

The local structure of compactified Jacobians

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ABSTRACT

This paper studies the local geometry of compactified Jacobians. The main result is a presentation of the completed local ring of the compactified Jacobian of a nodal curve as an explicit ring of invariants described in terms of the dual graph of the curve. The authors have investigated the geometric and combinatorial properties of these rings in previous work, and consequences for compactified Jacobians are presented in this paper. Similar results are given for the local structure of the universal compactified Jacobian over the moduli space of stable curves.

1. Introduction

This paper studies the local geometry of compactified Jacobians associated to nodal curves. These are projective varieties that play a role similar to that of the Jacobian variety for a non-singular curve. Recall that a Jacobian can be viewed as the moduli space of line bundles (of fixed degree) on a non-singular curve. A compactified Jacobian is an analogous parameter space associated to a nodal curve. A major barrier to constructing these spaces is that, while the moduli space of fixed-degree line bundles on a nodal curve exists, it typically does not have nice properties: often it has infinitely many connected components (that is, is not of finite type), and these components fail to be proper. To construct a well-behaved compactified Jacobian, one must modify the moduli problem. There are a number of different ways to do this, and the literature on the compactification problem is vast (for example, [4, 7, 14, 16, 18, 21, 22, 28, 36, 38, 41, 42]).

Geometric invariant theory (GIT) provides a general framework for these types of compactification problems, and in this approach, a compactified Jacobian $\bar{J}(X)$ of a nodal curve X is constructed as a coarse moduli space of certain line bundles together with their degenerations: rank 1, torsion-free sheaves. The sheaves parameterized by $\bar{J}(X) = \bar{J}_\phi(X)$ are those rank 1, torsion-free sheaves that are semistable with respect to a numerical polarization ϕ (see Definitions 2.1 and 2.2). As explained in [30, Section 2], this semistability condition generalizes the other known semistability conditions that appear in the work of Oda–Seshadri [36], Seshadri [41], Esteves [16] as well as the more familiar slope semistability condition with respect to an ample line bundle that appears in the work of Simpson [42]. In general, compactified Jacobians are non-fine moduli spaces because typically there are non-isomorphic semistable sheaves I and I' that correspond to the same point $[I] = [I'] \in \bar{J}(X)$. Indeed, this happens precisely when two Jordan–Hölder filtrations of I and I' have the same stable factors. If a Jordan–Hölder filtration of I splits, that is, if I is the direct sum of stable sheaves supported on subcurves of X , then we say that I is polystable. Therefore, given a point $x \in \bar{J}(X)$, there exists a unique polystable sheaf I such that $[I] = x \in \bar{J}(X)$; see §2.2 for more details.

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One motivation for constructing compactified Jacobians is that they provide degenerations of Jacobian varieties. Given a family of non-singular curves specializing to a nodal curve, the compactified Jacobian of the nodal fiber fits into a family that extends the family consisting of the Jacobians of the non-singular fibers. Note that because the coarse moduli space of stable curves \overline{M}_g does not admit a universal curve, this does not imply that the compactified Jacobians fit into a family over \overline{M}_g . However, Caporaso [7] (and later Pandharipande [38]) has constructed a family $\Phi : \overline{J}_{d,g} \rightarrow \overline{M}_g$ (which we call the universal compactified Jacobian) of projective schemes that extends the Jacobian of the generic genus g curve. The fiber of Φ over a stable curve X is isomorphic to a certain compactified Jacobian of X , modulo its automorphism group (see Fact 2.12).

The main result of this paper describes the local geometry of both a compactified Jacobian $\overline{J}(X)$ of a nodal curve X , and of the universal compactified Jacobian $\overline{J}_{d,g}$ at a point corresponding to an automorphism-free stable curve.

THEOREM A. *Let X be a nodal curve of arithmetic genus $g(X)$, let I be a rank 1, torsion-free sheaf on X , and let Σ be the set of nodes where I fails to be locally free. Set $\Gamma = \Gamma_X(\Sigma)$ to be the dual graph of any curve obtained from X by smoothing the nodes not in Σ . Fix an arbitrary orientation on Γ , and denote by $V(\Gamma)$, $E(\Gamma)$, and $s, t : E(\Gamma) \rightarrow V(\Gamma)$ the vertices, edges and source and target maps, respectively. Set $b_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1$. Let*

$$T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m, \quad \widehat{A}(\Gamma) := \frac{k[[X_e, Y_e : e \in E(\Gamma)]]}{(X_e Y_e : e \in E(\Gamma))} \quad \text{and} \quad \widehat{B}(\Gamma) := \frac{k[[X_e, Y_e, T_e : e \in E(\Gamma)]]}{(X_e Y_e - T_e : e \in E(\Gamma))}.$$

Define an action of the torus T_Γ on $\widehat{A}(\Gamma)$ and $\widehat{B}(\Gamma)$ by the rule that $\lambda = (\lambda_v)_{v \in V(\Gamma)} \in T_\Gamma$ acts as

$$\lambda \cdot X_e = \lambda_{s(e)} X_e \lambda_{t(e)}^{-1}, \quad \lambda \cdot Y_e = \lambda_{t(e)} Y_e \lambda_{s(e)}^{-1} \quad \text{and} \quad \lambda \cdot T_e = T_e.$$

Define complete local rings

$$R_I := \widehat{A}(\Gamma)[[W_1, \dots, W_{g(X)-b_1(\Gamma)}]] \quad \text{and} \quad R_{(X,I)} = \widehat{B}(\Gamma)[[W_1, \dots, W_{4g-3-b_1(\Gamma)-\#E(\Gamma)}]],$$

with actions of T_Γ induced by the actions on $\widehat{A}(\Gamma)$ and $\widehat{B}(\Gamma)$, and the trivial action on the remaining generators.

(i) Suppose that $\overline{J}(X)$ is a compactified Jacobian of X . If $[I] \in \overline{J}(X)$ with I polystable, then there is an isomorphism

$$\widehat{\mathcal{O}}_{\overline{J}(X),[I]} \cong R_I^{T_\Gamma}$$

between the completed local ring of $\overline{J}(X)$ at $[I]$ and the T_Γ -invariant subring of R_I .

(ii) If X is a stable curve with trivial automorphism group and $[(X, I)] \in \overline{J}_{d,g}$ with I polystable, then there is an isomorphism

$$\widehat{\mathcal{O}}_{\overline{J}_{d,g},[(X,I)]} \cong R_{(X,I)}^{T_\Gamma}$$

between the completed local ring of $\overline{J}_{d,g}$ at $[(X, I)]$ and the T_Γ -invariant subring of $R_{(X,I)}$.

