H}=p_H^B\circ\mu_B\circ(f\otimes i_H^B),$$

$$\varphi_{B_H}\circ(\eta_H\otimes B_H)=id_{B_H},$$

$$\varphi_{B_H}\circ(H\otimes\eta_{B_H})=\varphi_{B_H}\circ(\Pi_H^L\otimes\eta_{B_H}),$$

$$\mu_{B_H}\circ(\varphi_{B_H}\otimes B_H)\circ(H\otimes\eta_{B_H}\otimes B_H)=\varphi_{B_H}\circ(\Pi_H^L\otimes B_H),$$

$$\varphi_{B_H}\circ(H\otimes\mu_{B_H})=\mu_{B_H}\circ(\varphi_{B_H}\otimes\varphi_{B_H})\circ(H\otimes c_{H,B_H}\otimes B_H)\circ(\delta_H\otimes B_H\otimes B_H),$$

$$\mu_{B_H}\circ c_{B_H,B_H}\circ((\varphi_{B_H}\circ(H\otimes\eta_{B_H}))\otimes B_H)=\varphi_{B_H}\circ(\overline{\Pi}_H^L\otimes B_H).$$

and, if f is an algebra morphism, (B_H,φ_{B_H}) is a left H-module (Proposition 2.5 of [1]).

Moreover, in this setting, there exists an unique morphism $\sigma_{B_H}:H\otimes H\rightarrow B_H$ such that $i_H^B\circ\sigma_{B_H}=\sigma_B$ where $\sigma_B:H\otimes H\rightarrow B$ is the morphism defined by:

$$\sigma_B=\mu_B\circ((\mu_B\circ(f\otimes f))\otimes(\lambda_B\circ f\circ\mu_H))\circ\delta_{H\otimes H}.$$

Then, as a consequence, we have the equality $\sigma_{B_H}=p_H^B\circ\sigma_B$ (Proposition 2.6, [2]).

Now let $\omega_B:B_H\otimes H\rightarrow B$ be the morphism defined by $\omega_B=\mu_B\circ(i_H^B\otimes f)$. If we define $\omega_B':B\rightarrow B_H\otimes H$ by $\omega_B'=(p_H^B\otimes g)\circ\delta_B$ we have $\omega_B\circ\omega_B'=id_B$. Then, the morphism $\Omega_B=\omega_B'\circ\omega_B:B_H\otimes H\rightarrow B_H\otimes H$ is idempotent and there exists a diagram

$$\begin{tikzcd} & B \arrow[dr, "\omega_B'"] \arrow[dd, "b_B"] & \\ B_H\otimes H \arrow[ur, "\omega_B"] \arrow[rr, "\Omega_B"] \arrow[dr, "r_B"'] & & B_H\otimes H \\ & B_H\times H \arrow[ur, "s_B"'] & \end{tikzcd}$$

where $s_B\circ r_B=\Omega_B,\ r_B\circ s_B=id_{B_H\times H},\ b_B=r_B\circ\omega_B'$.

It is easy to prove that the morphism b_B is an isomorphism with inverse $b_B^{-1} = \omega_B \circ s_B$. Therefore, the object $B_H \times H$ is an algebra with unit and product defined by $\eta_{B_H \times H} = b_B \circ \eta_B$, $\mu_{B_H \times H} = b_B \circ \mu_B \circ (b_B^{-1} \otimes b_B^{-1})$ respectively. Also, $B_H \times H$ is a right H-comodule where $\rho_{B_H \times H} = (b_B \otimes H) \circ (B \otimes g) \circ \delta_B \circ b_B^{-1}$. Of course, with these structures b_B is an isomorphism of algebras and right H-comodules being $\rho_B = (B \otimes g) \circ \delta_B$.

On the other hand, we can define the following morphisms:

$$\eta_{B_H \sharp_{\sigma_{B_H}} H} : K \to B_H \times H, \; \mu_{B_H \sharp_{\sigma_{B_H}} H} : B_H \times H \otimes B_H \times H \to B_H \times H, \; \rho_{B_H \sharp_{\sigma_{B_H}} H} : B_H \to B_H \times H \otimes H$$

where

$$\eta_{B_H \sharp_{\sigma_{B_H}} H} = r_B \circ (\eta_{B_H} \otimes \eta_H),$$

$$\mu_{B_H \sharp_{\sigma_{B_H}} H} = r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_{H \otimes H}) \circ$$
$$(B_H \otimes H \otimes c_{H,B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),$$

$$\rho_{B_H \sharp_{\sigma_{B_H}} H} = (r_B \otimes H) \circ (B_H \otimes \delta_H) \circ s_B.$$

Finally, if we denote by $B_H \sharp_{\sigma_{B_H}} H$ (the crossed product of B_H and H) the triple

$$(B_H \times H, \eta_{B_H \sharp_{\sigma_{B_H}} H}, \mu_{B_H \sharp_{\sigma_{B_H}} H})$$

we have the following theorem.

Theorem 2.2. *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$ be a morphism of quantum groupoids and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. Then, $B_H \sharp_{\sigma_{B_H}} H$ is an algebra, $(B_H \times H, \rho_{B_H \sharp_{\sigma_{B_H}} H})$ is a right H-comodule and $b_B : B \to B_H \sharp_{\sigma_{B_H}} H$ is an isomorphism of algebras and right H-comodules.*

Proof: The proof of this Theorem is a consequence of the following identities (see Theorem 2.8 of [2] for the complete details)

$$\eta_{B_H \sharp_{\sigma_{B_H}} H} = \eta_{B_H \times H}, \quad \mu_{B_H \sharp_{\sigma_{B_H}} H} = \mu_{B_H \times H}, \quad \rho_{B_H \sharp_{\sigma_{B_H}} H} = \rho_{B_H \times H}.$$

Remark 2.3. We point out that if H and B are Hopf algebras, Theorem 2.2 is the result obtained by Blattner, Cohen and Montgomery in [6]. Moreover, if f is an algebra morphism, we have $\sigma_{B_H} = \varepsilon_H \otimes \varepsilon_H \otimes \eta_{B_H}$ and then $B_H \sharp_{\sigma_{B_H}} H$ is the smash product of B_H and H, denoted by $B_H \sharp H$. Observe that the product of $B_H \sharp H$ is

$$\mu_{B_H \sharp H} = (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)) \otimes H)$$

Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$, $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. In this case $\sigma_B = \Pi_L^B \circ f \circ \mu_H$ and then, using $\mu_B \circ (\Pi_B^L \otimes B) \circ \delta_B = id_B$, we obtain

$$\mu_{B_H \sharp_{\sigma_{B_H}} H} = r_B \circ (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)) \otimes H) \circ (s_B \otimes s_B)$$

As a consequence, for analogy with the Hopf algebra case, when $\sigma_B = \Pi_L^B \circ f \circ \mu_H$, we will denote the triple $B_H \sharp_{\sigma_{B_H}} H$ by $B_H \sharp H$ (the smash product of B_H and H).

Therefore, if f and g are morphisms of quantum groupoids, we have the following particular case of 2.2.

Corollary 2.4. *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$, $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then $B_H \sharp H$ is an algebra, $(B_H \times H, \rho_{B_H \sharp H})$ is a right H-comodule and $b_B : B \to B_H \sharp H$ is an isomorphism of algebras and right H-comodules.*

In a similar way we can obtain a dual theory. The arguments are similar to the ones used previously in this section, but passing to the opposite category. Let H, B be quantum groupoids in $\mathcal{C}$. Let $h : H \to B$ be a morphism of quantum groupoids and $t : B \to H$ be a morphism of algebras such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. The morphism $k_H^B : B \to B$ defined by

$$k_H^B = \mu_B \circ (B \otimes (h \circ t \circ \lambda_B)) \circ \delta_B$$

is idempotent in $\mathcal{C}$ and, as a consequence, we obtain that there exist an epimorphism l_H^B, a monomorphism n_H^B and an object B^H such that the diagram

$$\begin{tikzcd} B \arrow[rr, "k_H^B"] \arrow[dr, "l_H^B"'] & & B \\ & B^H \arrow[ur, "n_H^B"'] & \end{tikzcd}$$

commutes and $l_H^B \circ n_H^B = id_{B^H}$. Moreover, using the next coequalizer diagram in $\mathcal{C}$

$$\begin{tikzcd} B \otimes H \arrow[r, shift left, "\mu_B \circ (B \otimes h)"] \arrow[r, shift right, "\mu_B \circ (B \otimes (h \circ \Pi_H^L))"'] & B \arrow[r, "l_H^B"] & B^H \end{tikzcd}$$

it is possible to obtain a coalgebra structure for B^H. This structure is given by

$$(B^H, \varepsilon_{B^H} = \varepsilon_B \circ n_H^B, \delta_{B^H} = (l_H^B \otimes l_H^B) \circ \delta_B \circ n_H^B)).$$

Let $y^B : B \to H \otimes B^H$ be the morphism defined by:

$$y^B = (\mu_H \otimes l_H^B) \circ (t \otimes (t \circ \lambda_B) \otimes B) \circ (B \otimes (c_{B,B} \circ \delta_B)) \circ \delta_B.$$

The morphism y^B verifies that $y^B \circ \mu_B \circ (B \otimes h) = y^B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h))$ and then, there exists an unique morphism $r_{B^H} : B^H \to H \otimes B^H$ such that $r_{B^H} \circ l_H^B = y^B$.

Moreover the morphism ϱ_{B^H} satisfies:

$$\varrho_{B^H} = (t \otimes l_H^B) \circ \delta_B \circ n_H^B,$$

$$(\varepsilon_H \otimes B^H) \circ \varrho_{B^H} = id_{B^H},$$
$$(H \otimes \varepsilon_{B^H}) \circ \varrho_{B^H} = (\Pi_H^L \otimes \varepsilon_{B^H}) \circ \varrho_{B^H},$$
$$(H \otimes \varepsilon_{B^H} \otimes B^H) \circ (\varrho_{B^H} \otimes B^H) \circ \delta_{B_H} = (\Pi_H^L \otimes B^H) \circ \varrho_{B^H}$$
$$(H \otimes \delta_{B^H}) \circ \varrho_{B^H} = (\mu_H \otimes B^H \otimes B^H) \circ (H \otimes c_{B^H,H} \otimes B^H) \circ (\varrho_{B^H} \otimes \varrho_{B^H}) \circ \delta_{B^H},$$
$$(((H \otimes \varepsilon_{B^H}) \circ \varrho_{B^H}) \otimes B^H) \circ c_{B^H,B^H} \circ \delta_{B^H} = (\overline{\Pi}_H^L \otimes B^H) \circ \varrho_{B^H},$$

and, if t is a morphism of quantum groupoids, (B^H, ϱ_{B^H}) is a left H-comodule. Let $\gamma_B : B \to H \otimes H$ be the morphism defined by

$$\gamma_B = \mu_{H\otimes H} \circ (((t \otimes t) \circ \delta_B) \otimes (\delta_H \circ t \circ \lambda_B)) \circ \delta_B.$$

The morphism γ_B verifies that $\gamma_B \circ \mu_B \circ (B \otimes h) = \gamma_B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h))$ and then, there exists an unique morphism $\gamma_{B^H} : B^H \to H \otimes H$ such that $\gamma_{B^H} \circ l_H^B = \gamma_B$.

It is not difficult to see that the morphism $\Upsilon_B : B^H \otimes H \to B^H \otimes H$ defined by

$$\Upsilon_B = \varpi'_B \circ \varpi_B,$$

being $\varpi_B = \mu_B \circ (n_H^B \otimes h)$ and $\varpi'_B = (l_H^B \otimes t) \circ \delta_B$, is idempotent and there exists a diagram

$$\begin{tikzcd}
 & B \arrow[dr, "\varpi'_B"] \arrow[dd, "d_B"] & \\
B^H \otimes H \arrow[ur, "\varpi_B"] \arrow[rr, "\Upsilon_B"] \arrow[dr, "u_B"'] & & B^H \otimes H \\
 & B^H \boxdot H \arrow[ur, "v_B"'] &
\end{tikzcd}$$

where $v_B \circ u_B = \Upsilon_B$, $u_B \circ v_B = id_{B^H \boxdot H}$, $d_B = u_B \circ \varpi'_B$. Moreover, d_B is an isomorphism with inverse $d_B^{-1} = \varpi_B \circ v_B$ and the object $B^H \boxdot H$ is a coalgebra with counit and coproduct defined by

$$\varepsilon_{B^H \boxdot H} = \varepsilon_B \circ d_B^{-1}, \qquad \delta_{B^H \boxdot H} = (d_B \otimes d_B) \circ \delta_B \circ d_B^{-1}$$

respectively.

Also, $B^H \boxdot H$ is a right H-module where

$$\psi_{B^H \boxdot H} = d_B \circ \mu_B \circ (d_B^{-1} \otimes h).$$

With these structures d_B is an isomorphism of coalgebras and right H-modules being $\psi_B = \mu_B \circ (B \otimes h)$. Finally, we define the morphisms:

$$\varepsilon_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \to K, \;\; \delta_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \to B^H \boxdot H \otimes B^H \boxdot H,$$
$$\psi_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \otimes H \to B^H \boxdot H$$

where

$$\varepsilon_{B^H \ominus_{\gamma_{B^H}} H} = (\varepsilon_{B^H} \otimes \varepsilon_H) \circ v_B,$$

$$\delta_{B^H \ominus_{\gamma_{B^H}} H} = (u_B \otimes u_B) \circ (B^H \otimes \mu_H \otimes B^H \otimes H) \circ (B^H \otimes H \otimes c_{B^H,H} \otimes H) \circ$$
$$(B^H \otimes \varrho_{B^H} \otimes \mu_{H\otimes H}) \circ (\delta_{B^H} \otimes \gamma_{B^H} \otimes \delta_H) \circ (\delta_{B^H} \otimes H) \circ v_B,$$

$$\psi_{B^H \ominus_{\gamma_{B^H}} H} = u_B \circ (B^H \otimes \mu_H) \circ (v_B \otimes H).$$

If we denote by $B^H \ominus_{\gamma_{B^H}} H$ (the crossed coproduct of B^H and H) the triple

$$(B^H \boxdot H, \varepsilon_{B^H \ominus_{\gamma_{B^H}} H}, \delta_{B^H \ominus_{\gamma_{B^H}} H}),$$

we have the following theorem:

Theorem 2.5. *Let H, B be quantum groupoids in $\mathcal{C}$. Let $h : H \to B$ be a morphism of quantum groupoids and $t : B \to H$ be a morphism of algebras such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. Then, $B^H \ominus_{\gamma_{B^H}} H$ is a coalgebra, $(B^H \boxdot H, \psi_{B^H \ominus_{\gamma_{B^H}} H})$ is a right H-module and $d_B : B \to B^H \ominus_{\gamma_{B^H}} H$ is an isomorphism of coalgebras and right H-modules.*

Remark 2.6. In the Hopf algebra case (H and B Hopf algebras) Theorem 2.5 is the dual of the result obtained by Blattner, Cohen and Montgomery. In this case, if t is an algebra-coalgebra morphism, we have $\gamma_{B^H} = \varepsilon_{B^H} \otimes \eta_H \otimes \eta_H$ and then $B^H \ominus_{\gamma_{B^H}} H$ is the smash coproduct of B^H and H, denoted by $B^H \ominus H$. In $B^H \ominus H$ the coproduct is

$$\delta_{B^H \ominus H} = (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H,H}) \circ (\varrho_{B^H} \otimes H)) \otimes H) \circ (\delta_{B^H} \otimes \delta_H).$$

If t is a morphism of quantum groupoids we have $\gamma_B = \delta_H \circ \Pi_H^L \circ t$ and then the expression of $\delta_{B^H \ominus_{\gamma_{B^H}} H}$ is:

$$\delta_{B^H \ominus_{\gamma_{B^H}} H} = (u_B \otimes u_B) \circ (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H,H}) \circ (\varrho_{B^H} \otimes H)) \otimes H) \circ (\delta_{B^H} \otimes \delta_H) \circ v_B.$$

As a consequence, for analogy with the Hopf algebra case, when $\gamma_B = \delta_H \circ \Pi_L^H \circ t$, we will denote the triple $B^H \ominus_{\gamma_{B^H}} H$ by $B^H \ominus H$ (the smash coproduct of B^H and H).