Theorem A is a consequence of Theorems 5.10 and 6.1 (see also Remarks 5.9 and 6.2). We discuss the proof in more detail below. The rings $\widehat{A}(\Gamma)$ appearing above are further studied in [12]. In the notation of that paper, $\widehat{A}(\Gamma)$ is the completion of the ring $A(\Gamma)$ defined in [12, Theorem A], and the action of T_Γ in both papers is the same. It is shown in [12, Theorem A] that the invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic ring

$R(\Gamma)$ defined in [12, Definition 1.4]. In particular, the completed local ring of the compactified Jacobian is isomorphic to a power series ring over a completion of the cographic toric face ring $R(\Gamma)$. A number of geometric properties of cographic rings are established in [12], and some consequences for compactified Jacobians are discussed in Theorem B. The geometric and combinatorial properties of the rings $\widehat{B(\Gamma)}^{Tr}$ will be described in more detail by the authors in [13].

THEOREM B. *Let $\bar{J}(X)$ be a compactified Jacobian of a nodal curve X .*

(i) *The compactified Jacobian $\bar{J}(X)$ has Gorenstein, semi-log-canonical (slc) singularities. In particular, $\bar{J}(X)$ is seminormal.*

(ii) *Let $[I] \in \bar{J}(X)$ with I polystable. Then $[I]$ lies in the smooth locus of $\bar{J}(X)$ if and only if I fails to be locally free only at separating nodes of the dual graph of X .*

The proof is given at the end of §6. In [12], it is shown that a number of further properties of cographic rings can be determined from elementary combinatorics of the graph $\Gamma = \Gamma_X(\Sigma)$ introduced in Theorem A. For instance, that paper provides combinatorial formulas giving the embedding dimension and the multiplicity of $\widehat{\mathcal{O}}_{\bar{J}(X),x}$, as well as a description of the irreducible components and the normalization of this ring. The reader is directed to §7 and [12] for more details. We also point out that it is well known that the completed local ring of $\bar{J}(X)$ at a *stable* sheaf is isomorphic to a completed product of nodes and smooth factors. Using Theorem A and the results of [12] one can construct examples of compactified Jacobians whose structure at a strictly semistable point is more complicated (see, especially, §7.2).

We prove the theorems using deformation theory. The basic strategy is to show that the local structure of a compactified Jacobian is given by the subring of the miniversal deformation ring that consists of elements invariant under an action of the automorphism group. Let us sketch the proof for a compactified Jacobian $\bar{J}(X)$ of a nodal curve X (the case of the universal compactified Jacobian $\bar{J}_{d,g}$ is similar). To begin with, there is a well-known explicit description of the miniversal deformation ring R_I of a rank 1, torsion-free sheaf I on a nodal curve X (see Corollary 3.17), and we use that description to define an explicit action of $\text{Aut}(I)$ on R_I (see Theorems 5.10). We prove the main result by showing that, when $[I] \in \bar{J}(X)$ with I polystable, the ring of invariants $R_I^{\text{Aut}(I)}$ is isomorphic to the completed local ring of $\bar{J}(X)$ at $[I]$.

In order to establish this last point, we use the GIT construction of $\bar{J}(X)$ together with the Luna Slice Theorem and a theorem of Rim. Recall that the compactified Jacobian is constructed as a GIT quotient of a suitable Quot scheme $\text{Quot}(\mathcal{O}_X^{\oplus r})$ by the action of SL_r (see Corollary 2.10). We check that the complete local ring R_x of a slice (which exists by Luna Slice Theorem) at a polystable point $x = [\mathcal{O}_X^{\oplus r} \rightarrow I] \in \text{Quot}(\mathcal{O}_X^{\oplus r})$ is a miniversal deformation ring for I (Lemma 6.4). Thus R_x is (non-canonically) isomorphic to R_I . By definition, the stabilizer G_x of x (which is described in Lemma 6.6) acts on the ring R_x , and it follows from the definition of a slice that the invariant ring $R_x^{G_x}$ is isomorphic to the complete local ring of the GIT quotient at the image of the point x . We complete the proof by using a theorem of Rim (Fact 5.4) to identify the action of G_x on R_x to our explicit action of $\text{Aut}(I)$ on R_I , completing the proof (see Theorems 5.10 and 6.1).

Theorem B is one consequence of Theorem A. Other consequences will be found in the upcoming article [13]. There the authors will use Theorem A to describe the singularities of $\bar{J}_{d,g}$. More precisely, they will prove that $\bar{J}_{d,g}$ has canonical singularities provided $\text{char}(k) = 0$. When X does not admit a non-trivial automorphism, the authors will prove this result by using the explicit description of the completed local ring in Theorem A, and in general, they will reduce the proof to a similar argument using a generalization of the Reid–Tai–Shepherd–Barron criterion for toric singularities. The results in [13] will extend the work of Bini, Fontanari and the third author [6], where it is shown that $\bar{J}_{d,g}$ has canonical singularities when

$\gcd(d+1-g, 2g-2) = 1$, a condition equivalent to the condition that $\bar{J}_{d,g}$ has finite quotient singularities. Under the same assumption on d and g , the same authors computed the Kodaira dimension and the Itaka fibration of $\bar{J}_{d,g}$ ([6, Theorem 1.2]), and in [13], the present authors will extend that computation to all d, g .

The authors also hope to use the results of this paper to study the singularities of the theta divisor of a nodal curve. The theta divisor is an ample effective Cartier divisor on the canonical compactified Jacobian of degree $g-1$, parameterizing sheaves with a non-trivial section (see [2, 9, 10]). The case of integral nodal curves has been studied by the first two authors in [11], where an analog of the Riemann singularity theorem is proved. The authors are currently investigating how to extend the Riemann singularity theorem to non-integral nodal curves, based upon the explicit local description of the compactified Jacobian obtained in this paper.

This paper suggests two technical questions for future study. In Theorem 6.1(ii), the curve X is assumed to be automorphism-free. It is particularly difficult to describe the local structure of $\bar{J}_{d,g}$ when X admits an automorphism of order equal to p , the characteristic of k . When X admits such an automorphism, $\text{Aut}(X, I)$ is reductive but not linearly reductive. Linear reductivity is crucial in two places: in the proof of Theorem 6.1, which uses the Luna Slice Theorem, and Theorem 5.10, which uses a result of Rim. It would be interesting to know if suitable generalizations of Rim's Theorem and the Luna Slice Theorem hold for reductive groups such as $\text{Aut}(X, I)$. We discuss these issues after the proofs of the two theorems.

Positive characteristic issues also appear in Fact 2.12, which relates the fibers of $\bar{J}_{d,g} \rightarrow \bar{M}_g$ to compactified Jacobians. That result is only stated in characteristic 0, and it would be interesting to know if the result remains valid in positive characteristic. This is discussed in greater detail immediately after the proof of the fact.