Therefore, if h and t are morphisms of quantum groupoids, we have:

Corollary 2.7. *Let H, B be quantum groupoids in $\mathcal{C}$. Let $t : B \to H$, $h : H \to B$ be morphisms of quantum groupoids such that $t \circ h = id_H$. Then, $B^H \ominus H$ is a coalgebra, $(B^H \boxdot H, \psi_{B^H \ominus H})$ is a right H-module and $d_B : B \to B^H \ominus H$ is an isomorphism of coalgebras and right H-modules.*

3 Quantum groupoids, projections and Hopf algebras in ${}^H_H\mathcal{YD}$

In this section we give the connection between projection of quantum groupoids an Hopf algebras in the category ${}^H_H\mathcal{YD}$. The results presented here can be found in [3].

Suppose that $g : B \to H$ and $f : H \to B$ are morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then $q_H^B = k_H^B$ and therefore $B_H = B^H$, $p_H^B = l_H^B$ and $i_H^B = n_H^B$. Thus

$$\begin{tikzcd} B_H \arrow[r, "i_H^B"] & B \arrow[r, shift left, "(B\otimes g)\circ\delta_B"] \arrow[r, shift right, "(B\otimes(\Pi_H^L\circ g))\circ\delta_B"'] & B\otimes H \end{tikzcd}$$

is an equalizer diagram and

$$\begin{tikzcd} B\otimes H \arrow[r, shift left, "\mu_B\circ(B\otimes f)"] \arrow[r, shift right, "\mu_B\circ(B\otimes(f\circ\Pi_H^L))"'] & B \arrow[r, "p_B^H"] & B_H \end{tikzcd}$$

is a coequalizer diagram.

Then $(B_H, \eta_{B_H} = p_H^B \circ \eta_B, \mu_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B))$ is an algebra in $\mathcal{C}$, $(B_H, \varepsilon_{B_H} = \varepsilon_B \circ i_H^B, \delta_{B_H} = (p_H^B \otimes p_H^B) \circ \delta_B \circ i_H^B))$ is a coalgebra in $\mathcal{C}$, (B_H, φ_{B_H}) is a left H-module and (B_H, ϱ_{B_H}) is a left H-comodule.

Also, $\omega_B = \varpi_B$, $\omega'_B = \varpi'_B$ and then $B_H \times H = B^H \boxdot H$. Moreover, the morphism $\Omega_B = \omega'_B \circ \omega_B$ admits a new formulation. Note that by the usual arguments in the quantum groupoid calculus, we have

$$\begin{aligned}
\Omega_B &= (p_H^B \otimes \mu_H) \circ (\mu_B \otimes H \otimes g) \circ (B \otimes c_{H,B} \otimes B) \circ (((B \otimes g) \circ \delta_B \circ i_H^B) \otimes (\delta_B \circ f)) \\
&= (p_H^B \otimes \mu_H) \circ (\mu_B \otimes H \otimes H) \circ (B \otimes c_{H,B} \otimes H) \circ (((B \otimes (\overline{\Pi}_H^R \circ g)) \circ \delta_B \circ i_H^B) \otimes ((f \otimes H) \circ \delta_H))) \\
&= (p_H^B \otimes \varepsilon_H \otimes H) \circ (\mu_{B\otimes H} \otimes H) \circ (((B \otimes g) \circ \delta_B \circ i_H^B) \otimes ((f \otimes \delta_H) \circ \delta_H))) \\
&= (p_H^B \otimes (\varepsilon_H \circ g) \otimes H) \circ (\mu_{B\otimes B} \otimes H) \circ (\delta_B \otimes \delta_B \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= ((p_H^B \circ \mu_B) \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= ((p_H^B \circ \mu_B \circ (B \otimes q_H^B)) \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= (p_H^B \otimes H) \circ ((\mu_B \circ (B \otimes (\Pi_B^L \circ f)) \otimes H) \circ (i_H^B \otimes \delta_H) \\
&= (p_H^B \otimes H) \circ ((\mu_B \circ c_{B,B} \circ ((\Pi_B^L \circ f) \otimes i_H^B)) \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \delta_H) \\
&= ((p_H^B \circ i_H^B \circ \varphi_{B_H} \circ (\overline{\Pi}_H^L \otimes B_H)) \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \delta_H) \\
&= (\varphi_{B_H} \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \overline{\Pi}_H^L \otimes H) \circ (B_H \otimes \delta_H) \\
&= (\varphi_{B_H} \otimes \mu_H) \circ (H \otimes c_{H,B_H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes B_H \otimes H). \\
&= \nabla_{B_H \otimes H}.
\end{aligned}$$

Therefore, the object $B_H \times H$ is the tensor product of B_H and H in the representation category of H, i.e. the category of left H-modules, studied in [8] and [21].

Proposition 3.1. *Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, $(B_H, \varphi_{B_H}, \varrho_{B_H})$ belongs to ${}^H_H\mathcal{YD}$.*

Proof: In Proposition 2.8 of [1] we prove that $(B_H, \varphi_{B_H}, \varrho_{B_H})$ satisfy

$$(\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ ((\varrho_{B_H} \circ \varphi_{B_H}) \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$

$$= (\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H) \circ (\delta_H \otimes \varrho_{B_H} \otimes \Pi^R_H) \circ$$

$$(H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H).$$

Moreover, the following identity

$$(\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H) \circ (\delta_H \otimes \varrho_{B_H} \otimes \Pi^R_H) \circ$$

$$(H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$

$$= (\mu_H \otimes \varphi_{B_H}) \circ (H \otimes c_{H,H} \otimes (\varphi_{B_H} \circ ((\overline{\Pi}^L_H \circ \overline{\Pi}^R_H) \otimes B_H) \circ \varrho_{B_H})) \circ (\delta_H \otimes \varrho_{B_H}).$$

is true because B_H is a left H-module and a left H-comodule. Then, using the identity

$$\varphi_{B_H} \circ ((\overline{\Pi}^L_H \circ \overline{\Pi}^R_H) \otimes B_H) \circ \varrho_{B_H} = id_{B_H}$$

we prove (b1). The prove for (b2) is easy and we leave the details to the reader.

3.2. As a consequence of the previous proposition we obtain $\nabla_{B_H \otimes B_H} = \nabla'_{B_H \otimes B_H}$ and $\nabla_{B_H \otimes H} = \nabla'_{B_H \otimes H} = \Omega_B$.

3.3. Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Put $u_{B_H} = p^B_H \circ f \circ i_L : H_L \to B_H$ and $e_{B_H} = p_L \circ g \circ i^B_H : B_H \to H_L$. This morphisms belong to ${}^H_H\mathcal{YD}$ and we have the same for $m_{B_H \times B_H} : B_H \times B_H \to B_H$ defined by

$$m_{B_H \times B_H} = \mu_{B_H} \circ i_{B_H,B_H}$$

and $\Delta_{B_H} : B_H \to B_H \times B_H$ defined by $\Delta_{B_H} = p_{B_H,B_H} \circ \delta_{B_H}$.

Then, we have the following result.

Proposition 3.4. *Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, we have the following:*

(i) (B_H, u_{B_H}, m_{B_H}) *is an algebra in* ${}^H_H\mathcal{YD}$.

(ii) $(B_H, e_{B_H}, \Delta_{B_H})$ *is a coalgebra in* ${}^H_H\mathcal{YD}$.

Proof: See Proposition 2.6 in [3].

3.5. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Let Θ_H^B be the morphism $\Theta_H^B = ((f \circ g) \wedge \lambda_B) \circ i_H^B : B_H \to B$. Following Proposition 2.9 of [1] we have that $(B \otimes g) \circ \delta_B \circ \Theta_H^B = (B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ \Theta_H^B$ and, as a consequence, there exists an unique morphism $\lambda_{B_H} : B_H \to B_H$ such that $i_H^B \circ \lambda_{B_H} = \Theta_H^B$. Therefore, $\lambda_{B_H} = p_H^B \circ \Theta_H^B$ and λ_{B_H} belongs to the category of left-left Yetter-Drinfeld modules.

The remainder of this section will be devoted to the proof of the main Theorem of this paper.

Theorem 3.6. *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras satisfying the equality $g \circ f = id_H$ and suppose that the antipode of H is an isomorphism. Let u_{B_H}, m_{B_H}, e_{B_H}, Δ_{B_H}, λ_{B_H} be the morphisms defined in 3.3 and 3.5 respectively. Then $(B_H, u_{B_H}, m_{B_H}, e_{B_H}, \Delta_{B_H}, \lambda_{B_H})$ is a Hopf algebra in the category of left-left Yetter-Drinfeld modules.*

Proof: By Proposition 3.4 we know that (B_H, u_{B_H}, m_{B_H}) is an algebra and $(B_H, e_{B_H}, \Delta_{B_H})$ is a coalgebra in ${}_H^H\mathcal{YD}$.

First we prove that m_{B_H} is a coalgebra morphism. That is:

(c1) $$\Delta_{B_H} \circ m_{B_H} = (m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H \times B_H} \circ (B_H \times a^{-1}_{B_H,B_H,B_H}) \circ$$
$$(B_H \times (\tau_{B_H,B_H} \times B_H)) \circ (B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H \times B_H} \circ (\Delta_{B_H} \times \Delta_{B_H}),$$

(c2) $$e_{B_H} \circ m_{B_H} = l_{H_L} \circ (e_{B_H} \times e_{B_H}).$$

Indeed:

$$(m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H \times B_H} \circ (B_H \times a^{-1}_{B_H,B_H,B_H}) \circ (B_H \times (\tau_{B_H,B_H} \times B_H)) \circ$$
$$(B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H \times B_H} \circ (\Delta_{B_H} \times \Delta_{B_H})$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes i_{B_H,B_H} \otimes B_H) \circ (\nabla_{B_H \otimes (B_H \times B_H)} \otimes B_H) \circ$$
$$(B_H \otimes \nabla_{(B_H \times B_H) \otimes B_H}) \circ (B_H \otimes (p_{B_H,B_H} \circ t_{B_H,B_H} \circ i_{B_H,B_H}) \otimes B_H) \circ (B_H \otimes \nabla_{(B_H \times B_H) \otimes B_H})$$
$$(\nabla_{B_H \otimes (B_H \times B_H)} \otimes B_H) \circ (B_H \otimes p_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes (\nabla_{B_H \otimes B_H} \circ t_{B_H,B_H} \circ \nabla_{B_H \otimes B_H}) \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ$$
$$i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes t_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ \delta_{B_H} \circ \mu_{B_H} \circ i_{B_H,B_H}$$

$$= \Delta_{B_H} \circ m_{B_H}.$$

In the last computations, the first and the second equalities follow from Lemma 1.7 of [3] and by $\mu_{B_H} \circ \nabla_{B_H \otimes B_H} = \mu_{B_H}$, $\nabla_{B_H \otimes B_H} \circ \delta_{B_H} = \delta_{B_H}$. In the third one we use the following result: if M is a left-left Yetter-Drinfeld module then $t_{M,M} \circ \nabla_{M \otimes M} = t_{M,M}$, $\nabla_{M \otimes M} \circ t_{M,M} = t_{M,M}$. The fourth equality follows from Proposition 2.9 of [1] and, finally, the fifth one follows by definition.

On the other hand,

$$l_{H_L} \circ (e_{B_H} \times e_{B_H})$$

$$= p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L \otimes H_L} \circ (p_L \otimes p_L) \circ ((g \circ i_H^B) \otimes (g \circ i_H^B)) \circ i_{B_H, B_H}$$

$$= p_L \circ \mu_H \circ ((\Pi_H^L \circ g \circ i_H^B) \otimes (\Pi_H^L \circ g \circ i_H^B)) \circ i_{B_H, B_H}$$

$$= p_L \circ \mu_H \circ ((g \circ q_H^B \circ i_H^B) \otimes (g \circ q_H^B \circ i_H^B)) \circ i_{B_H, B_H}$$

$$= p_L \circ \mu_H \circ ((g \circ i_H^B) \otimes (g \circ i_H^B)) \circ i_{B_H, B_H}$$

$$= p_L \circ g \circ i_H^B \circ \mu_{B_H} \circ i_{B_H, B_H}$$

$$= e_{B_H} \circ m_{B_H}.$$

The first equality follows from definition, the second one from

$$p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L \otimes H_L} \circ (p_L \otimes p_L) = p_L \circ \mu_H \circ (\Pi_H^L \otimes \Pi_H^L)$$

and the third one from $\Pi_H^L \circ g = g \circ q_H^B$. Finally, the fourth one follows from the idempotent character of q_H^B, the fifth one from the properties of g and the definition of μ_{B_H} and the sixth one from definition.

To finish the proof we only need to show

$$m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H} = l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H}.$$

We begin by proving $l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = u_{B_H} \circ e_{B_H}$. Indeed:

$$l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1}$$

$$= p_H^B \circ \mu_B \circ (f \otimes B) \circ (i_L \otimes i_H^B) \circ \nabla_{H_L \otimes B_H} \circ (p_L \otimes p_H^B) \circ (g \otimes f) \circ (i_H^B \otimes i_L) \circ \nabla_{B_H \otimes H_L} \circ (p_H^B \otimes p_L) \circ$$

$$((\mu_B \circ (f \otimes i_H^B)) \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \eta_H) \otimes B_H)$$

$$= p_H^B \circ \mu_B \circ ((\Pi_B^L \wedge \Pi_B^L) \otimes \Pi_B^L) \circ ((f \circ g \circ q_H^B) \otimes (\mu_B \circ (\Pi_B^L \otimes (f \circ g \circ \Pi_B^L)))) \circ (\delta_B \otimes B) \circ$$

$$\delta_B \circ i_H^B$$

$$= p_H^B \circ \mu_B \circ ((\Pi_B^L \circ f \circ g \circ q_H^B) \otimes (f \circ \Pi_H^L \circ g \circ \Pi_B^L)) \circ \delta_B \circ i_H^B$$

$$= p_H^B \circ f \circ \mu_H \circ (\Pi_H^L \otimes \Pi_H^L) \circ \delta_H \circ g \circ i_H^B$$

$$= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B$$

$$= u_{B_H} \circ e_{B_H}.$$

The first equality follows from definition, the second one from

$$((\mu_B \circ (f \otimes i_H^B)) \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \eta_H) \otimes B_H) = (B \otimes (g \circ \Pi_B^L)) \circ \delta_B \circ i_H^B,$$

$$(i_H^B \otimes i_L) \circ \nabla_{B_H \otimes H_L} \circ (p_H^B \otimes p_L) = (q_H^B \otimes (\Pi_H^L \circ g \circ \mu_B)) \circ (B \otimes \Pi_B^L \otimes f) \circ (\delta_B \otimes H)$$

and

$$(i_L \otimes i_H^B) \circ \nabla_{H_L \otimes B_H} \circ (p_L \otimes p_H^B) = (\Pi_H^L \circ g) \otimes (q_H^B \circ \mu_B)) \circ (B \otimes \Pi_B^L \otimes B) \circ ((\delta_B \circ f) \otimes B).$$

In the third one we use $\Pi_B^L \wedge \Pi_B^L = \Pi_B^L$. The fourth one follows from $\Pi_H^L \circ g = g \circ q_H^B$ and from the idempotent character of Π_H^L. Finally, in the fifth one we apply (75) for $\Pi_H^L \wedge \Pi_H^L = \Pi_H^L$.