There are approaches to describing the local structure of a compactified Jacobian different from the approach taken here. Alexeev has proved in [2, Theorem 5.1] that compactified Jacobians are stable semiabelic varieties in the sense of [1], and consequently can be described using Mumford's construction [31] of degenerations of abelian varieties. In Mumford's approach (that has been further developed in [1, 3, 33, 34]), one compactifies a semiabelian variety by first forming the projectivization of a (non-finitely generated) graded algebra and then quotienting out by a lattice. This procedure provides direct access to the local structure of the compactification, and thus Alexeev's work provides another approach to studying the local structure of compactified Jacobians. It would be interesting to compare the descriptions arising from this approach to the descriptions given in this paper, but we do not pursue this topic here.

The results of Theorem B are related to some results in the literature. Specifically, it was known that $\bar{J}(X)$ is seminormal [2, Theorem 5.1] and Gorenstein [3, Lemma 4.1]. In personal correspondence, Alexeev has explained to the authors that the techniques of those papers can also be used to establish the fact that $\bar{J}(X)$ is semi-log canonical. The description of the smooth locus of $\bar{J}(X)$ is certainly well known to the experts (see, for example, [7, Theorem 6.1(3)]; [8, Theorem 7.9(iii)]; [9, Fact 4.1.5(iv)]; [30, Fact 1.19(ii)]); however, it seems that a proof has not appeared in print.

This paper is organized as follows. We review the definition and the GIT construction of (universal) compactified Jacobians in §2 with the goal of collecting the facts needed to prove Theorems A and B. The proofs of the main theorems begin in §3, where we develop the deformation theory needed to compute deformation rings parameterizing deformations of a rank 1, torsion-free sheaf. These rings admit natural actions of automorphism groups, which are described in the next two sections. The structure of the automorphism groups is studied in §4, and then those results are used in §5 to compute group actions. Finally, in §6 we prove the main results of this paper by using the Luna Slice Theorem to relate the local structure of a compactified Jacobian to a deformation ring. In §7, we describe some examples using results of [12].

Conventions

1.1. We denote by k an algebraically closed field (of arbitrary characteristic). All schemes are k -schemes, and all morphisms are implicitly assumed to respect the k -structure.

1.2. A *curve* is a connected, complete, reduced scheme (over k) of pure dimension 1. We denote by ω_X the *dualizing sheaf* of X . The *genus* $g(X)$ of a curve X is $g(X) := h^1(X, \mathcal{O}_X)$.

1.3. A *subcurve* Y of a curve X is a closed k -scheme $Y \hookrightarrow X$ that is reduced and of pure dimension 1 (but possibly disconnected). A subcurve $Y \subseteq X$ is said to be *proper* if it is non-empty and different from X . Given a subcurve Y , the *complementary subcurve* Y^c is defined to be $\overline{X \setminus Y}$, or, in other words, Y^c is the subcurve that is the union of all the irreducible components of X that are not contained in Y .

1.4. A *family of curves* is a proper, flat morphism $X \rightarrow T$ whose geometric fibers are curves.

1.5. A *family of coherent sheaves* on a family of curves $X \rightarrow T$ is a \mathcal{O}_T -flat, finitely presented \mathcal{O}_X -module I .

1.6. A coherent sheaf I on a curve X is said to be:

- (i) of *rank 1* if I has rank 1 at every generic point;
- (ii) *pure* if, for every non-zero subsheaf $J \subseteq I$, the dimension of the support of J is equal to the dimension of the support of I ;
- (iii) *torsion-free* if it is pure and the support of I is X .

The *degree* of a torsion-free, rank 1 sheaf I on a curve X is defined to be $\deg I := \chi(I) - \chi(\mathcal{O}_X)$, where χ denotes the Euler characteristic.

2. Preliminaries on GIT and compactified Jacobians

Here we review the definition and the construction of compactified Jacobians of a fixed nodal curve as well as of the universal compactified Jacobian, with the goal of collecting the results needed to prove Theorem 6.1.

2.1. Geometric invariant theory

The (universal) compactified Jacobians are coarse moduli spaces of sheaves constructed using GIT and, in the proof of our results, we will need to make use of their construction, and not just the fact of their existence. Therefore, we will quickly review some background from GIT.

Recall that GIT is a tool for constructing a quotient of a quasi-projective variety Q by the action of a reductive group G . Given an auxiliary ample line bundle $\mathcal{O}(1)$ together with a lift of the action of G on Q to an action on $\mathcal{O}(1)$ (that is, a linearization), there is distinguished open subscheme Q^{ss} of Q that consists of points that are semistable with respect to the linearized action. The significance of Q^{ss} is that it admits a *categorical quotient* that we define to be the GIT quotient of Q , written as $Q//G$; that is, there exists a pair $(Q^{\text{ss}}/G, \pi)$ consisting of a quasi-projective variety Q^{ss}/G and a G -invariant map

$$\pi: Q^{\text{ss}} \longrightarrow Q^{\text{ss}}/G$$

with the property that π is universal among all G -invariant maps out of Q^{ss} . When the characteristic of k is 0, the pair (Q^{ss}, π) is actually a *universal categorical quotient*, that is,

for any morphism $T \rightarrow Q^{\text{ss}}/G$ the base change morphism $\pi_T: Q^{\text{ss}} \times_{Q^{\text{ss}}/G} T \rightarrow T$ is again a categorical quotient.

The local structure of $Q//G$ is described by the Luna Slice Theorem, which compares $Q//G$ to the quotient of a certain model G -space. For the remainder of §2.1, we assume that Q is affine, so $Q = Q^{\text{ss}}$ and $Q//G$ is the categorical quotient [32, Theorem 1, p. 27]. The model scheme is $G \times_H V$, whose definition we now review. Suppose $H \subset G$ is a reductive subgroup and V is a scheme with a left H -action. The product $G \times V$ carries an H -action defined by

$$h \cdot (g, x) := (gh^{-1}, h \cdot x),$$

and we write $G \times_H V$ for the categorical quotient. This quotient admits a left action of G defined by the translation action on the first factor. The two projections out of $G \times V$ induce morphisms

$$p: G \times_H V \longrightarrow V/H \quad \text{and} \quad q: G \times_H V \longrightarrow G/H.$$

The first map is G -invariant and realizes V/H as the quotient of $G \times_H V$ by G . The map q is equivariant and can often be described as a contraction onto an orbit. To be precise, suppose that we are given an element $v_0 \in V$ fixed by H . One may verify that the image of $(e, v_0) \in G \times V$ in $G \times_H V$ has stabilizer H , and the associated orbit map defines a section of q .

The Luna Slice Theorem provides sufficient conditions for $Q//G$ to be étale locally isomorphic to V/H for a suitable H and V . More precisely, let x be a point of Q with stabilizer H . Given any affine, locally closed subscheme $V \subset Q$ that contains x and is stabilized by H (that is, $H \cdot V \subset V$), the action map induces a G -equivariant morphism $G \times_H V \rightarrow Q$. We say that V is a *slice* at x if the following conditions are satisfied.

- (1) The morphism $G \times_H V \rightarrow Q$ is étale.
- (2) The image of $G \times_H V \rightarrow Q$ is an open affine $U \subset Q$ that is π -saturated (that is, for each $u \in U$, $\pi^{-1}(\pi(u)) \subseteq U$).
- (3) The induced morphism $(G \times_H V)/G \rightarrow U/G$ is étale.
- (4) The induced morphism $G \times_H V \rightarrow U \times_{U/G} V/H$ is an isomorphism.

Note in particular that condition (3) together with the observation above on the map p implies that there is an étale morphism

$$V/H \xrightarrow{\text{ét}} Q//G. \tag{2.1}$$

The original Luna Slice Theorem [24, p. 97] states that in characteristic zero a slice exists, provided that x is (GIT-)polystable, that is, the orbit of x is closed. When x has a closed orbit, Matsushima’s criterion implies that the stabilizer H is reductive ([27] for $k = \mathbb{C}$; [39] for k arbitrary). Bardsley and Richardson have extended the Luna Slice Theorem to arbitrary characteristic. With no assumptions on $\text{char}(k)$, they prove that a slice exists, provided the orbit of x is closed and the stabilizer H is reduced and linearly reductive ([5, Proposition 7.6]; the condition in [5] that the orbit is separable is equivalent to our condition that H is reduced).

2.2. Compactified Jacobians of nodal curves

In this subsection, we review the definition of compactified Jacobians of a nodal curve X .

Any (known) compactified Jacobian of X parameterizes torsion-free, rank-1 sheaves on X that are semistable with respect to some polarization. There are several ways to define a polarization and the associated semistability condition on X . The most general definition is stated in terms of numerical polarizations (following [30, Section 2.4]): all the other known semistability conditions are special cases of this one; see [30, Section 2].

DEFINITION 2.1. Let X be a nodal curve with irreducible components $\{X_1, \dots, X_\gamma\}$. A numerical polarization on X is a γ -tuple of rational numbers $\phi = \{\phi_i = \phi_{X_i}\}_{i=1}^\gamma \in \mathbb{Q}^\gamma$, one for each irreducible component of X , such that $|\phi| := \sum_i \phi_i \in \mathbb{Z}$. For any subcurve Y of X , we set $\phi_Y = \sum_{X_i \subseteq Y} \phi_{X_i} \in \mathbb{Q}$. For any subcurve $Y \subset X$ such that $\phi_Y - \#(Y \cap Y^c)/2 \in \mathbb{Z}$, we define a numerical polarization ϕ^Y on Y by setting

$$(\phi^Y)_{Y_i} := \phi_{Y_i} - \frac{\#(Y_i \cap Y^c)}{2} \text{ for any irreducible component } Y_i \text{ of } Y.$$

The semistability of a torsion-free, rank 1 sheaf on X with respect to a numerical polarization ϕ is defined as follows.

DEFINITION 2.2. Let X be a nodal curve and let $\phi = (\phi_i)$ be a numerical polarization on X .

- (i) A torsion-free, rank 1 sheaf I on X is said to be ϕ -semistable if $\deg I = |\phi|$ and

$$\deg(I_Y) \geq \phi_Y - \frac{\#(Y \cap Y^c)}{2}, \tag{2.2}$$

for any subcurve $Y \subseteq X$, where I_Y denotes the biggest torsion-free quotient of the restriction $I|_Y$ of I to Y .

- (ii) A torsion-free, rank 1 sheaf I on X is said to be ϕ -stable if it is ϕ -semistable and the inequality (2.2) is strict for every proper subcurve $\emptyset \neq Y \subsetneq X$.

- (iii) A torsion-free, rank 1 sheaf I on X is said to be ϕ -polystable if it is ϕ -semistable and, for all subcurves Y for which inequality (2.2) is an equality, it holds that $I = I_Y \oplus I_{Y^c}$.

The ϕ -semistability condition is stated as a lower bound on the multidegree of I . However, this implies also an upper bound on the multidegree of I , as we observe in the following remark.

REMARK 2.3. Let ϕ be a numerical polarization on a nodal curve X and let I be a torsion-free rank 1 sheaf on X of degree $\deg I = |\phi|$. Then I is ϕ -semistable if and only if

$$\deg(I_Y) \leq \phi_Y + \frac{\#(Y \cap Y^c)}{2} - \#\{p \in Y \cap Y^c : I \text{ fails to be locally free at } p\} \tag{2.3}$$

holds for every subcurve Y of X . Indeed, this follows from the two easily checked formulas

$$\begin{cases} \deg(I_Y) + \deg(I_{Y^c}) = \deg I - \#\{p \in Y \cap Y^c : I \text{ fails to be locally free at } p\}, \\ \phi_Y + \phi_{Y^c} = |\phi|. \end{cases}$$

The condition of being polystable is better understood in terms of the *Jordan–Hölder* filtration. Recall that, given a ϕ -semistable sheaf I , a *Jordan–Hölder* filtration of I is a filtration

$$0 = I_{q+1} \subsetneq I_q \subsetneq \dots \subsetneq I_1 \subsetneq I_0 = I,$$

with the following properties:

- (1) The sheaf I_k is a rank 1, torsion-free sheaf supported on a subcurve $Z_k \subset X$ and ϕ^{Z_k} -semistable for every $0 \leq k \leq q$.
- (2) The quotient sheaf I_k/I_{k+1} is a rank 1, torsion-free sheaf supported on the subcurve $Y_k = Z_k \setminus Z_{k+1}$ and ϕ^{Y_k} -stable for every $0 \leq k \leq q$.

Jordan–Hölder filtrations exist for every ϕ -semistable sheaf I but they are not unique; however, the graded sheaf

$$\text{Gr}(I) := I_0/I_1 \oplus \dots \oplus I_q/I_{q+1}$$

depends only on I (see, for example, [16, § 1.3; 30, § 2.5]). Then it is easy to check that I is polystable if and only if $I \cong \text{Gr}(I)$. Moreover, we say that two ϕ -semistable sheaves I and I' are *S-equivalent* (or Jordan–Hölder equivalent) if $\text{Gr}(I) \cong \text{Gr}(I')$. Therefore, every ϕ -semistable sheaf is *S-equivalent* to a unique ϕ -polystable sheaf, namely $\text{Gr}(I)$.

With the above definitions, we can now introduce the ϕ -compactified Jacobian functor

$$\bar{J}_\phi^\sharp(X) : k\text{-Sch} \longrightarrow \text{Sets},$$

which associates to a k -scheme T the set of families of coherent sheaves on $X \times_k T \rightarrow T$ that are fiberwise rank 1, torsion-free and ϕ -semistable.