On the other hand,

$$m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H}$$

$$= \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H}$$

$$= \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H}$$

$$= ((\varepsilon_{B_H} \circ \mu_{B_H}) \otimes B_H) \circ (B_H \otimes t_{B_H,B_H}) \circ ((\delta_{B_H} \circ \eta_{B_H}) \otimes B_H)$$

$$= ((\varepsilon_B \circ q_H^B \circ \mu_B) \otimes p_H^B) \circ ((\mu_B \circ (q_H^B \otimes (f \circ g)) \circ \delta_B) \otimes c_{B,B}) \circ ((\delta_B \circ q_H^B \circ \eta_B) \otimes i_H^B)$$

$$= p_H^B \circ \Pi_B^L \circ i_H^B$$

$$= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B$$

$$= u_{B_H} \circ e_{B_H}.$$

In these computations, the first equality follows from definition, the second one from $\mu_{B_H} \circ \nabla_{B_H \otimes B_H} = \mu_{B_H}$ and $\nabla_{B_H \otimes B_H} \circ \delta_{B_H} = \delta_{B_H}$, the third one from (4-1) of Proposition 2.9 of [1] and the fourth one is a consequence of the coassociativity of δ_B. The fifth equality follows from $\mu_B \circ (q_H^B \otimes (f \circ g)) \circ \delta_B = id_B$ and $q_H^B \circ \eta_B = \eta_B$, $\varepsilon_B \circ q_H^B = \varepsilon_B$. In the sixth one we use $f \circ \Pi_H^L \circ g = \Pi_B^L$ and the last one follows from definition.

Finally, using similar arguments and (4-2) of Proposition 2.9 of [1] we obtain:

$$m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H}$$

$$= \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (B_H \otimes \lambda_{B_H}) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H}$$

$$= \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H}$$

$$= (B_H \otimes (\varepsilon_{B_H} \circ \mu_{B_H})) \circ (t_{B_H,B_H} \otimes B_H) \circ (B_H \otimes (\delta_{B_H} \circ \eta_{B_H}))$$

$$= p_H^B \circ \mu_B \circ ((f \circ g) \otimes \Pi_B^R) \circ \delta_B \circ i_H^B$$

$$= p_H^B \circ \mu_B \circ ((f \circ g) \otimes (f \circ \Pi_H^R \circ g) \circ \delta_B \circ i_H^B$$

$$= p_H^B \circ f \circ (id_H \wedge \Pi_H^R) \circ g \circ i_H^B$$

$$= p_H^B \circ f \circ g \circ i_H^B$$

$$= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B$$

$$= u_{B_H} \circ e_{B_H}.$$

Finally, using the last theorem and Theorem 4.1 of [2] we obtain the complete version of Radford's Theorem linking weak Hopf algebras with projection and Hopf algebras in the category of Yetter-Drinfeld modules over H.

Theorem 3.7. *Let H, B be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$ and suppose that the antipode of H is an isomorphism. Then there exists a Hopf algebra B_H living in the braided monoidal category ${}_H^H\mathcal{YD}$ such that B is isomorphic to $B_H \times H$ as weak Hopf algebras, being the (co)algebra structure in $B_H \times H$ the smash (co)product, that is the (co)product defined in 2.3, 2.6. The expression for the antipode of $B_H \times H$ is*

$$\lambda_{B_H \times H} := p_{B_H,H} \circ (\varphi_{B_H} \otimes H) \circ$$

$$(H \otimes c_{H,B_H}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_{B_H}) \circ (H \otimes c_{B_H,H}) \circ$$

$$(\varrho_{B_H} \otimes H) \circ i_{B_H,H}.$$

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Shifted determinants over universal enveloping algebra

Natasha Rozhkovskaya*

Abstract

We present a family of polynomials with coefficients in the universal enveloping algebra. These polynomials are shifted analogues of a determinant of a certain non-commutative matrix, labeled by irreducible representations of $\mathfrak{gl}_n(\mathbb{C})$. We show plethysm relation with Capelli polynomials and compute the polynomials explicitly for $\mathfrak{gl}_2(\mathbb{C})$.

Keywords: Casimir element, universal enveloping algebra, irreducible representation, determinant, characteristic polynomial, Capelli polynomial, shifted symmetric functions.

1 Introduction

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ with the standard basis $\{E_{ij}\}$, let V_λ be an irreducible representation of $\mathfrak{g}$. The matrix Ω_λ, defined by

$$\Omega_\lambda = \sum_{i,j=1,\dots,n} E_{ij} \otimes \pi_\lambda(E_{ji}),$$

naturally appears in many problems of representation theory. We will call it braided Casimir element.

Consider the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. The algebra $S(\mathfrak{g})$ is commutative. So if we think of Ω_λ as an element of $S(\mathfrak{g}) \otimes \mathrm{End}V_\lambda$, the determinant $p_\lambda(u) = det(\Omega_\lambda - u)$ is a well-defined polynomial. The coefficients of $D_\lambda(u)$ are invariant under the ajoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$. The determinant also serves as a characteristic polynomial for Ω_λ (namely, $p_\lambda(\Omega_\lambda) = 0$).

Now let us consider Ω_λ as a matrix with coefficients in the universal enveloping algebra $U(\mathfrak{g})$. This is non-commutative algebra, and it can be viewed as a deformation of $S(\mathfrak{g})$. Due to B. Kostant's theorem [6], the matrix Ω_λ (now as a matrix with coefficients in $U(\mathfrak{g})$), satisfies a characteristics equation with coefficients in the center of the universal enveloping algebra. We also define in Section 2 a (shifted) analogue of determinant $\Omega(\lambda)$. Thus, we have two deformations of the polynomial $D_\lambda(u)$: a shifted determinant and a characteristic polynomial. It is well-known, that in case of V_λ - vector representation of $\mathfrak{gl}(n, \mathbb{C})$, both deformations coincide: the shifted determinant is a characteristic polynomial of Ω_λ. But

*Department of Mathematics, Harvard Univeristy. E-mail: `rozhkovs@math.harvard.edu`

it turns out, that in general, these two deformations are different, which follows from the results of Section 3. There we prove the centrality of shifted determinant and find the explicit formula in the case of $\mathfrak{gl}_2(\mathbb{C})$.

With every new nice example of central elements it is natural to ask, how it is related to the known ones. One of the most well-studied families of central polynomials is the set of Capelli polynomials. These polynomials are parametrized by dominant weights of $\mathfrak{gl}_n(\mathbb{C})$. Their theory is developed in the series of works [5], [8],[11],[12],[13], etc. Again, the shifted determinant coincides with a Capelli polynomial only in the case of vector representations, when $\lambda = (1)$. In Section 5 we prove that there is a plethysm-like relation between Capelli polynomials and shifted determinants.

In Section 2 we define shifted determinant for a non-commutative matrix. In Section 3 we study the case of $\mathfrak{gl}_2(\mathbb{C})$. In Section 4 we discuss relations with characteristic polynomials and state two conjectures about centrality of shifted determinants in general. In Section 5 we give a formula for relation with Capelli polynomials.

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2 Definitions

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ with the universal enveloping algebra $U(\mathfrak{g})$. Denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. Fix the basis $\{E_{ij}\}$ of $\mathfrak{gl}_n(\mathbb{C})$, which consists of standard unit matrices.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$, for $i = 1, \ldots, n-1$, be a dominant weight of $\mathfrak{gl}_n(\mathbb{C})$. Denote by π_λ be the corresponding irreducible rational $\mathfrak{gl}_n(\mathbb{C})$-representation, and by V_λ the space of this representation. We assume that $\dim V_\lambda = m+1$. Then we define an element Ω_λ of $U(\mathfrak{g}) \otimes \mathrm{End}(V_\lambda)$, which we call *braided Casimir element.*

Definition.

$$\Omega_\lambda = E_{ij} \otimes \sum_{i,j=1,\ldots,n} \pi_\lambda(E_{ji}).$$

We will think of Ω_λ as a matrix of size $(m+1)\times(m+1)$ with coefficients in $U(\mathfrak{gl}_n(\mathbb{C}))$. Next we define a shifted determinant of Ω_λ.

Let A be an element of $\mathcal{A} \otimes \mathrm{End}\,(\mathbb{C}^{m+1})$, where $\mathcal{A}$ is a non-commutative algebra. We again think of A as a non-commutative matrix of size $(m+1)\times(m+1)$ with coefficients $A_{ij} \in \mathcal{A}$.

Definition. The (column)-determinant of A is the following element of $\mathcal{A}$:

$$\det(A) = \sum_{\sigma \in S_{m+1}} (-1)^\sigma A_{\sigma(1)1} A_{\sigma(2)2} \ldots A_{\sigma(m+1)(m+1)}. \tag{1}$$

Here the sum is taken over all elements σ of the symmetric group S_{m+1}, and $(-1)^\sigma$ is the sign of the permutation σ.

Put $\Omega_\lambda(u) = \Omega_\lambda + u \otimes id$. Define L as a diagonal matrix of the size $(m+1)\times(m+1)$ of the form:

$$L = \mathrm{diag}(m, m-1, \ldots, 0).$$

Definition. The shifted determinant of $\Omega_\lambda(u)$ is the column-determinant $\det(\Omega_\lambda(u) - L)$. We will use notation $D_\lambda(u)$ for this polynomial with coefficients in $U(\mathfrak{gl}_n(\mathbb{C}))$:

$$D_\lambda(u) = \det(\Omega_\lambda(u) - L).$$

There is another way to define the same determinant. Let $A_1, \ldots, A_s$ be a set of matrices of size $(m+1)\times(m+1)$ with coefficients in some associative (non-commutative) algebra $\mathcal{A}$. Let μ be the multiplication in $\mathcal{A}$. Consider an element of $\mathcal{A}\otimes \mathrm{End}(\mathbb{C}^{m+1})$

$$\Lambda^s(A_1\otimes\cdots\otimes A_s) = (\mu^{(s)}\otimes Asym_s)(A_1\otimes\cdots\otimes A_s),$$

where $Asym_s = \frac{1}{s!}\sum_{\sigma\in S_s}(-1)^\sigma\sigma$. By Young's construction, the antisymmetrizer can be realized as an element of $\mathrm{End}\left((\mathbb{C}^{m+1})^{\otimes s}\right)$.

Lemma 2.1. *For $s = m+1$*

$$\Lambda^{m+1}(A_1\otimes\cdots\otimes A_{m+1}) = \alpha(A_1,\ldots,A_{m+1})\otimes Asym_{m+1}, \tag{2}$$

where $\alpha(A_1,\ldots,A_{m+1})\in\mathcal{A}$,

$$\alpha(A_1,\ldots,A_{m+1}) = \sum_{\sigma\in S_{m+1}}(-1)^\sigma[A_1]_{\sigma(1),1}\ldots[A_{m+1}]_{\sigma(m+1),m+1},$$

and $[A_k]_{i,j}$ are matrix elements of A_k.

Proof. (cf [9].) Let $\{e_i\}$, $(i=1,\ldots,m{+}1)$, be a basis of $V=\mathbb{C}^{m+1}$. Observe that $Asym_{(m+1)}$ is a one-dimensional projector to

$$v = \frac{1}{(m+1)!}\sum_{\sigma\in S_{m+1}}(-1)^\sigma e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(m+1)}.$$

We apply $\Lambda^{m+1}(A_1\otimes\cdots\otimes A_{m+1})$ to $e_1\otimes\cdots\otimes e_{m+1}\in V^{\otimes m+1}$:

$$\begin{aligned}&\Lambda^{m+1}(A_1\otimes\cdots\otimes A_{m+1})(e_1\otimes\cdots\otimes e_m)\\ &= \sum_{i_1,\ldots i_k}(A_1)_{i_1,1}\ldots(A_{m+1})_{i_{m+1},m+1}\,Asym_{(m+1)}(e_{i_1}\otimes\cdots\otimes e_{i_{m+1}}).\end{aligned}\tag{3}$$

The vector $Asym_{(m+1)}(e_{i_1}\otimes\cdots\otimes e_{i_{m+1}})\neq 0$ only if all indices $\{i_1,\ldots,i_{m+1}\}$ are pairwise distinct. In this case denote by σ be a permutation defined by $\sigma(k)=i_k$. Then

$$Asym_{(m+1)}(e_{i_1}\otimes\cdots\otimes e_{i_{m+1}}) = (-1)^\sigma v,$$

and (3) gives

$$\Lambda^{m+1}(A_1 \otimes \cdots \otimes A_{m+1})\,(e_1 \otimes \cdots \otimes e_{m+1}) = \alpha(A_1,\, \ldots,\, A_{m+1})\, v$$
$$= \alpha(A_1,\, \ldots,\, A_{m+1})\, Asym_{\,(m+1)}\,(e_1 \otimes \cdots \otimes e_{m+1}).$$

□

Corollary 2.2.

$$\alpha\,(A,\, \ldots,\, A) = det(A),$$
$$\alpha\,(\Omega_\lambda(u-m),\, \ldots,\, \Omega_\lambda(u)) = D_\lambda(u).$$

3 Case of $\mathfrak{gl}_2(\mathbb{C})$

In this section we prove the centrality of polynomials $D_\lambda(u)$ for $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$ an write them explicitly.