FACT 2.4 (Oda–Seshadri, Seshadri). There exists a projective variety $\bar{J}_\phi(X)$, called the ϕ -compactified Jacobian or simply *compactified Jacobian*, that co-represents the functor $\bar{J}_\phi^\sharp(X)$. Moreover, two sheaves $I, I' \in \bar{J}_\phi^\sharp(X)(k)$ define the same k -point $[I] = [I'] \in \bar{J}_\phi(X)$ if and only if I and I' are S-equivalent. In particular, every k -point of $\bar{J}_\phi(X)$ is equal to $[I]$ for a unique ϕ -polystable sheaf I .

Recall that the fact that $\bar{J}_\phi(X)$ co-represents $\bar{J}_\phi^\sharp(X)$ means that there exist a natural transformation of functors $\pi : \bar{J}_\phi^\sharp(X) \rightarrow \text{Hom}(-, \bar{J}_\phi(X))$ which is universal with respect to natural transformations from $\bar{J}_\phi^\sharp(X)$ to the functor of points of k -schemes. Given a point $I \in \bar{J}_\phi^\sharp(X)(k)$, we set $[I] := \pi(I) \in \text{Hom}(\text{Spec } k, \bar{J}_\phi(X)) = \bar{J}_\phi(X)(k)$.

Proof. This is proved by Oda–Seshadri [36, Theorems 11.4 and 12.14] and Seshadri [41, Theorem 15]. Note that in [41] the authors use two different definitions of ϕ -(semi)stability, which are, however, equivalent to our definition as discussed in [2, § 2.1] and [30, § 2]. \square

REMARK 2.5. If the numerical polarization ϕ is such that $\phi_Y - \#(Y \cap Y^c)/2 \notin \mathbb{Z}$ for every proper subcurve $\emptyset \neq Y \subsetneq X$ (in which case we say that ϕ is *general*), then it follows from Definition 2.2 that every ϕ -semistable sheaf is also ϕ -stable. Hence, $\bar{J}_\phi(X)$ is a fine moduli space parameterizing ϕ -stable sheaves and we say that $\bar{J}_\phi(X)$ is a *fine compactified Jacobian*. Such compactified Jacobians are studied in [30] and in [29].

We now compare the ϕ -semistability condition introduced above with the (more familiar) notion of slope semistability. Recall that, given a nodal curve X and a polarization L on X , that is, an ample line bundle on X , the *slope* $\mu_L(I)$ of a coherent sheaf I with respect to L is defined to be a/r , where a and r are coefficients of the Hilbert polynomial $P_L(I, t) := r \cdot t + a$ of I with respect to L .

DEFINITION 2.6. Let X be a nodal curve and L be a polarization on X .

(i) The sheaf I is said to be *slope semistable* (respectively, *slope stable*) with respect to the polarization L if it is pure and satisfies $\mu_L(I) \leq \mu_L(J)$ (respectively, $<$) for all pure non-trivial quotients $I \twoheadrightarrow J$ with one-dimensional support $\text{Supp}(J)$.

(ii) The sheaf I is said to be *slope polystable* if it is slope semistable and isomorphic to a direct sum of slope stable sheaves.

With the above definitions, we can now introduce the *Simpson Jacobian functor* of degree d to be the functor

$$\bar{J}_{L,d}^\sharp(X) : S\text{-Sch} \longrightarrow \text{Sets}$$

which sends a k -scheme T into the set of families of coherent sheaves on $X \times_k T \rightarrow T$ that are fiberwise rank 1, torsion-free of degree d and slope semistable with respect to the polarization L .

FACT 2.7 (Simpson). There exists a projective scheme $\bar{J}_{L,d}(X)$, called the *Simpson compactified Jacobian*, that co-represents the functor $\bar{J}_{L,d}^\sharp(X)$.

Proof. This follows easily from the work of Simpson [42]. However, for later use, we need to review the explicit GIT construction. Consider the polynomial

$$P_d(t) := \deg(L) \cdot t + d + 1 - g(X),$$

which is the Hilbert polynomial, with respect to the polarization $\mathcal{O}_X(1) := L$, of any rank 1, torsion-free sheaf of degree d . Let $M^\sharp(\mathcal{O}_X, P_d) : k\text{-Sch.} \rightarrow \text{Sets}$ to be the functor that associates to a k -scheme T the set of isomorphism classes of coherent sheaves on $X \times_k T$, flat over T , that are fiberwise slope semistable with Hilbert polynomial $P_d(t)$ with respect to L . Note that $\bar{J}_{d,L}^\sharp(X)$ is the subfunctor of $M^\sharp(\mathcal{O}_X, P_d)$ parameterizing families of torsion-free, rank 1 sheaves.

Following Simpson’s construction [42] (note that Simpson [42] works over $k = \mathbb{C}$ but his construction has been extended over an arbitrary base field k by Maruyama [26] and Langer [23]), choose b sufficiently large and set $r := P_d(b) = d \cdot b + 1 - g(X)$. Consider the Quot scheme $\text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_d(t))$ parameterizing quotients $\mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I$, where I is a coherent sheaf on X of Hilbert polynomial $P_d(t)$ with respect to $\mathcal{O}(1)$ (see [20, 35] for details on Quot schemes).

There is a closed and open subscheme [42, p. 66] $Q^\circ \subseteq \text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_d)$ that parameterizes quotient maps

$$q : \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I$$

satisfying the following additional conditions:

- (1) $H^1(X, I(b)) = 0$;
- (2) $q \otimes 1 : H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I(b))$ is an isomorphism;
- (3) $I(b)$ is generated by its global section.

The natural linearized action of SL_r on the Quot scheme $\text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_d(t))$ restricts to a linearized action on Q° and the GIT stability for this action is naturally related to slope stability. Specifically, a point of the Quot scheme corresponding to $q : \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I$ is GIT (respectively, semi, poly)stable if and only if I is (respectively, semi, poly)stable with respect to the polarization L (see [42, Corollary 1.20, Theorem 1.19, Pf. of Theorem 1.21]). Therefore, the projective GIT quotient $M(\mathcal{O}_X, P_d) := Q^\circ // \text{SL}_r = (Q^\circ)^{\text{ss}} / \text{SL}_r$ naturally co-represents the functor $M^\sharp(\mathcal{O}_X, P_d)$.

Consider now the locus $Q^{\circ\circ} \subseteq Q^\circ$ parameterizing quotients $q : \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I$ such that

- (1) The sheaf I is a rank 1, torsion-free sheaf on X .