The center $Z(\mathfrak{gl}_2(\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{gl}_2(\mathbb{C}))$ is generated by two elements:

$$\Delta_1 = E_{11} + E_{22}, \qquad \Delta_2 = (E_{11} - 1)E_{22} - E_{12}E_{21}.$$

Let $\lambda = (\lambda_1 \geq \lambda_2)$ be a dominant weight. Put $m = \lambda_1 - \lambda_2, \quad d = \lambda_1 + \lambda_2$. Then $\dim V_\lambda = m+1$ and Ω_λ is a "tridiagonal" matrix: all entries $[\Omega_\lambda]_{ij}$ of the matrix Ω_λ are zeros, except

$$[\Omega_\lambda]_{k,k} = (\lambda_1 - k + 1)E_{11} + (\lambda_2 + k - 1)E_{22}, \qquad k = 1,\, \ldots,\, m+1,$$
$$[\Omega_\lambda]_{k,k+1} = (m + 1 - k)E_{21}, \qquad k = 1,\, \ldots,\, m,$$
$$[\Omega_\lambda]_{k+1,k} = kE_{12}, \qquad k = 1,\, \ldots,\, m.$$

Proposition 3.1. *a) Polynomial $D_\lambda(u)$ is central.*

b) Let $\mu = (\mu_1 \geq \mu_2)$ be another dominant weight of $\mathfrak{gl}_n(\mathbb{C})$. The image of $D_\lambda(u)$ under Harish - Chandra isomorphism χ is the following function of μ:

$$\chi(D_\lambda(u)) = \prod_{k=0}^{m} (u + (\lambda_1 - k)\mu_1 + (\lambda_2 + k)\mu_2 - k). \tag{4}$$

Proof. a) Let $X = X(a, b, c)$ be a matrix of size $(m+1) \times (m+1)$ of the form

$$\begin{pmatrix} a_m & b_m & 0 & \ldots & 0 & 0 & 0 \\ c_m & a_{m-1} & b_{m-1} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & c_2 & a_1 & b_1 \\ 0 & 0 & 0 & \ldots & 0 & c_1 & a_0 \end{pmatrix},$$

where a_i, b_j, c_k are elements of some (noncommutative) algebra. Define $\det X$ as in Section 2. Using the principle k-minors of X, the determinant of X can be computed by recursion formula. Denote by X_k the matrix obtained from X by deleting the first $(m-k+1)$ rows and the first $(m-k+1)$ columns, and by $I^{(k)}$ the determinant of X_k. Then $\det X = I^{(m+1)}$, and we have the following recursion:

Lemma 3.2. *The determinants $I^{(k)}$ satisfy the recursion realtion*

$$I^{(k+1)} = a_k I^{(k)} - c_k b_k \, I^{(k-1)}, \qquad (k = 2, ..., n-1) \tag{5}$$

with initial conditions $I^{(1)} = a_0$, $I^{(2)} = a_1 a_0 - c_1 b_1$.

Lemma 3.3. *If $X(a,b,c)$ is a tri-diagonal matrix with coefficients $\{a_i, b_j, c_k\}$ and $X(a',b',c')$ is another tri-diagonal matrix with coefficients $\{a'_i, b'_j, c'_k\}$ with the property*

$$a_i = a'_i, \qquad c_j \, b_j = c'_j \, b'_j$$

for all $i = 0, ..., n$ $j = 1, ..., n$, then $det\, X(a,b,c) = det\, X(a',b',c')$.

Proof. Follows from the recursion relation. ☐

We apply these observations to compute the determinant of the matrix $X(a,b,c) = \Omega_\lambda(u) - L$.

The following obvious lemma allows to reduce the determinant of non-commutative matrix $(\Omega_\lambda(u) - L)$ to a determinant of a matrix with commutative coefficients.

Lemma 3.4. *The subalgebra of $U(\mathfrak{gl}_2(\mathbb{C}))$ generated by $\{E_{11}, E_{22}, (E_{12}\, E_{21})\}$ is commutative.*

Put $h = E_{11} - E_{22}$, $a = E_{12} E_{21}$. Due to Lemma 3.3 the tridiagonal matrix $X(a',b',c')$ with coefficients

$$\begin{aligned} a'_k = a_k = \lambda_1 E_{11} + \lambda_2 E_{22} + u - m + (k-m)(h-1), &\qquad k = 0, \ldots, m, \\ b'_k = k\, a, \qquad c'_k = (m-k+1), &\qquad k = 1, \ldots, m, \end{aligned}$$

has the same determinant as $(\Omega_\lambda(u) - L)$. By Lemma 3.4, $X(a',b',c')$ has commutative coefficients. Hence $\det(\Omega_\lambda(u) - L)$ equals $\det(\lambda_1 E_{11} + \lambda_2 E_{22} + u - m + A_m)$ where A_m is the following matrix:

$$A_m = \begin{pmatrix} 0 & ma & 0 & \ldots & 0 & 0 \\ 1 & -(h-1) & (m-1)a & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & (1-m)(h-1) & a \\ 0 & 0 & 0 & \ldots & m & -m(h-1) \end{pmatrix}$$

with h and a as above.

Lemma 3.5.

$$det\, A_m = \prod_{k=0}^{m} \left(\frac{-m(h-1)}{2} + \frac{(m-2k)}{2} \sqrt{(h-1)^2 + 4a} \right).$$

Proof. By Lemma 3.3 $\det A_m = (h-1)^{m+1} \det A'_m$ with

$$A'_m = \begin{pmatrix} 0 & ms & 0 & \dots & 0 & 0 \\ 1(s-1) & -1 & (m-1)s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1-m & s \\ 0 & 0 & 0 & \dots & m(s-1) & -m \end{pmatrix},$$

and s is such that $s(s-1) = a/(h-1)^2$. This reduces to

$$s = \frac{1 \pm \sqrt{1 + 4a/(h-1)^2}}{2}. \tag{6}$$

The determinant of A'_m is a variant of Sylvester determinant ([1], [4]). It equals

$$\det A'_m = \prod_{k=0}^{m} ((m-2k)s - m + k).$$

With s as in (6) we have:

$$(h-1)((m-2k)s - m + k) = \frac{-m(h-1)}{2} \pm \frac{(m-2k)}{2}\sqrt{((h-1)^2 + 4a)}$$

and lemma follows. Note that both values of s give the same value of $\det A_m$. $\square$

We obtain from calculations above

$$\det(\Omega_\lambda(u) - L) = \prod_{k=0}^{m} \left(u + \frac{d}{2}(E_{11} + E_{22}) - \frac{m}{2} + \frac{(m-2k)}{2}\big((E_{11} - E_{22} - 1)^2 + 4E_{12}E_{21}\big)^{\frac{1}{2}} \right). \tag{7}$$

Observe that $\big((E_{11} - E_{22} - 1)^2 + 4E_{12}E_{21}\big)^{\frac{1}{2}} = \big((\Delta_1 - 1)^2 - 4\Delta_2\big)^{\frac{1}{2}}$, and finally we get

$$D_\lambda(u) = \prod_{k=0}^{m} \left(u + \frac{d\,\Delta_1}{2} - \frac{m}{2} + \frac{(m-2k)}{2}\big((\Delta_1 - 1)^2 - 4\Delta_2\big)^{\frac{1}{2}} \right). \tag{8}$$

The quantity in (8) has coefficients in W_τ-extension of $Z(\mathfrak{gl}_2(\mathbb{C}))$, where W_τ is the translated Weyl group. But it is easy to see that after expanding the product, we get a polynomial in u with coefficients in $Z(\mathfrak{gl}_n(\mathbb{C})$. We proved the first part of the proposition.

b) The images of the generators of $Z(\mathfrak{gl}_n(\mathbb{C}))$ under Harish-Chandra homomorphism χ are

$$\chi(\Delta_1) = \mu_1 + \mu_2, \qquad \chi(\Delta_2) = \mu_1(\mu_2 - 1).$$

This together with (8) implies (4). $\square$

4 Characteristic polynomial

Proposition 4.1. *For any dominant weight λ of $\mathfrak{gl}_n(\mathbb{C})$ there exists a polynomial*

$$P_\lambda(u) = \sum_{k=0}^{m} z_k u^k$$

with coefficients $z_k \in Z(\mathfrak{gl}_n(\mathbb{C}))$, such that $P_\lambda(-\Omega_\lambda) = 0$.

This proposition follows from the similar statements for central elements of semisimple Lie algebras, proved in [6],[2]. In [2] the Harish-Chandra-image of the polynomial $P_\lambda(u)$ is also obtained:

$$\chi(P_\lambda(u)) = \prod(u + (\mu, \lambda_i) + \frac{1}{2}(2\rho + \lambda_i, \lambda_i) - \frac{1}{2}(2\rho + \lambda, \lambda)),$$

where ρ is the half-sum of positive roots, and the product is taken over all weights $\{\lambda_i\}$ of V_λ.

Corollary 4.2. *In case of $\mathfrak{gl}_2(\mathbb{C})$ the image of the polynomial $P_\lambda(u)$ under Harish-Chandra homomorphism is*

$$\chi(P_\lambda(u)) = \prod_{k=0}^{m} \left(u + (\lambda_1 - k)\mu_1 + (\lambda_2 + k)\mu_2 - k(m + 1 - k)\right), \tag{9}$$

where $m = \lambda_1 - \lambda_2$.

Comparing (9) with (4), we can see that the polynomials $P_\lambda(u)$ and $D_\lambda(u)$ are different.

Recall that for a semi-simple Lie algebra $\mathfrak{g}$, the algebra $U(\mathfrak{g})$ is a deformation of the symmetric algebra $S(\mathfrak{g})$. Hence, the central polynomials $P_\lambda(u)$ and $D_\lambda(u)$ are the deformations of a polynomial $p_\lambda(u)$, which has coefficients in the ring of invariants $I(\mathfrak{g})$ of the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$. Since $I(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^W$, where W is a Weyl group, and $\mathfrak{h}^*$ is a dual to the Cartan subalgebra $\mathfrak{h}$, the polynomial $p_\lambda(u)$ can be represented as a polynomial of u with coefficients in the ring $\mathbb{C}[\mathfrak{h}^*]^W$. We can extend these observations for the case of $\mathfrak{gl}_n(\mathbb{C})$. Let $\{\lambda_1, \ldots, \lambda_{m+1}\}$ be the set of weights of V_λ. Then we can write that

$$p_\lambda(u) = \prod_{\lambda_i}(u + (\mu, \lambda_i)),$$

as a (symmetric) function of $\mu \in \mathfrak{h}^*$.

Let us summarize the facts about $D_\lambda(u)$ and $P_\lambda(u)$:

a) $D_\lambda(u)$ and $P_\lambda(u)$ are deformations of $p_\lambda(u)$, and the case of $\mathfrak{gl}_2(\mathbb{C})$ shows that in general these are different deformations.

b) In case of vector representation $\lambda = (1)$, $D_{(1)}(u) = P_{(1)}(u)$ for any $\mathfrak{gl}_n(\mathbb{C})$. This is proved, for example, in [7].

c) $P_\lambda(u)$ is known to be a central polynomial for all λ for any $\mathfrak{gl}_n(\mathbb{C})$.

d) The centrality of $D_\lambda(u)$ is proved in two cases: for all λ for $\mathfrak{gl}_2(\mathbb{C})$, and for $\lambda = (1)$ for $\mathfrak{gl}_n(\mathbb{C})$. However, we state the following two conjectures.

Conjecture 4.3. *For any dominant weight λ there exists a basis of the vector space V_λ such that the polynomial $D_\lambda(u)$ has coefficients in the center $Z(\mathfrak{gl}_n(\mathbb{C}))$.*

Conjecture 4.4. *The Harish-Chandra image of $D_\lambda(u)$ is given by polynomial*

$$\prod (u + (\mu + \rho, \lambda_i) - \frac{m}{2}),$$

where the product is taken over all weights $\{\lambda_i\}$ of the representation V_λ, and $\dim V_\lambda = m+1$.

5 Capelli elements

In this section we show that the shifted determinant can be viewed as some sort of plethysm applied to Capelli elements.

Consider an element S of $U(\mathfrak{g}) \otimes \operatorname{End} V$, defined by $S = \sum_{ij} E_{ij} \otimes E_{ij}$. Using the abbreviated notation $S = S(1) \otimes S(2)$, put

$$S_{1j} = S(1) \otimes 1^{\otimes(j-1)} \otimes S(2) \otimes 1^{\otimes(m-j)}$$

(viewed as an element of $U(\mathfrak{g}) \otimes \operatorname{End}(\mathbb{C}^n)^{\otimes m}$).

Let $\{c_1, \ldots, c_M\}$ be the set of contents of the standard Young tableau of shape λ, and let F_λ be a Young symmetrizer that corresponds to the diagram λ:

$$F_\lambda : (\mathbb{C}^n)^{\otimes M} \to V_\lambda, \qquad V_\lambda \subset (\mathbb{C}^n)^{\otimes M}.$$

Following [11], define an element $S_\lambda(u)$ of $U(\mathfrak{gl}_n(\mathbb{C})) \otimes \operatorname{End} V_\lambda$ by

$$S_\lambda(u) = ((S_{12} - u - c_1) \ldots (S_{1\,M+1} - u - c_M))\ (id \otimes F_\lambda). \tag{10}$$

Then

$$c_\lambda(u) = tr\,(S_\lambda(u))$$

is the *Capelli polynomial*, associated to λ. It has coefficients in $Z(\mathfrak{gl}_n(\mathbb{C}))$. The theory of Capelli elements is developed in the papers, mentioned in the Introduction.

Now let π_0 be a vector representation, let $\tilde{S}_\lambda(u) = (\pi_0 \otimes id) S_\lambda(u)$, and let $Asym_s$ be the antisymmetrizer, defined as in Section 2.

Proposition 5.1. *For $\lambda \vdash M$, $\dim V_\lambda = (m+1)$,*

$$D_\lambda(u) = f(u)\, tr \left(\tilde{S}_\lambda(m-u)_{12} \tilde{S}_\lambda(m-1-u)_{13} \ldots \tilde{S}_\lambda(-u)_{1m+2} \cdot Asym_{m+1} \right), \tag{11}$$

where

$$f(u) = \prod_{s=0}^{m} \prod_{k=1}^{M} \frac{(u-s)}{u-s-c_k}.$$

Remark. The formula (11) tells that in order to obtain a shifted determinant for dominant weight λ, one has to do the following:

1) Take $(m+1)$ copies of elements $\tilde{S}_\lambda(u)$.
2) Shift the variable u in each copy by $(m-s)$, $s=0,...,m$.
3) Apply antisymmetrizer of order $m+1$ to these shifted $(m+1)$ copies.
4) Multiply by certain rational function and take the trace.

The result is a shifted analogue of plethysm of two Schur functors: F_λ and $F_{(1^{m+1})}$ (compare with the definition of S_λ (10)).

Proof. Let $(\pi_0, V_0=\mathbb{C}^n)$ be the vector representation of $\mathfrak{gl}_n(\mathbb{C})$. Using the Young symmetrizer F_λ, we construct the element $(1\otimes F_\lambda)$ of End (V_0^M), which is a projector from $V_0^{\otimes M}$ to $V_0\otimes V_\lambda$. We use Proposition 2.12 from [10]:

Proposition 5.2. *Let $P=\sum E_{ij}\otimes E_{ji}$ be a permutation matrix on $\mathbb{C}^n\otimes\mathbb{C}^n$. Let $P_{k,l}$ denote the action of this operator on the kth and lth components of the tensor product $(\mathbb{C}^n)^{\otimes M}$. Then*

$$\prod_{k=1}^{M}\left(1-\frac{P_{1,\,k+1}}{u-c_k}\right)(1\,\otimes\,F_\lambda)=\left(1-\sum_{k=1}^{M}\frac{P_{1,\,k+1}}{u}\right)(1\,\otimes\,F_\lambda), \tag{12}$$

where F_λ is the Young symmertrizer, and $\{c_1,\ldots c_n\}$ are the contents of the standard tableau of shape λ.