This is an SL_r -invariant subset that is closed and open in Q° by [38, Lemma 8.1.1]. Therefore, the image of $Q^{\circ\circ}$ in the GIT quotient $Q^\circ // \text{SL}_r$, which we set to be equal to $\bar{J}_{L,d}(X/S)$, must be closed-and-open in $M(\mathcal{O}_X, P_d)$ by [32, p. 8, Remark 6]. By construction, the projective scheme $\bar{J}_{L,d}(X/S)$ co-represents the functor $\bar{J}_{L,d}^\sharp(X/S)$. □

Simpson compactified Jacobians are a special case of Oda–Seshadri ϕ -compactified Jacobians.

FACT 2.8 (Alexeev). Let X be a nodal curve endowed with a polarization L and fix $d \in \mathbb{Z}$. Consider the numerical polarization ϕ such that

$$\phi_{X_i} := \frac{\deg(L|_{X_i})}{\deg(L)} \left(d - \frac{\deg(\omega_X)}{2} \right) + \frac{\deg(\omega_X|_{X_i})}{2}, \tag{2.4}$$

for each irreducible component X_i of X . Then a rank 1, torsion-free sheaf I of degree d on X is slope semistable (respectively, stable, polystable) with respect to L if and only if it is ϕ -semistable (respectively, ϕ -stable, ϕ -polystable).

In particular, we have that $\bar{J}_{L,d}^\#(X) = \bar{J}_\phi^\#(X)$, which implies $\bar{J}_{L,d}(X) \cong \bar{J}_\phi(X)$.

Proof. This is proved by Alexeev [2], where it is shown that a torsion-free, rank 1 sheaf I of degree d is slope (semi)stable (with respect to the polarization L) if and only if, for any subcurve $i: Y \hookrightarrow X$, we have that $\mu_L(I) \leq \mu_L(i_*(I_Y))$, where I_Y is the biggest torsion-free quotient the restriction $i^*(I) = I|_Y$. By the definition of the slope μ_L , we obtain

$$\frac{\deg(I) - 1/2 \deg(\omega_X)}{\deg(L)} = \mu_L(I) \leq \mu_L(i_*(I_Y)) = \frac{\deg(I_Y) - 1/2 \deg(\omega_X|_Y) + 1/2 \#(Y \cap Y^c)}{\deg(L|_Y)}, \tag{2.5}$$

where we used the formula $\omega_X|_Y = \omega_Y(Y \cap Y^c)$. Equation (2.5) can be rewritten as

$$\deg(I_Y) \geq \frac{\deg(L|_Y)}{\deg(L)} \left(\deg(I) - \frac{\deg(\omega_X)}{2} \right) + \frac{\deg(\omega_X|_Y)}{2} - \frac{\#(Y \cap Y^c)}{2} = \phi_Y - \frac{\#(Y \cap Y^c)}{2}, \tag{2.6}$$

which says that I is ϕ -(semi)stable with respect to the numerical polarization defined by (2.4). The fact that slope polystability corresponds to ϕ -polystability follows easily from the above. \square

REMARK 2.9. Let X be a nodal curve of genus g and consider the compactified Jacobians of X .

(i) *There are ϕ -compactified Jacobians of degree d that are not Simpson Jacobians of degree d .*

The most extreme case is $d = g - 1$. As it follows from (2.4), every Simpson compactified Jacobian of degree $g - 1$ is isomorphic to the ϕ -compactified Jacobian $\bar{J}_\phi(X)$ such that $\phi_{X_i} = \deg(\omega_X|_{X_i})/2$ for every irreducible component X_i of X (a very special compactified Jacobian, called the canonical compactified Jacobian of degree $g - 1$, that was studied in detail in [2, Section 3; 9; 10]. However, there are many ϕ -compactified Jacobians of degree $d = g - 1$ (indeed, as many as in the other degrees).

Also in degree $d \neq g - 1$, there are, in general, ϕ -compactified Jacobians that are not Simpson compactified Jacobians. For example, let X be the genus 2 nodal curve that consists of two rational components meeting in three nodes. For a numerical polarization $\phi = (\phi_1, \phi_2)$ such that $|\phi| = \phi_1 + \phi_2 = 0$, then one can compute that $\bar{J}_\phi(X)$ has two irreducible components if $\phi_1, \phi_2 \in \frac{1}{2} + \mathbb{Z}$ and three irreducible components otherwise. On the other hand, given an ample line bundle L with bidegree (a, b) , the associated ϕ -parameter (see (2.4)) is

$$\phi = \left(\frac{1}{2} - b/(a + b), \frac{1}{2} - a/(a + b) \right),$$

and ϕ_1, ϕ_2 cannot belong to $\frac{1}{2} + \mathbb{Z}$ because $a, b > 0$. Therefore, every Simpson compactified Jacobian of degree 0 has three irreducible components.

(ii) *Every ϕ -compactified Jacobian of degree d is isomorphic to a Simpson Jacobian of degree d' for some $d' \gg d$.*

Indeed given a numerical polarization ϕ , pick a line bundle M of sufficiently small degree on each irreducible component X_i of X in such a way that

$$a_i := \phi_{X_i} - \deg(M|_{X_i}) - \frac{\deg(\omega_X|_{X_i})}{2} > 0.$$

Moreover, pick a sufficiently divisible natural number $e \in \mathbb{N}$ such that

$$b_i := e \frac{a_i}{d + g - 1} \in \mathbb{Z} \text{ for every } i.$$

Finally, choose a line bundle L of total degree e such that $\deg(L|_{X_i}) = b_i$ and observe that L is ample since $\deg(L|_{X_i}) = b_i > 0$. With the above choices, we obtain that

$$\psi_{X_i} := \phi_{X_i} - \deg(M|_{X_i}) = \left(d - \frac{\deg(\omega_X)}{2} \right) \frac{\deg(L|_{X_i})}{\deg(L)} + \frac{\deg(\omega_X|_{X_i})}{2}. \tag{2.7}$$

Therefore, using Fact 2.8, we obtain the isomorphism

$$\begin{aligned} \bar{J}_\phi(X) &\xrightarrow{\cong} \bar{J}_\psi(X) \cong \bar{J}_{L,d'}(X), \\ I &\longmapsto I \otimes M^{-1}, \end{aligned}$$

where $d' = |\psi| = |\phi| - \deg M$.

We record in the following corollary a presentation of any compactified Jacobian of a nodal curve as a GIT quotient of an open subset of a suitable Quot scheme. Such a GIT description will be crucial in proving Theorem A(i).