With the standard coproduct δ in $U(\mathfrak{gl}_n(\mathbb{C}))$ we obtain:

$$\Omega_\lambda=\sum_{ij}\pi_0(E_{ij})\otimes\pi_\lambda(E_{ji})=\left(\sum_{ij}E_{ij}\otimes\delta^{(M)}(E_{ji})\right)(1\otimes F_\lambda)=\left(\sum_{l=1\ldots M,}P_{1,l+1}\right)(1\otimes F_\lambda),$$

This implies

$$\Omega_\lambda(u)=u\prod_{k=1}^{M}\left(1+\frac{P_{1,\,k+1}}{u+c_k}\right)(1\otimes F_\lambda)\,\in\,\mathrm{End}V_0\otimes\mathrm{End}V_\lambda. \tag{13}$$

Let $\phi:\,\mathfrak{gl}_n(\mathbb{C})\to\mathfrak{gl}_n(\mathbb{C})$ be an automorphism, defined by $\phi(X)=-X^\top$. Put $\pi_{\lambda^\star}=\pi_\lambda\circ\phi$. Observe that $(\phi\otimes\pi_\lambda)\,\Omega=id\otimes(\pi_\lambda\circ\phi)\,\Omega$, so

$$\Omega_{\lambda^\star}(u)=u\prod_{k=1}^{M}\left(1-\frac{S_{1,\,k+1}}{u+c_k}\right)(1\otimes F_\lambda)=u\prod_{k=1}^{M}\frac{(-1)}{(u+c_k)}\,(\pi_0\otimes Id)\,S_\lambda(u). \tag{14}$$

In other words, $\Omega_{\lambda^\star}(u)$ is proportional to the image of $S_\lambda(u)$ under the map $(\pi_0\otimes Id)$: $U(\mathfrak{gl}_n(\mathbb{C}))\otimes\mathrm{End}\ V_\lambda\to\mathrm{End}V_0\otimes\mathrm{End}V_\lambda$.

The representation $X\to-(\pi_{\lambda^\star}(X))^\top$, $X\in\mathfrak{gl}_n(\mathbb{C})$ is isomorphic to π_λ. Thus we can write in some basis

$$\Omega_\lambda^\top(u)=-\Omega_{\lambda^\star}(-u). \tag{15}$$

Recall from Section 2 that

$$D_\lambda(u) = \alpha\,(\Omega_\lambda(u-m),\,\ldots,\,\Omega_\lambda(u)) = tr(Asym_{m+1}\,\Omega_\lambda(u-m)_{12}\,\ldots\,\Omega_\lambda(u)_{1m+2}).$$

It is easy to see that

$$tr(Asym_s A_1\otimes\cdots\otimes A_s) = tr(A_1^\top\otimes\cdots\otimes A_s^\top\,Asym_s)$$

Hence,

$$\begin{aligned} D_\lambda &= tr\left(\Omega_\lambda^\top(u-m)_{1\,2}\,\ldots\,\Omega_\lambda^\top(u)_{1\,m+2}\,Asym_{m+1}\right)\\ &= (-1)^{m+1}\,tr\,(\,(\Omega_{\lambda^\star}(-u+m)_{12}\,\ldots\,\Omega_{\lambda^\star}(-u)_{1\,m+2})\,Asym_{m+1}\,)\\ &= tr\left(\prod_{s=0}^{m}(u-s)\prod_{k=1}^{M}\left(\frac{S_{1,sM+k+1}(u+s-m-c_k)}{u-s-c_k}\right)(\pi_0\otimes\,F_\lambda)^{\otimes(m+1)}\cdot\,Asym_{m+1}\right)\end{aligned}$$

Comparing the last formula with the definition of $\tilde{S}_\lambda(u)$, we obtain (11) $\square$

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Graded-simple Lie algebras of type B_2 and Jordan systems covered by a triangle

Erhard Neher* Maribel Tocón†

Abstract

We announce a classification of graded-simple Jordan systems covered by a compatible triangle, under some natural assumptions on the abelian group, in order to get the corresponding classification of graded-simple Lie algebras of type B_2.

Keywords: Root-graded Lie algebra, Jordan system, idempotent.

1 Introduction

Graded-simple Lie algebras which also have second compatible grading by a root system appear in the structure theory of extended affine Lie algebras, which generalize affine Lie algebras and toroidal Lie algebras. If the root system in question is 3-graded, these Lie algebras are Tits-Kantor-Koecher algebras of Jordan pairs covered by a grid.

In this note we will consider the case of the root system B_2. A centreless B_2-graded Lie algebra is the Tits-Kantor-Koecher algebra of a Jordan pair covered by a triangle. Such a Lie algebra is graded-simple with respect to a compatible Λ-grading if and only if the Jordan pair is graded-simple with respect to a Λ-grading which is compatible with the covering triangle [8]. In [10] we give a classification of graded-simple Jordan systems covered by a triangle that is compatible with the grading, under some natural assumptions on the abelian group, as well as the corresponding classification of graded-simple Lie algebras of type B_2. Our work generalizes earlier results of Allison-Gao [1] and Benkart-Yoshii [2], and is an extension of the structure theory of simple Jordan pairs and Jordan triple systems covered by a triangle due to McCrimmon-Neher [7].

*Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, Ontario K1N 6N5, Canada. E-mail: neher@uottawa.ca. Supported in part by Discovery Grant #8836 of the Natural Sciences and Research Council of Canada.

†Departamento de Estadística, Econometría, Inv. Operativa y Org. Empresa, Facultad de Derecho, Puerta Nueva, 14071, Córdoba, Spain. E-mail: td1tobam@uco.es. Support from Discovery Grant #8836 of the NSRC of Canada, MEC and fondos FEDER (Becas posdoctorales and MTM2004-03845), and Junta de Andalucía (FQM264) is gratefully acknowledged. This paper was written during the second author's stay at Department of Mathematics and Statistics, University of Ottawa, Canada.

The aim of this note is to provide an outline of our results for Jordan systems. The details of proofs will appear in [10]. For unexplained notation we refer the reader to [3] and [4].

2 Graded-simple Lie algebras of type B_2

The motivation for our research is two-fold. On the one hand, we would like to advance the theory of graded Jordan structures, and on the other hand we are interested in certain types of root-graded Lie algebras. In this section we will describe the second part of our motivation and how it is related to the first.

Let R be a reduced root system. In the following only the case $R = \mathrm{B}_2$ will be of interest, but the definition below works for any finite, even locally finite reduced root system. We will assume that $0 \in R$ and denote by $\mathcal{Q}(R) = \mathbb{Z}[R]$ the root lattice of R. We will consider Lie algebras defined over a ring of scalars k containing $\frac{1}{2}$ and $\frac{1}{3}$. Let Λ be an arbitrary abelian group.

Definition 2.1. ([9]) A Lie algebra L over k is called (R, Λ)-*graded* if

(1) L has a compatible $\mathcal{Q}(R)$- and Λ-gradings,

$$L = \oplus_{\lambda \in \Lambda} L^{\lambda} \quad \text{and} \quad L = \oplus_{\alpha \in \mathcal{Q}(R)} L_{\alpha},$$

i.e., using the notation $L_{\alpha}^{\lambda} = L^{\lambda} \cap L_{\alpha}$ we have

$$L_{\alpha} = \oplus_{\lambda \in \Lambda} L_{\alpha}^{\lambda}, \quad L^{\lambda} = \oplus_{\alpha \in \mathcal{Q}(R)} L_{\alpha}^{\lambda}, \quad \text{and} \quad [L_{\alpha}^{\lambda}, L_{\beta}^{\kappa}] \subseteq L_{\alpha+\beta}^{\lambda+\kappa},$$

for $\lambda, \kappa \in \Lambda$, $\alpha, \beta \in \mathcal{Q}(R)$,

(2) $\{\alpha \in \mathcal{Q}(R) : L_{\alpha} \neq 0\} \subseteq R$,

(3) $L_0 = \sum_{0 \neq \alpha \in R} [L_{\alpha}, L_{-\alpha}]$, and

(4) for every $0 \neq \alpha \in R$ the homogeneous space L_{α}^{0} contains an element e_{α} that is *invertible*, i.e., there exists $f_{-\alpha} \in L_{-\alpha}^{0}$ such that $h_{\alpha} := [e_{\alpha}, f_{-\alpha}]$ acts on L_{β}, $\beta \in R$, by

$$[h_{\alpha}, x_{\beta}] = \langle \beta, \alpha^{\vee} \rangle x_{\beta}, \quad x_{\beta} \in L_{\beta}.$$

In particular, $(e_{\alpha}, h_{\alpha}, f_{\alpha})$ is an $\mathfrak{sl}_2$-triple.

An (R, Λ)-graded Lie algebra is said to be *graded-simple* if it does not contain proper nontrivial Λ-graded ideals and *graded-division* if every nonzero element in L_{α}^{λ}, $\alpha \neq 0$, is invertible.

Let now L be a centerless $(B_2, \{0\})$-graded Lie algebra. It then follows from [8] that L is the Tits-Kantor-Koecher algebra of a Jordan pair V *covered by a triangle*: i.e.,

$$V = V_1 \oplus M \oplus V_2,$$

where $V_i = V_2(e_i)$, $i = 1, 2$, and $M = V_1(e_1) \cap V_1(e_2)$, for a triangle $(u; e_1, e_2)$. Recall that a triple $(u; e_1, e_2)$ of nonzero idempotents of V is a *triangle* if

$$e_i \in V_0(e_j),\ i \neq j, \quad e_i \in V_2(u),\ i = 1, 2, \quad \text{and} \quad u \in V_1(e_1) \cap V_1(e_2),$$

and the following multiplication rules hold for $\sigma = \pm$:

$$Q(u^\sigma)e_i^{-\sigma} = e_j^\sigma,\ i \neq j, \quad \text{and} \quad Q(e_1^\sigma, e_2^\sigma)u^{-\sigma} = u^\sigma.$$

If moreover, L is (B_2, Λ)-graded, then V is also Λ-graded, i.e., as k-module $V^\sigma = \bigoplus_{\lambda \in \Lambda} V^\sigma[\lambda]$, $\sigma = \pm$, with

$$Q(V^\sigma[\lambda])V^{-\sigma}[\mu] \subseteq V^\sigma[2\lambda + \mu] \quad \text{and} \quad \{V^\sigma[\lambda], V^{-\sigma}[\mu], V^\sigma[\nu]\} \subseteq V^\sigma[\lambda + \mu + \nu]$$

for all $\lambda, \mu, \nu \in \Lambda$, $\sigma = \pm$. A Jordan pair that is Λ-graded and covered by a triangle which lies in the homogeneous 0-space $V[0]$ is called Λ-*triangulated*. It therefore follows from the above that if L is a centerless (B_2, Λ)-graded Lie algebra, then L is the Tits-Kantor-Koecher algebra of a Λ-triangulated Jordan pair V. Moreover, L is graded-simple if and only if V is graded-simple.

Therefore, one can get a description of graded-simple (B_2, Λ)-graded Lie algebras from the corresponding classification of graded-simple Λ-triangulated Jordan pairs. However, the classification of graded-simple Λ-triangulated Jordan pairs is only known for $\Lambda = \{0\}$ [7]. In what follows, we extend this classification to more general Λ. In doing so, we work with Jordan structures over arbitrary rings of scalars k. This generality is of independent interest from the point of view of Jordan theory. Moreover, the simplifications that would arise from assuming $\frac{1}{2}$ and $\frac{1}{3} \in k$ are minimal, e.g., we could avoid working with ample subspaces in our two basic examples 3.1 and 3.2.

3 Graded-simple triangulated Jordan triple systems

Let k be an arbitrary ring of scalars and let J be a Jordan triple system over k. Recall that a triple of nonzero tripotents $(u; e_1, e_2)$ is called a *triangle* if $e_i \in J_0(e_j)$, $i \neq j$, $e_i \in J_2(u)$, $i = 1, 2$, $u \in J_1(e_1) \cap J_1(e_2)$, and the following multiplication rules hold: $P(u)e_i = e_j$, $i \neq j$, and $P(e_1, e_2)u = u$. In this case, $e := e_1 + e_2$ is a tripotent such that e and u have the same Peirce spaces. A Jordan triple system with a triangle $(u; e_1, e_2)$ is said to be *triangulated* if $J = J_2(e_1) \oplus \big(J_1(e_1) \cap J_1(e_2)\big) \oplus J_2(e_2)$ which is equivalent to $J = J_2(e)$. In this case, we will use the notation $J_i = J_2(e_i)$ and $M = J_1(e_1) \cap J_1(e_2)$. Hence

$$J = J_1 \oplus M \oplus J_2.$$

Note that $^* := P(e)P(u) = P(u)P(e)$ is an automorphism of J of period 2 such that $u^* = u$, $e_i^* = e_j$, and so $J_i^* = J_j$.

Let Λ be an abelian group. We say that J is Λ-*graded* if the underlying module is Λ-graded, say $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$, and the family $(J^\lambda : \lambda \in \Lambda)$ of k-submodules satisfies $P(J^\lambda)J^\mu \subseteq J^{2\lambda+\mu}$ and $\{J^\lambda, J^\mu, J^\nu\} \subseteq J^{\lambda+\mu+\nu}$ for all $\lambda, \mu, \nu \in \Lambda$. We call J Λ-*triangulated* if it is Λ-graded and triangulated by $(u; e_1, e_2) \subseteq J^0$ and *faithfully* Λ-*triangulated* if any $x_1 \in J_1$ with $x_1 \cdot u = 0$ vanishes, where the product $\cdot$ is defined as follows:

$$J_i \times M \to M : (x_i, m) \mapsto x_i \cdot m = L(x_i)m := \{x_i, e_i, m\}.$$

There are two basic models for Λ-triangulated Jordan triple systems:

Example 3.1. *Λ-triangulated hermitian matrix systems* $H_2(A, A_0, \pi, {}^-)$. A Λ-*graded (associative) coordinate system* $(A, A_0, \pi, {}^-)$ consists of a unital associative Λ-graded k-algebra $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$, a graded submodule $A_0 = \bigoplus_{\lambda\in\Lambda} A_0^\lambda$ for $A_0^\lambda = A_0 \cap A^\lambda$, an involution π and an automorphism ${}^-$ of period 2 of A. These data satisfy the following conditions: π and ${}^-$ commute and are both of degree 0, i.e., $(A^\lambda)^\pi = A^\lambda = \overline{A^\lambda}$ for all $\lambda \in \Lambda$, A_0 is ${}^-$-stable and π-ample in the sense that $\overline{A_0} = A_0 \subseteq H(A, \pi)$, $1 \in A_0$ and $aa_0a^\pi \subseteq A_0$ for all $a \in A$ and $a_0 \in A_0$.

To a Λ-graded coordinate system $(A, A_0, \pi, {}^-)$ we associate the Λ-*triangulated hermitian matrix system* $H = H_2(A, A_0, \pi, {}^-)$ which, by definition, is the Jordan triple system of 2×2-matrices over A which are hermitian ($X = X^{\pi t}$) and have diagonal entries in A_0, with triple product $P(X)Y = X\overline{Y}^{\pi t}X = X\overline{Y}X$. This system is clearly Λ-graded: $H = \bigoplus_{\lambda\in\Lambda} H^\lambda$, where $H^\lambda = \operatorname{span}\{a_0^\lambda E_{ii}, a^\lambda E_{12} + (a^\lambda)^\pi E_{21} : a_0^\lambda \in A_0^\lambda, a^\lambda \in A^\lambda, i = 1, 2\}$, and Λ-triangulated by $(u = E_{12} + E_{21};\ e_1 = E_{11},\ e_2 = E_{22}) \subseteq H^0$.