COROLLARY 2.10 (GIT presentation of compactified Jacobians). *Let X be a nodal curve of genus g and let $\bar{J}(X)$ be any compactified Jacobian of X . There exists a Quot scheme $\text{Quot}(\mathcal{O}_X^{\oplus r}, P_d(t))$, parameterizing quotients $q : \mathcal{O}_X^{\oplus r} \twoheadrightarrow I$ with Hilbert polynomial $P_d(t) = d \cdot t + 1 - g$ with respect to some ample line bundle $\mathcal{O}_X(1)$, with an open and closed SL_r -invariant subscheme $U \subseteq \text{Quot}(\mathcal{O}_X^{\oplus r}, P_d(t))$ parameterizing the quotients $q : \mathcal{O}_X^{\oplus r} \twoheadrightarrow I$ with the property that*

- (1) $H^1(X, I) = 0$,
- (2) $q : H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I)$ is an isomorphism,
- (3) I is generated by the global sections,
- (4) I is a torsion-free, rank 1 sheaf,

in such a way that

$$\bar{J}(X) \cong U // \text{SL}_r = U^{\text{ss}} / \text{SL}_r,$$

where the GIT quotient on the right-hand side is taken with respect to the natural linearized action of SL_r .

Proof. By Remark 2.9(ii), it is enough to prove the Corollary for a Simpson compactified Jacobian $\bar{J}_{L,e}(X)$ (with $e \gg 0$). In the proof of Fact 2.7, we have seen that $\bar{J}_{L,e}(X)$ admits a GIT description as $Q^{\circ\circ} // \text{SL}_r$, where $Q^{\circ\circ}$ is the open and closed subscheme of a suitable Quot scheme $\text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_e(t))$, parameterizing the quotients $q : \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I$ such that

- (1) $H^1(X, I(b)) = 0$;
- (2) $q \otimes 1 : H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I(b))$ is an isomorphism;
- (3) I is a rank 1, torsion-free sheaf on X .

The isomorphism

$$\begin{aligned} \Phi : \text{Quot}(\mathcal{O}_X(-b)^{\oplus r}, P_e(t)) &\xrightarrow{\cong} \text{Quot}(\mathcal{O}_X^{\oplus r}, P_{e+b \deg \mathcal{O}_X(1)}(t)), \\ [q : \mathcal{O}_X(-b)^{\oplus r} \twoheadrightarrow I] &\longmapsto [q : \mathcal{O}_X^{\oplus r} \twoheadrightarrow I(b)], \end{aligned}$$

sends $Q^{\circ\circ}$ isomorphically onto the open subset $U \subseteq \text{Quot}(\mathcal{O}_X^{\oplus r}, P_{e+b \deg \mathcal{O}_X(1)}(t))$ parameterizing quotients $q : \mathcal{O}_X^{\oplus r} \twoheadrightarrow I$ satisfying the three conditions (1), (2), (4) and, moreover,

$$\bar{J}_{L,e}(X) \cong Q^{\circ\circ} // \text{SL}_r \xrightarrow{\cong} U // \text{SL}_r. \quad \square$$

2.3. *The universal compactified Jacobian*

In this subsection, we review the definition and the construction of the universal degree d compactified Jacobian $\bar{J}_{d,g} \rightarrow \bar{M}_g$ over the moduli space of stable curves \bar{M}_g of genus $g \geq 2$.

This construction of $\bar{J}_{d,g}$ is originally due to Caporaso [7] in terms of balanced line bundles on quasi-stable curves. Later, Pandharipande [38] re-interpreted $\bar{J}_{d,g}$ in terms of rank 1, torsion-free semistable sheaves on stable curves. We will focus on Pandharipande’s later construction because this description most naturally relates to the other compactified Jacobians we discuss here. For a description of Caporaso’s approach, we direct the interested reader to [7] and [38, § 10].

Given integers d and $g \geq 2$, the *universal compactified Jacobian functor*

$$\bar{J}_{d,g}^\sharp : k\text{-Sch.} \longrightarrow \text{Sets}$$

is defined to be the functor sending a k -scheme T to the set of isomorphism classes of families $\mathcal{X} \rightarrow T$ of stable curves of genus g together with a family of coherent sheaves which is fiberwise torsion-free, rank 1 of degree d and slope semistable with respect to the relative dualizing line bundle.

FACT 2.11 (Pandharipande [38]). The functor $\bar{J}_{d,g}^\sharp$ is co-representable by a projective scheme $\bar{J}_{d,g}$, called the *universal compactified Jacobian*, which is endowed with a forgetful projective morphism $\Phi : \bar{J}_{d,g} \rightarrow \bar{M}_g$.

Proof. This follows from the work of Pandharipande [38], where the projective scheme $\bar{J}_{d,g}$ is constructed via GIT. Since we will need this GIT description in the proof of Theorem A(ii), we will now review the relevant GIT set-up.

To begin, we may assume that d is sufficiently large because tensoring with the dualizing sheaf defines a canonical isomorphism between $\bar{J}_{d,g}^\sharp$ and $\bar{J}_{d+2g-2,g}^\sharp$. Thus, let d be large and fixed. Set $N := 10(2g - 2) - g$ and $e := 10(2g - 2)$. Consider the polynomial $P(t) := e \cdot t + d + 1 - g$ and set $r := P(0)$.

Inside the Hilbert scheme of degree e curves in \mathbb{P}^N , we can consider the locally closed subscheme H_g parameterizing non-degenerate, 10-canonically embedded stable curves. The product $H_g \times \mathbb{P}^N$ contains the universal 10-canonically embedded curve X_g , and associated to this family is the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r}, P(t))$, parameterizing quotients $q : \mathcal{O}_{X_g}^{\oplus r} \twoheadrightarrow E$ such that E is a coherent sheaf on X_g , flat over H_g , with the property that, on each fiber of $X_g \rightarrow H_g$, the Hilbert polynomial (with respect to the polarization given the embedding $X_g \hookrightarrow H_g \times \mathbb{P}^N$) of E is equal to $P(t)$.

The product group $\text{SL}_r \times \text{SL}_{N+1}$ acts on this Quot scheme by making SL_r act on $\mathcal{O}_{X_g}^{\oplus r}$ by changing bases, SL_{N+1} act on \mathbb{P}^N by changing projective coordinates, and then making $\text{SL}_r \times \text{SL}_{N+1}$ act on the Quot scheme by the product action. The action of $\text{SL}_r \times \text{SL}_{N+1}$ admits a natural linearization coming from the construction of the relative Quot scheme (see [35]).

Inside $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r}, P(t))$, there is an invariant closed-and-open subset \mathcal{Q}° parameterizing torsion-free, rank 1 quotients [38, Lemma 8.1.1]. It is shown in [38, Theorem 8.2.1, Theorem 9.1.1] that a point $[q : \mathcal{O}_{X_g}^{\oplus r} \twoheadrightarrow E] \in \mathcal{Q}^\circ$ is GIT semistable if and only if E is relatively semistable with respect to the relative dualizing sheaf. Therefore, the GIT (projective) quotient

$$\bar{J}_{d,g} := \mathcal{Q}^\circ // \text{SL}_r \times \text{SL}_{N+1}. \tag{2.8}$$

co-represents $\bar{J}_{d,g}^\sharp$ and, by construction, it is endowed with a forgetful projective morphism $\Phi : \bar{J}_{d,g} \rightarrow \bar{M}_g$. □

The fibers of the forgetful morphism $\Phi : \bar{J}_{d,g} \rightarrow \bar{M}_g$ are related to compactified Jacobians of stable curves with respect to their canonical polarization.