One can prove that $H_2(A, A_0, \pi, {}^-)$ is graded-simple if and only if $(A, \pi, {}^-)$ is graded-simple. In this case, $H_2(A, A_0, \pi, {}^-)$ is graded isomorphic to one of the following:

(I) $H_2(A, A_0, \pi, {}^-)$ for a graded-simple associative unital A;

(II) $\mathrm{Mat}_2(B)$ for a graded-simple associative unital B with graded automorphism ${}^-$, where $\overline{(b_{ij})} = (\overline{b_{ij}})$ for $(b_{ij}) \in \mathrm{Mat}_2(B)$ and $P(x)y = x\overline{y}x$;

(III) $\mathrm{Mat}_2(B)$ for a graded-simple associative unital B with graded involution ι, where $\overline{(b_{ij})} = (b_{ij}^\iota)$ for $(b_{ij}) \in \mathrm{Mat}_2(B)$ and $P(x)y = x\overline{y}^t x$;

(IV) polarized $H_2(B, B_0, \pi) \oplus H_2(B, B_0, \pi)$ for a graded-simple B with graded involution π;

(V) polarized $\mathrm{Mat}_2(B) \oplus \mathrm{Mat}_2(B)$ for a graded-simple associative unital B and $P(x)y = xyx$.

The examples (IV) and (V) are special cases of polarized Jordan triple systems. Recall that a Jordan triple system T is called *polarized* if there exist submodules $T^\pm$ such that $T = T^+ \oplus T^-$ and for $\sigma = \pm$ we have $P(T^\sigma)T^\sigma = 0 = \{T^\sigma, T^\sigma, T^{-\sigma}\}$ and $P(T^\sigma)T^{-\sigma} \subseteq T^\sigma$. In this case, $V = (T^+, T^-)$ is a Jordan pair. Conversely, to any Jordan pair $V = (V^+, V^-)$

we can associate a polarized Jordan triple system $T(V) = V^+ \oplus V^-$ with quadratic map P defined by $P(x)y = Q(x^+)y^- \oplus Q(x^-)y^+$ for $x = x^+ \oplus x^-$ and $y = y^+ \oplus y^-$. In particular, for any Jordan triple system T the pair (T, T) is a Jordan pair and hence it has an associated polarized Jordan triple system which we denote $T \oplus T$. It is clear that if T is a Λ-triangulated Jordan triple system then so is $T \oplus T$.

Example 3.2. *Λ-triangulated ample Clifford systems $AC(q, S, D_0)$.* This example is a subtriple of a full Clifford system which we will define first. It is given in terms of

(i) a Λ-graded unital commutative associative k-algebra $D = \bigoplus_{\lambda \in \Lambda} D^\lambda$ endowed with an involution $^-$ of degree 0,

(ii) a Λ-graded D-module $M = \bigoplus_{\lambda \in \Lambda} M^\lambda$,

(iii) a Λ-graded D-quadratic form $q : M \to D$, hence $q(M^\lambda) \subset D^{2\lambda}$ and $q(M^\lambda, M^\mu) \subset M^{\lambda+\mu}$,

(iv) a hermitian isometry $S : M \to M$ of q of order 2 and degree 0, i.e., $S(dx) = \overline{d}S(x)$ for $d \in D$, $q\big(S(x)\big) = \overline{q(x)}$, $S^2 = Id$ and $S(M^\lambda) = M^\lambda$, and

(v) $u \in M^0$ with $q(u) = 1$ and $S(u) = u$.

Given these data, we define

$$V := De_1 \oplus M \oplus De_2,$$

where $De_1 \oplus De_2$ is a free Λ-graded D-module with basis (e_1, e_2) of degree 0. Then V is a Jordan triple system, called a *full Clifford system* and denoted by $FC(q, S)$, with respect to the product

$$P(c_1e_1 \oplus m \oplus c_2e_2)\,(b_1e_1 \oplus n \oplus b_2e_2) = d_1e_1 \oplus p \oplus d_2e_2, \quad \text{where}$$

$$\begin{aligned} d_i &= c_i^2\,\overline{b_i} + c_i\,q\big(m, S(n)\big) + \overline{b_j}\,q(m) \\ p &= [c_1\overline{b_1} + c_2\overline{b_2} + q\big(m, S(n)\big)]m + [c_1c_2 - q(m)]S(n). \end{aligned}$$

It is easily seen that $FC(q, S)$ is Λ-triangulated by $(u; e_1, e_2)$.

But in general we need not take the full Peirce spaces De_i in order to get a Λ-triangulated Jordan triple system. Indeed, let us define a *Clifford-ample subspace of* $(D, ^-, q)$ as above as a Λ-graded k-submodule $D_0 = \bigoplus_{\lambda \in \Lambda}(D_0 \cap D^\lambda)$ such that $D_0 = \overline{D_0}$, $1 \in D_0$ and $D_0 q(M) \subseteq D_0$. Then

$$AC(q, S, D_0) := D_0\,e_1 \oplus M \oplus D_0\,e_2,$$

also denoted $AC(q, M, S, D, ^-, D_0)$ if more precision is helpful, is a Λ-graded subsystem of the full Clifford system $FC(q, S)$ which is triangulated by $(u; e_1, e_2)$. It is called a *Λ-triangulated ample Clifford system.*

One can prove that $AC(q, S, D_0)$ is graded-simple if and only if q is graded-nondegenerate (in the obvious sense) and $(D, ^-)$ is graded-simple. In this case, either $D = F$ is a graded-division algebra or D is graded isomorphic to $F \boxplus F$ for a graded-division algebra F with $^-$ the exchange automorphism. In the latter case $AC(q, S, D_0) = AC(q, S, F_0) \oplus AC(q, S, F_0)$ is polarized with a Clifford ample subspace $F_0 \subseteq F$.

It is an important fact that one can find the above two examples of Λ-triangulated Jordan triple systems inside any faithfully Λ-triangulated Jordan triple system. More precisely, let $J = J_1 \oplus M \oplus J_2$ be faithfully Λ-triangulated by $(u; e_1, e_2)$, put $C_0 = L(J_1)$ and let C be the subalgebra of $\mathrm{End}_k(M)$ generated by C_0. Then C is naturally Λ-graded, $c \mapsto \overline{c} = P(e) \circ c \circ P(e)$ is an automorphism of C of degree 0 and $L(x_1) \cdots L(x_n) \mapsto (L(x_1) \cdots L(x_n))^\pi = L(x_n) \cdots L(x_1)$ induces a (well-defined) involution of C of degree 0. One can prove that the Λ-graded subsystem

$$J_h = J_1 \oplus Cu \oplus J_2$$

is Λ-triangulated by $(u; e_1, e_2)$ and graded isomorphic to $H_2(A, A_0, \pi, ^-)$ under the map

$$x_1 \oplus cu \oplus y_2 \mapsto \begin{pmatrix} L(x_1) & c \\ c^\pi & L(y_2^*) \end{pmatrix}$$

for $A = C|_{Cu}$, $A_0 = C_0|_{Cu}$. Moreover, J has a Λ-graded subsystem

$$J_q = K_1 \oplus N \oplus K_2$$

for appropriately defined submodules $K_i \subset J_i$ and $N \subset M$, which is Λ-triangulated by $(u; e_1, e_2)$ and graded isomorphic to $AC(q, S, D_0)$ under the map

$$x_1 \oplus n \oplus x_2 \mapsto L(x_1) \oplus n \oplus L(x_2^*)$$

where $D_0 = L(K_1)$, D is the subalgebra of $\mathrm{End}_k(N)$ generated by D_0, $q(n) = L(P(n)e_2)$ and $S(n) = P(e)n$. (Roughly speaking, J_q is the biggest graded subsystem of J where the identity $(x_1 - x_1^*) \cdot N \equiv 0$ holds). Moreover, the two isomorphisms above map the triangle $(u; e_1, e_2)$ of J onto the standard triangle of $H_2(A, A_0, \pi, ^-)$ or $AC(q, S, D_0)$, respectively. We note that for $\Lambda = \{0\}$ these two partial coordinatization theorems were proven in [7].

A question that arises naturally is the following: *Let J be faithfully Λ-triangulated by $(u; e_1, e_2)$. When is J_h or J_q the whole J?*

(i) If $M = Cu$, then $J = J_h$ and thus J is graded isomorphic to a hermitian matrix system,

(ii) If u is C-faithful and $(x_1 - x_1^*) \cdot m = 0$ for all $x_1 \in J_1$ and $m \in M$, then $J = J_q$ and thus J is graded isomorphic to an ample Clifford system.

One can show that (ii) holds whenever C is commutative and $^-$-simple. In fact, (i) or (ii) above holds if $(C, \pi, ^-)$ is graded-simple.

Proposition 3.3. *Let J be a graded-simple Λ-triangulated Jordan triple system satisfying one of the following conditions*

(a) *every $m \in M$ is a linear combination of invertible homogeneous elements, or*

(b) *Λ is torsion-free.*

Then $(C, \pi, ^-)$ is graded-simple. In this case, u is C-faithful and $M = Cu$ or C is commutative.

All together, we have the following result:

Theorem 3.4. *A graded-simple Λ-triangulated Jordan triple system satisfying* (a) *or* (b) *of* Prop. *3.3 is graded isomorphic to one of the following:*
a non polarized Jordan triple system

(I) *$H_2(A, A_0, \pi, ^-)$ for a graded-simple A with graded involution π and automorphism $^-$;*

(II) $\mathrm{Mat}_2(B)$ *with $P(x)y = x\overline{y}x$ for a graded-simple associative unital B with graded automorphism $^-$ and $\overline{(y_{ij})} = (\overline{y_{ij}})$ for $(y_{ij}) \in \mathrm{Mat}_2(B)$;*

(III) $\mathrm{Mat}_2(B)$ *with $P(x)y = x\overline{y}^t x$ for a graded-simple associative unital B with graded involution ι and $\overline{(y_{ij})} = (y_{ij}^\iota)$ for $(y_{ij}) \in \mathrm{Mat}_2(B)$;*

(IV) *$AC(q, S, F_0)$ for a graded-nondegenerate q over a graded-division F with Clifford-ample subspace F_0;*

or a polarized Jordan triple system

(V) *$H_2(B, B_0, \pi) \oplus H_2(B, B_0, \pi)$ for a graded-simple B with graded involution π;*

(VI) $\mathrm{Mat}_2(B) \oplus \mathrm{Mat}_2(B)$ *for a graded-simple associative unital B with $P(x)y = xyx$;*

(VII) *$AC(q, S, F_0) \oplus AC(q, S, F_0)$ for $AC(q, S, F_0)$ as in* (IV).

Conversely, all Jordan triple systems in (I)-(VII) *are graded-simple Λ-triangulated.*

Since $\Lambda = \{0\}$ is a special case of our assumption (b), the theorem above generalizes [7, Prop. 4.4].

4 Graded-simple triangulated Jordan pairs and algebras

We consider Jordan algebras and Jordan pairs over arbitrary rings of scalars. In order to apply our results, we will view Jordan algebras as Jordan triple systems with identity elements. Thus, to a Jordan algebra J we associate the Jordan triple system $T(J)$ defined on the k-module J with Jordan triple product $P_x y = U_x y$. The element $1_J \in J$ satisfies $P(1_J) = \mathrm{Id}$.

Conversely, every Jordan triple system T containing an element $1 \in T$ with $P(1) = \mathrm{Id}$ is a Jordan algebra with unit element 1 and multiplication $U_x y = P_x y$. A Λ-graded Jordan algebra J is called Λ-*triangulated* by $(u; e_1, e_2)$ if $e_i = e_i^2 \in J^0$, $i = 1, 2$, are supplementary orthogonal idempotents and $u \in J_1(e_1)^0 \cap J_1(e_2)^0$ with $u^2 = 1$ and $u^3 = u$. Thus, with our definition of a triangle in a Jordan algebra, J is Λ-triangulated by $(u; e_1, e_2)$ iff $T(J)$ is Λ-triangulated by $(u; e_1, e_2)$.

This close relation to Λ-triangulated Jordan triple systems also indicates how to get examples of Λ-triangulated Jordan algebras: We take a Jordan triple system which is Λ-triangulated by $(u; e_1, e_2)$ and require $P(e) = \mathrm{Id}$ for $e = e_1 + e_2$. Doing this for our two basic examples 3.1 and 3.2, yields the following examples of Λ-triangulated Jordan algebras.

(A) *Hermitian matrix algebra*: This is the Jordan triple system $H_2(A, A_0, \pi, ^-)$ with $^- = \mathrm{Id}$, which we will write as $H_2(A, A_0, \pi)$. Note that this is a Jordan algebra with product $U(x)y = P(x)y = xyx$ and identity element $E = E_{11} + E_{22}$. If, for example, $A = B \boxplus B^{\mathrm{op}}$ with π the exchange involution, then $H_2(A, A_0, \pi)$ is graded isomorphic to $\mathrm{Mat}_2(B)$ where $\mathrm{Mat}_2(B)$ is the Jordan algebra with product $U_x y = xyx$.

(B) *Quadratic form Jordan algebra*: This is the ample Clifford system $AC(q, S, D, ^-, D_0)$ with $^- = \mathrm{Id}$ and $S|_M = \mathrm{Id}$. Since then $P(e) = \mathrm{Id}$ we get indeed a Λ-triangulated Jordan algebra denoted $AC_{\mathrm{alg}}(q, D, D_0)$. Note that this Jordan algebra is defined on $D_0 e_1 \oplus M \oplus D_0 e_2$ and has product $U_x y = q(x, \tilde{y})x - q(x)\tilde{y}$ where $q(d_1 e_1 \oplus m \oplus d_2 e_2) = d_1 d_2 - q(m)$ and $(d_1 e_1 \oplus m \oplus d_2 e_2)\tilde{} = d_2 e_1 \oplus -m \oplus d_1 e_1$. (If $\frac{1}{2} \in k$ it is therefore a reduced spin factor in the sense of [6, II, §3.4].)

From the classification given in Th. 3.4 we get:

Theorem 4.1. *A graded-simple Λ-triangulated Jordan algebra satisfying*

(a) *every $m \in M$ is a linear combination of invertible homogeneous elements of M, or*

(b) *Λ is torsion-free,*

is graded isomorphic to one of the following Jordan algebras:

(I) *$H_2(A, A_0, \pi)$ for a graded-simple A with graded involution π;*

(II) *$\mathrm{Mat}_2(B)$ for a graded-simple associative unital B;*

(III) *$AC_{\mathrm{alg}}(q, F, F_0)$ for a graded-nondegenerate $q : M \to F$ over a commutative graded-division algebra F and a Clifford-ample subspace F_0.*

Conversely, all Jordan algebras in (I)–(III) *are graded-simple Λ-triangulated.*

Note that for $\Lambda = \{0\}$ this theorem generalizes the well known Capacity Two Theorem for Jordan algebras.

With the above algebra classification at hand and taking into account that a Λ-triangulated Jordan pair can be viewed as a disguised Λ-triangulated Jordan algebra, we get the following classification of Λ-triangulated Jordan pairs:

Theorem 4.2. *A graded-simple Λ-triangulated Jordan pair satisfying*

(a) *every $m \in M^\sigma$ is a linear combination of invertible homogeneous elements of M^σ, or*

(b) *Λ is torsion-free,*

is graded isomorphic to a Jordan pair (J, J) where

(I) *$J = H_2(A, A_0, \pi)$ is the hermitian matrix algebra of a graded-simple A with graded involution π;*

(II) *$J = \mathrm{Mat}_2(B)$ for a graded-simple associative unital B;*

(III) *$J = AC(q, \mathrm{Id}, F_0)$ for a graded-nondegenerate q over a graded-division algebra F with Clifford-ample subspace F_0.*

Conversely, all Jordan pairs described above are graded-simple Λ-triangulated.

Note that a Jordan pair V satisfies assumption (a) if it is the Jordan pair associated to a graded-division (B_2, Λ)-graded Lie algebra for an arbitrary Λ (cf. §1).

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Restricted simple Lie algebras and their infinitesimal deformations

Filippo Viviani*

Abstract

In the first two sections, we review the Block-Wilson-Premet-Strade classification of restricted simple Lie algebras. In the third section, we compute their infinitesimal deformations. In the last section, we indicate some possible generalizations by formulating some open problems.

Keywords: Restricted simple Lie algebras, Deformations.

1 Restricted Lie algebras

We fix a field F of characteristic $p > 0$ and we denote with $\mathbb{F}_p$ the prime field with p elements. All the Lie algebras that we will consider are of finite dimension over F. We are interested in particular class of Lie algebras, called restricted (or p-Lie algebras).

Definition 1.1 (Jacobson [JAC37]). *A Lie algebra L over F is said to be restricted (or a p-Lie algebra) if there exits a map (called p-map), $[p] : L \to L$, $x \mapsto x^{[p]}$, which verifies the following conditions:*

1. $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^{[p]}$ *for every* $x \in L$.
2. $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ *for every* $x \in L$ *and every* $\alpha \in F$.
3. $(x_0 + x_1)^{[p]} = x_0^{[p]} + x_1^{[p]} + \sum_{i=1}^{p-1} s_i(x_0, x_1)$ *for every* $x, y \in L$, *where the element* $s_i(x_0, x_1) \in L$ *is defined by*

$$s_i(x_0, x_1) = -\frac{1}{r}\sum_u \mathrm{ad}x_{u(1)} \circ \mathrm{ad}x_{u(2)} \circ \cdots \circ \mathrm{ad}x_{u(p-1)}(x_1),$$

the summation being over all the maps $u : [1, \cdots, p-1] \to \{0, 1\}$ *taking* r*-times the value* 0.

*Universitá degli studi di Roma Tor Vergata, Dipartimento di Matematica, via della Ricerca Scientifica 1, 00133 Rome. E-mail: `viviani@mat.uniroma2.it`. The author was supported by a grant from the Mittag-Leffler Institute of Stockholm.

Example. 1. *Let A an associative F-algebra. Then the Lie algebra $\mathrm{Der}_F A$ of F-derivations of A is a restricted Lie algebra with respect to the p-map $D \mapsto D^p := D \circ \cdots \circ D$.*

2. *Let G a group scheme over F. Then the Lie algebra $\mathrm{Lie}(G)$ associated to G is a restricted Lie algebra with respect to the p-map given by the differential of the homomorphism $G \to G$, $x \mapsto x^p := x \circ \cdots \circ x$.*

One can naturally ask when a F-Lie algebra can acquire the structure of a restricted Lie algebra and how many such structures there can be. The following criterion of Jacobson answers to that question.

Proposition 1.2 (Jacobson)**.** *Let L be a Lie algebra over F. Then*

1. *It is possible to define a p-map on L if and only if, for every element $x \in L$, the p-th iterate of $\mathrm{ad}(x)$ is still an inner derivation.*
2. *Two such p-maps differ by a semilinear map from L to the center $Z(L)$ of L, that is a map $f : L \to Z(L)$ such that $f(\alpha x) = \alpha^p f(x)$ for every $x \in L$ and $\alpha \in F$.*

Proof. See [JAC62, Chapter V.7]. □

Many of the modular Lie algebras that arise "in nature" are restricted. As an example of this principle, we would like to recall the following two results from the theory of finite group schemes and the theory of inseparable field extensions.

Theorem 1.3. *There is a bijective correspondence*

$$\{\textit{Restricted Lie algebras}/F\} \longleftrightarrow \{\textit{Finite group schemes}/F \textit{ of height } 1\},$$

where a finite group scheme G has height 1 if the Frobenius $F : G \to G^{(p)}$ is zero. Explicitly to a finite group scheme G of height 1, one associates the restricted Lie algebra $\mathrm{Lie}(G) := T_0 G$. Conversely, to a restricted Lie algebra $(L, [p])$, one associates the finite group scheme corresponding to the dual of the restricted enveloping Hopf algebra $U^{[p]}(L) := U(L)/(x^p - x^{[p]})$.

Proof. See [DG70, Chapter 2.7]. □

Theorem 1.4. *Suppose that $[F : F^p] < \infty$. There is a bijective correspondence*

$$\{\textit{Inseparable subextensions of exponent } 1\} \longleftrightarrow \{\textit{Restricted subalgebras of } Der(F)\}$$

where the inseparable subextensions of exponent 1 are the subfields $E \subset F$ such that $F^p \subset E \subset F$ and $\mathrm{Der}(F) := \mathrm{Der}_{\mathbb{F}_p}(F) = \mathrm{Der}_{F^p}(F)$. Explicitly to any field $F^p \subset E \subset F$ one associates the restricted subalgebra $\mathrm{Der}_E(F)$. Conversely, to any restricted subalgebra $L \subset \mathrm{Der}(F)$, one associates the subfield $E_L := \{x \in F \mid D(x) = 0 \text{ for all } D \in L\}$.

Proof. See [JAC80, Chapter 8.16]. □

2 Classification of restricted simple Lie algebras

Simple Lie algebras over an algebraically closed field of **characteristic zero** were classified at the beginning of the XIX century by Killing and Cartan. The classification proceeds as follows: first the non-degeneracy of the Killing form is used to establish a correspondence between simple Lie algebras and irreducible root systems and then the irreducible root systems are classified by mean of their associated Dynkin diagrams. It turns out that there are four infinite families of Dynkin diagrams, called A_n, B_n, C_n, D_n, and five exceptional Dynkin diagram, called E_6, E_7, E_8, F_4 and G_2. The four infinite families correspond, respectively, to the the special linear algebra $\mathfrak{sl}(n+1)$, the special orthogonal algebra of odd rank $\mathfrak{so}(2n+1)$, the symplectic algebra $\mathfrak{sp}(2n)$ and the special orthogonal algebra of even rank $\mathfrak{so}(2n)$. For the simple Lie algebras corresponding to the exceptional Dynkin diagrams, see the book [JAC71] or the nice account in [BAE02].

These simple Lie algebras admits a model over the integers via the (so-called) Chevalley bases. Therefore, via reduction modulo a prime p, one obtains a restricted Lie algebra over $\mathbb{F}_p$, which is simple up to a quotient by a small ideal. For example $\mathfrak{sl}(n)$ is not simple if p divide n, but its quotient $\mathfrak{psl}(n) = \mathfrak{sl}(n)/(I_n)$ by the unit matrix I_n becomes simple. There are similar phenomena occuring only for $p = 2, 3$ for the other Lie algebras (see [STR04, Page 209] or [SEL67]). The restricted simple algebras obtained in this way are called algebras of **classical type**. Their Killing form is non-degenerate except at a finite number of primes. Moreover, they can be characterized as those restricted simple Lie algebras admitting a projective representation with nondegenerate trace form (see [BLO62], [KAP71]).

However, there are restricted simple Lie algebras which have no analogous in characteristic zero and therefore are called nonclassical. The first example of a nonclassical restricted simple Lie algebra is due to E. Witt, who in 1937 realized that the derivation algebra $W(1) := \mathrm{Der}_F(F[X]/(X^p))$ over a field F of characteristic $p > 3$ is simple with a degenerate Killing form. In the succeeding three decades, many more nonclassical restricted simple Lie algebras have been found (see [JAC43], [FRA54], [AF54], [FRA64]). The first comprehensive conceptual approach to constructing these nonclassical restricted simple Lie algebras was proposed by Kostrikin and Shafarevich in 1966 (see [KS66]). They showed that all the known examples can be constructed as finite-dimensional analogues of the four classes of infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups (see [CAR09]). These restricted simple Lie algebras, called of **Cartan-type**, are divided into four families, called Witt-Jacobson, Special, Hamiltonian and Contact algebras.

Definition 2.1. *Let $A(n) := F[x_1, \cdots, x_n]/(x_1^p, \cdots, x_n^p)$ the algebra of p-truncated polynomials in n variables. Then the Witt-Jacobson Lie algebra $W(n)$ is the derivation algebra of $A(n)$:*

$$W(n) = \mathrm{Der}_F A(n).$$

For every $j \in \{1, \ldots, n\}$, we put $D_j := \frac{\partial}{\partial x_j}$. The Witt-Jacobson algebra $W(n)$ is a free $A(n)$-module with basis $\{D_1, \ldots, D_n\}$. Hence $\dim_F W(n) = np^n$ with a basis over F given by $\{x^a D_j \mid 1 \leq j \leq n,\ x^a \in A(n)\}$.

The other three families are defined as m-th derived algebras of the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. More precisely, consider the natural action of $W(n)$ on the exterior algebra of differential forms in $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$ over $A(n)$. Define the following three forms, called volume form, Hamiltonian form and contact form:

$$\begin{cases} \omega_S = \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n, \\ \omega_H = \displaystyle\sum_{i=1}^{m} \mathrm{d}x_i \wedge \mathrm{d}x_{i+m} \ \text{ if } n = 2m, \\ \omega_K = \mathrm{d}x_{2m+1} + \displaystyle\sum_{i=1}^{m} (x_{i+m}\mathrm{d}x_i - x_i \mathrm{d}x_{i+m}) \ \text{ if } n = 2m+1. \end{cases}$$

Definition 2.2. *Consider the following three subalgebras of $W(n)$:*

$$\begin{cases} \widetilde{S}(n) = \{D \in W(n) \mid D\omega_S = 0\}, \\ \widetilde{H}(n) = \{D \in W(n) \mid D\omega_H = 0\}, \\ \widetilde{K}(n) = \{D \in W(n) \mid D\omega_K \in A(n)\omega_K\}. \end{cases}$$

Then the Special algebra $S(n)$ $(n \geq 3)$ is the derived algebra of $\widetilde{S}(n)$, while the Hamiltonian algebra $H(n)$ $(n = 2m \geq 2)$ and the Contact algebra $K(n)$ $(n = 2m+1 \geq 3)$ are the second derived algebras of $\widetilde{H}(n)$ and $\widetilde{K}(n)$, respectively.

We want to describe more explicitly the above algebras, starting from the Special algebra $S(n)$. For every $1 \leq i, j \leq n$ consider the following maps

$$D_{ij} = -D_{ji} : \begin{cases} A(n) \longrightarrow W(n) \\ \qquad f \mapsto D_j(f)D_i - D_i(f)D_j. \end{cases}$$

Proposition 2.3. *The algebra $S(n)$ has F-dimension equal to $(n-1)(p^n-1)$ and is generated by the elements $D_{ij}(x^a)$ for $x^a \in A(n)$ and $1 \leq i < j \leq n$.*

Proof. See [FS88, Chapter 4.3]. □

Suppose now that $n = 2m \geq 2$ and consider the map $D_H : A(n) \to W(n)$ defined by

$$D_H(f) = \sum_{i=1}^{m} \left[D_i(f)D_{i+m} - D_{i+m}(f)D_i\right],$$

where, as before, $D_i := \frac{\partial}{\partial x_i} \in W(n)$. Then the Hamiltonian algebra can be described as follows:

Proposition 2.4. *The above map D_H induces an isomorphism*

$$D_H : A(n)_{\neq 1, x^\sigma} \xrightarrow{\cong} H(n),$$

where $A(n)_{\neq 1,x^\sigma} = \{x^a \in A(n) \mid x^a \neq 1, x^a \neq x^\sigma := x_1^{p-1} \cdots x_n^{p-1}\}$. *Therefore* $H(n)$ *has dimension* $p^n - 2$.

Proof. See [FS88, Chapter 4.4]. □

Suppose finally that $n = 2m + 1 \geq 3$. Consider the map $D_K : A(n) \to K(n)$ defined by

$$D_K(f) = \sum_{i=1}^{m} [D_i(f)D_{i+m} - D_{i+m}(f)D_i] + \sum_{j=1}^{2m} x_j \, [D_n(f)D_j - D_j(f)D_n] + 2fD_n.$$

Then the Contact algebra can be described as follows:

Proposition 2.5. *The above map D_K induces an isomorphism*

$$K(n) \cong \begin{cases} A(n) & \text{if } p \not| \, (m+2), \\ A(n)_{\neq x^\tau} & \text{if } p \mid (m+2), \end{cases}$$

where $A(n)_{\neq x^\tau} := \{x^a \in A(n) \mid x^a \neq x^\tau := x_1^{p-1} \cdots x_n^{p-1}\}$. *Therefore* $K(n)$ *has dimension* p^n *if* $p \not| \, (m+2)$ *and* $p^n - 1$ *if* $p \mid (m+2)$.

Proof. See [FS88, Chapter 4.5]. □

Kostrikin and Shafarevich (in the above mentioned paper [KS66]) conjectured that a restricted simple Lie algebras (that is a restricted algebras without proper ideals) over an algebraically closed field of characteristic $p > 5$ is either of classical or Cartan type. The **Kostrikin-Shafarevich conjecture** was proved by Block-Wilson (see [BW84] and [BW88]) for $p > 7$, building upon the work of Kostrikin-Shafarevich ([KS66] and [KS69]), Kac ([KAC70] and [KAC74]), Wilson ([WIL76]) and Weisfailer ([WEI78]).

Recently, Premet and Strade (see [PS97], [PS99], [PS01], [PS04]) proved the Kostrikin-Shafarevich conjecture for $p = 7$. Moreover they showed that for $p = 5$ there is only one exception, the Melikian algebra ([MEL80]), whose definition is given below.

Definition 2.6. *Let* $p = \mathrm{char}(F) = 5$. *Let* $\widetilde{W(2)}$ *be a copy of* $W(2)$ *and for an element* $D \in W(2)$ *we indicate with* $\widetilde{D}$ *the corresponding element inside* $\widetilde{W(2)}$. *The Melikian algebra* M *is defined as*

$$M = A(2) \oplus W(2) \oplus \widetilde{W(2)},$$

with Lie bracket defined by the following rules (for all $D, E \in W(2)$ *and* $f, g \in A(2)$*):*

$$\begin{cases} [D, \widetilde{E}] := \widetilde{[D, E]} + 2\,\mathrm{div}(D)\widetilde{E}, \\ [D, f] := D(f) - 2\,\mathrm{div}(D)f, \\ [f_1\widetilde{D_1} + f_2\widetilde{D_2}, g_1\widetilde{D_1} + g_2\widetilde{D_2}] := f_1g_2 - f_2g_1, \\ [f, \widetilde{E}] := fE, \\ [f, g] := 2\,(gD_2(f) - fD_2(g))\widetilde{D_1} + 2\,(fD_1(g) - gD_1(f))\widetilde{D_2}, \end{cases}$$

where $\mathrm{div}(f_1D_1 + f_2D_2) := D_1(f_1) + D_2(f_2) \in A(2)$.

In characteristic $p = 2, 3$, there are many exceptional restricted simple Lie algebras (see [STR04, page 209]) and the classification seems still far away.

3 Infinitesimal Deformations

An infinitesimal deformation of a Lie algebra L over a field F is a Lie algebra L' over $F[\epsilon]/(\epsilon^2)$ such that $L' \times_{F[\epsilon]/(\epsilon^2)} F \cong L$. Explicitly, $L' = L + \epsilon L$ with Lie bracket $[-,-]'$ defined by (for any two elements $X, Y \in L \subset L'$):

$$[X,Y]' = [X,Y] + \epsilon f(X,Y),$$

where $[-.-]$ is the Lie bracket of L and $f(-,-)$ is an 2-alternating function from L to L, considered a module over itself via adjoint representation. The Jacobi identity for $[-,-]'$ forces f to be a cocycle and moreover one can check that two cocycles differing by a coboundary define isomorphic Lie algebras. Therefore the infinitesimal deformations of a Lie algebra L are parametrized by the second cohomology $H^2(L,L)$ of the Lie algebra with values in the adjoint representation (see [GER64] for a rigorous treatment).

It is a classical result that simple Lie algebras in characteristic zero are rigid. We want to give a sketch of the proof of the following Theorem (see [HS97] for details).

Theorem 3.1. *Let L be a simple Lie algebra over a field F of characteristic* 0*. Then, for every $i \geq 0$, we have that*

$$H^i(L,L) = 0.$$

Sketch of the Proof. Since the Killing form $\beta(x,y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ is non-degenerate (by Cartan's criterion), we can choose two bases $\{e_i\}$ and $\{e_i'\}$ of L such that $\beta(e_i, e_j') = \delta_{ij}$. Consider the Casimir element $C := \sum_i e_i \otimes e_i'$ inside the enveloping algebra $\mathfrak{U}(L)$. One can check that:

1. C belongs to the center of the enveloping algebra and therefore it induces an L-homomorphism $C : L \to L$, where L is a consider a module over itself via adjoint action. Moreover since $\mathrm{tr}_L(C) = \dim(L) \neq 0$, C is non-zero and hence is an isomorphism by the simplicity of L.

2. The map induced by C on the exact complex $\{\mathfrak{U}(L) \otimes_k \bigwedge^n L\}_n \to F$ is homotopic to 0.

Therefore the induced map on cohomology $C_* : H^*(L,L) \to H^*(L,L)$ is an isomorphism by (1) and the zero map by (2), which implies that $H^*(L,L) = 0$. ◻

The above proof uses the non-degeneracy of the Killing form and the non-vanishing of the trace of the Casimir element, which is equal to the dimension of the Lie algebra. Therefore the same proof works also for the restricted simple Lie algebras of **classical type** over a field of characteristic not dividing the determinant of the Killing form and the dimension of the Lie algebra. Actually Rudakov (see [RUD71]) showed that such Lie algebras are rigid if the characteristic of the base field is greater or equal to 5 while in characteristic 2 and 3 there are non-rigid classical Lie algebras (see [CHE05], [CK00], [CKK00]).

It was already observed by Kostrikin and Džumadildaev ([DK78], [DZU80], [DZU81] and [DZU89]) that Witt-Jacobson Lie algebras admit infinitesimal deformations. More precisely: in [DK78] the authors compute the infinitesimal deformations of the Jacobson-Witt algebras of rank 1, while in [DZU80, Theorem 4], [DZU81] and [DZU89] the author describes the infinitesimal deformations of the Jacobson-Witt algebras of any rank but without a detailed proof.

In the papers [VIV1], [VIV2] and [VIV3], we computed the infinitesimal deformations of the restricted simple Lie algebras of **Cartan type** in characteristic $p \geq 5$, showing in particular that they are non-rigid. Before stating the results, we need to recall the definition of the Squaring operators ([GER64]). The Squaring of a derivation $D : L \to L$ is the 2-cochain defined, for any $x, y \in L$ as it follows

$$\mathrm{Sq}(D)(x,y) = \sum_{i=1}^{p-1} \frac{[D^i(x), D^{p-i}(y)]}{i!(p-i)!}, \tag{3.1}$$

where D^i is the i-iteration of D. Using the Jacobi identity, it is straightforward to check that $\mathrm{Sq}(D)$ is a 2-cocycle and therefore it defines a class in the cohomology group $H^2(L,L)$, which we will continue to call $\mathrm{Sq}(D)$ (by abuse of notation). Moreover for an element $\gamma \in L$, we define $\mathrm{Sq}(\gamma) := \mathrm{Sq}(\mathrm{ad}(\gamma))$.

Theorem 3.2. *We have that*

$$H^2(W(n), W(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(D_i) \rangle_F.$$

Theorem 3.3. *We have that*

$$H^2(S(n), S(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(D_i) \rangle_F \bigoplus \langle \Theta \rangle_F,$$

where Θ is defined by $\Theta(D_i, D_j) = D_{ij}(x^\tau)$.

Theorem 3.4. *Let $n = 2m + 1 \geq 3$. Then we have that*

$$H^2(K(n), K(n)) = \bigoplus_{i=1}^{2m} \langle \mathrm{Sq}(D_K(x_i)) \rangle_F \bigoplus \langle \mathrm{Sq}(D_K(1)) \rangle_F.$$

Before stating the next theorem, we need some notations about n-tuples of natural numbers. We consider the order relation inside $\mathbb{N}^n$ given by $a = (a_1, \cdots, a_n) \leq b = (b_1, \cdots, b_n)$

if $a_i \leq b_i$ for every $i = 1, \cdots, n$. We define the degree of $a \in \mathbb{N}^n$ as $|a| = \sum_{i=1}^n a_i$ and the factorial as $a! = \prod_{i=1}^n a_i!$. For two multindex $a, b \in \mathbb{N}^n$ such that $b \leq a$, we set $\binom{a}{b} := \prod_{i=1}^n \binom{a_i}{b_i} = \frac{a!}{b!(a-b)!}$. For every integer $j \in \{1, \cdots, n\}$ we call ϵ_j the n-tuple having 1 at the j-th entry and 0 outside. We denote with σ the multindex $(p-1, \cdots, p-1)$.

Assuming now that $n = 2m$, we define the sign $\sigma(j)$ and the conjugate j' of $1 \leq j \leq 2m$ as follows:

$$\sigma(j) = \begin{cases} 1 & \text{if } 1 \leq j \leq m, \\ -1 & \text{if } m < j \leq 2m, \end{cases} \qquad \text{and} \qquad j' = \begin{cases} j+m & \text{if } 1 \leq j \leq m, \\ j-m & \text{if } m < j \leq 2m. \end{cases}$$

Given a multindex $a = (a_1, \cdots, a_{2m}) \in \mathbb{N}^{2m}$, we define the sign of a as $\sigma(a) = \prod \sigma(i)^{a_i}$ and the conjugate of a as the multindex $\hat{a}$ such that $\hat{a}_i = a_{i'}$ for every $1 \leq i \leq 2m$.

Theorem 3.5. *Let $n = 2m \geq 2$. Then if $n \geq 4$ we have that*

$$H^2(H(n), H(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(D_H(x_i)) \rangle_F \bigoplus_{\substack{i<j \\ j \neq i'}} \langle \Pi_{ij} \rangle_F \bigoplus_{i=1}^{m} \langle \Pi_i \rangle_F \bigoplus \langle \Phi \rangle_F,$$

where the above cocycles are defined (and vanish outside) by

$$\begin{cases} \Pi_{ij}(D_H(x^a), D_H(x^b)) = D_H(x_{i'}^{p-1} x_{j'}^{p-1}[D_i(x^a)D_j(x^b) - D_i(x^b)D_j(x^a)]), \\ \Pi_i(D_H(x_i x^a), D_H(x_{i'} x^b)) = D_H(x^{a+b+(p-1)\epsilon_i+(p-1)\epsilon_{i'}}), \\ \Pi_i(D_H(x_k), D_H(x^{\sigma-(p-1)\epsilon_i-(p-1)\epsilon_{i'}})) = -\sigma(k) D_H(x^{\sigma-\epsilon_{k'}}) \ \textit{for } 1 \leq k \leq n, \\ \Phi(D_H(x^a), D_H(x^b)) = \displaystyle\sum_{\substack{\delta \leq a, \hat{b} \\ |\delta|=3}} \binom{a}{\delta} \binom{b}{\hat{\delta}} \sigma(\delta)\, \delta!\, D_H(x^{a+\hat{b}-\delta-\hat{\delta}}). \end{cases}$$

If $n = 2$ then we have that

$$H^2(H(2), H(2)) = \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_H(x_i)) \rangle_F \bigoplus \langle \Phi \rangle_F.$$

Theorem 3.6. *We have that*

$$H^2(M, M) = \langle \mathrm{Sq}(1) \rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i) \rangle_F \bigoplus_{i=1}^{2} \mathrm{Sq}(\widetilde{D_i}) \rangle_F.$$

4 Open Problems

Simple Lie algebras (not necessarily restricted) over an algebraically closed field F of characteristic $p \neq 2, 3$ have been classified by Strade and Wilson for $p > 7$ (see [SW91], [STR89], [STR92], [STR91], [STR93], [STR94], [STR98]) and by Premet-Strade for $p = 5, 7$ (see [PS97], [PS99], [PS01], [PS04]). The classification says that for $p \geq 7$ a simple Lie algebra is of classical type (and hence restricted) or of generalized Cartan type. Those latter are generalizations of the Lie algebras of Cartan type, obtained by considering higher truncations of divided power algebras (not just p-truncated polynomial algebras) and by considering only the subalgebra of (the so called) special derivations (see [FS88] or [STR04] for the precise definitions). Again in characteristic $p = 5$, the only exception is represented by the generalized Melikian algebras. Therefore an interesting problem would be the following:

Problem 1. *Compute the infinitesimal deformations of the simple Lie algebras.*

Note that there is an important distinction between **restricted simple** Lie algebras and **simple restricted** Lie algebras. The former algebras are the restricted Lie algebras which do not have any nonzero proper ideal, while the second ones are the restricted Lie algebras which do not have any nonzero proper restricted ideal (or p-ideal), that is an ideal closed under the p-map. Clearly every restricted simple Lie algebra is a simple restricted Lie algebra, but a simple restricted Lie algebra need not be a simple Lie algebras. Indeed we have the following

Proposition 4.1. *There is a bijection*

$$\{\textit{Simple restricted Lie algebras}\} \longleftrightarrow \{\textit{Simple Lie algebras}\}.$$

Explicitly to a simple restricted Lie algebra $(L, [p])$ *we associates its derived algebra* $[L, L]$*. Conversely to a simple Lie algebra* M *we associate the restricted subalgebra* $M^{[p]}$ *of* $\mathrm{Der}_F(M)$ *generated by* $\mathrm{ad}(M)$ *(which is called the universal* p*-envelope of* M*).*

Proof. We have to prove that the above maps are well-defined and are inverse one of the other.

• Consider a simple restricted Lie algebra $(L, [p])$. The derived subalgebra $[L, L] \lhd L$ is a non-zero ideal (since L can not be abelian) and therefore $[L, L]_p = L$, where $[L, L]_p$ denotes the p-closure of $[L, L]$ inside L.

Take a non-zero ideal $0 \neq I \lhd [L, L]$. Since $[L, L]_p = L$, we deduce from [FS88, Chapter 2, Prop. 1.3] that I is also an ideal of L and therefore $I_p = L$ by restricted simplicity of $(L, [p])$. From loc. cit., it follows also that $[L, L] = [I_p, I_p] = [I, I] \subset I$ from which we deduce that $I = L$. Therefore $[L, L]$ is simple.

Since $\mathrm{ad} : L \to \mathrm{Der}_F(L)$ is injective and $[L, L]_p = L$, it follows by loc. cit. that $\mathrm{ad} : L \to \mathrm{Der}_F([L, L])$ is injective. Therefore we have that $[L, L] \subset L \subset \mathrm{Der}_F([L, L])$ and hence $[L, L]^{[p]} = [L, L]_p = L$.

• Conversely, start with a simple Lie algebra M and consider its universal p-envelop $M < M^{[p]} < \mathrm{Der}_F(M)$.

Take any restricted ideal $I \lhd_p M^{[p]}$. By loc. cit., we deduce $[I, M^{[p]}] \subset I \cap [M^{[p]}, M^{[p]}] = I \cap [M, M] = I \cap M \lhd M$. Therefore, by the simplicity of M, either $I \cap M = M$ or $I \cap M = 0$. In the first case, we have that $M \subset I$ and therefore $M^{[p]} = I$. in the second case, we have that $[I, M^{[p]}] = 0$ and therefore $I = 0$ because $M^{[p]}$ has trivial center. We conclude that $M^{[p]}$ is simple restricted.

Moreover, by loc. cit., we have that $[M^{[p]}, M^{[p]}] = [M, M] = M$.

□

Therefore the preceding classifications of simple Lie algebras (for $p \neq 2, 3$) give a classification of simple restricted Lie algebras.

Problem 2. *Compute the infinitesimal deformations of the simple restricted Lie algebras.*

There is an important connection between simple restricted Lie algebras and **simple finite group schemes**.

Proposition 4.2. *Over an algebraically closed field F of characteristic $p > 0$, a simple finite group scheme is either the constant group scheme associated to a simple finite group or it is the finite group scheme of height 1 associated to a simple restricted Lie algebra.*

Proof. Let G be a simple finite group scheme. The kernel of the Frobenius map $F : G \to G^{(p)}$ is a normal subgroup and therefore, by the simplicity of G, we have that either $\mathrm{Ker}(F) = 0$ or $\mathrm{Ker}(F) = G$. In the first case, the group G is constant (since $F = \overline{F}$), and therefore it corresponds to an (abstract) simple finite group. In the second case, the group G is of height 1 and therefore the result follows from Proposition 1.3. □

The following problem seems very interesting.

Problem 3. *Compute the infinitesimal deformations of the simple finite group schemes.*

Since constant finite group schemes (or more generally étale group schemes) are rigid, one can restrict to the simple finite group schemes of height 1 associated to the simple restricted Lie algebras. Moreover, if $(L, [p])$ is the simple restricted Lie algebra corresponding to the simple finite group scheme G, then the infinitesimal deformations of G correspond to restricted infinitesimal deformations of $(L, [p])$, that are infinitesimal deformations that admit a restricted structure. These are parametrized by the second restricted cohomology group $H^2_*(L, L)$ (defined in [HOC54]). Therefore the above Problem 3 is equivalent to the following:

Problem 4. *Compute the restricted infinitesimal deformations of the simple restricted Lie algebras.*

The above Problem 4 is closely related to Problem 2 because of the following spectral sequence relating the restricted cohomology to the ordinary one (see [FAR91]):

$$E_2^{p,q} = \mathrm{Hom}_{\mathrm{Frob}}\left(\bigwedge^q L, H^p_*(L, L)\right) \Rightarrow H^{p+q}(L, L),$$

where $\mathrm{Hom}_{\mathrm{Frob}}$ denote the homomorphisms that are semilinear with respect to the Frobenius.

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