FACT 2.12. Assume $\text{char}(k) = 0$. Then the fiber of $\Phi : \bar{J}_{d,g} \rightarrow \bar{M}_g$ over a stable curve X is

$$\Phi^{-1}(X) \cong \bar{J}_{\omega_X, d}(X) / \text{Aut}(X). \tag{2.9}$$

Proof. This fact is surely well-known (see, for example, [2, § 1.8]); however, we sketch a proof for the lack of a suitable reference.

Since $\text{char}(k) = 0$, the GIT quotient (2.8) is a universal categorical quotient (see § 2.1), which implies that the variety $\bar{J}_{d,g}$ co-represents the functor $\bar{J}_{d,g}^\sharp$ *universally*, that is, for any scheme $T \rightarrow \bar{J}_{d,g}$ the base change functor $\bar{J}_{d,g}^\sharp \times_{\text{Hom}(-, \bar{J}_{d,g})} \text{Hom}(-, T)$ is co-represented by T . Applying this property to the inclusion $\Phi^{-1}(X) \hookrightarrow \bar{J}_{d,g}$, we deduce that $\Phi^{-1}(X)$ co-represents the functor

$$(\bar{J}_{d,g}^\sharp)_{|X} := \bar{J}_{d,g}^\sharp \times_{\text{Hom}(-, \bar{J}_{d,g})} \text{Hom}(-, \Phi^{-1}(X)) : k\text{-Sch.} \rightarrow \text{Sets}$$

which associates to a k -scheme S the set of isomorphism classes of iso-trivial families $p : \mathcal{X} \rightarrow S$ with fiber X together with a coherent sheaf \mathcal{I} , flat over S , which is fiberwise torsion-free, rank 1 and $\omega_{\mathcal{X}/S}$ -semistable of degree d . Therefore, there is a natural transformation of functors $\bar{J}_{\omega_X, d}^\sharp \rightarrow (\bar{J}_{d,g}^\sharp)_{|X}$ that factors through the natural action of $\text{Aut}(X)$ on $\bar{J}_{\omega_X, d}^\sharp$

$$\eta : \bar{J}_{\omega_X, d}^\sharp / \text{Aut}(X) \longrightarrow (\bar{J}_{d,g}^\sharp)_{|X}. \tag{2.10}$$

It is easily seen that η is a local isomorphism in the étale topology (using that every iso-trivial family becomes trivial after an étale base change); therefore, passing to the varieties that co-represent the above functors, we obtain an isomorphism

$$\bar{\eta} : \bar{J}_{\omega_X, d} / \text{Aut}(X) \xrightarrow{\cong} \Phi^{-1}(X). \tag{2.11}$$

□

REMARK 2.13. If $\text{char}(k) \gg g$, then the stabilizers of the GIT quotient (2.8) are linearly reductive (by Lemma 6.6(ii) and Corollary 4.3), which implies that the above GIT quotient is a universal categorical quotient, and so the proof of Fact 2.12 still goes through. We ignore the question of whether, in small characteristic, the GIT quotient (2.8) remains universal and Fact 2.12 still holds true.

3. Deformation theory

In the previous section, we studied the representability properties of global moduli functors parameterizing rank 1, torsion-free sheaves on a (fixed or varying) nodal curve. This section focuses on the analogous local topic: the pro-representability properties of deformation functors parameterizing infinitesimal deformations of a rank 1, torsion-free sheaf on a (fixed or varying) nodal curve. The main result is Corollary 3.17, which explicitly describes miniversal deformations rings parameterizing such deformations. The corollary is used in § 6 to prove Theorem A by relating the deformation rings to the completed local rings of (universal) compactified Jacobians (Theorem 6.1).

3.1. The deformation functors

We begin by reviewing the deformation functors of interest.

DEFINITION 3.1. Suppose we are given a k -scheme S , a finitely presented \mathcal{O}_S -module F , and a local k -algebra A with residue field k . A *deformation of the pair* (S, F) over A is a quadruple (S_A, F_A, i, j) that consists of

- (1) a flat A -scheme S_A ;
- (2) a A -flat, finitely presented \mathcal{O}_{S_A} -module F_A ;
- (3) an isomorphism $i: S_A \otimes_A k \xrightarrow{\sim} S$;
- (4) an isomorphism $j: i_*(F_A \otimes_A k) \xrightarrow{\sim} F$ of \mathcal{O}_S -modules.

The *trivial deformation* of a pair (S, F) over A is defined to be the quadruple $(S \otimes_k A, F \otimes_k A, i_{\text{can}}, j_{\text{can}})$. Here i_{can} and j_{can} are defined to be the canonical maps. If (S'_A, F'_A, i', j') is a second deformation of the pair (S, F) , then an *isomorphism* from (S_A, F_A, i, j) to (S'_A, F'_A, i', j') is defined to be a pair (ϕ, ψ) that consists of

- (1) an isomorphism $\phi: S_A \xrightarrow{\sim} S'_A$ over A such that $i' \circ (\phi \otimes 1) = i$;
- (2) an isomorphism $\psi: \phi_*(F_A) \xrightarrow{\sim} F'_A$ of $\mathcal{O}_{S'_A}$ -modules such that $j' \circ i'_*(\psi \otimes 1) = j$.

A *deformation of the scheme* S over A is defined by omitting the data of F_A and j from the definition of a deformation of a pair. Similarly, an *isomorphism* from one deformation (S_A, i) of S to another (S'_A, i') is defined by omitting ψ from Definition 3.1. The scheme S always admits the *trivial deformation* over A given by the pair $(S \otimes_k A, i_{\text{can}})$.

A *deformation of a sheaf* F over A is defined to be a pair (F_A, j) such that the quadruple $(S \otimes_k A, F_A, i_{\text{can}}, j)$ is a deformation of the pair (S, F) . An *isomorphism* from one deformation of F to another is defined to be a deformation of the associated deformations of the pair (S, F) . The trivial deformation of the pair (S, F) may be considered as a *trivial deformation of* F .

Let Art_k be the category of artin local k -algebras with residue field k . Recall that a *deformation functor* is a functor $F: \text{Art}_k \rightarrow \text{Sets}$ of artin rings with the property that $F(k)$ is a singleton set. We study the following deformation functors.

DEFINITION 3.2. Define functors $\text{Def}_S, \text{Def}_F, \text{Def}_{(S,F)}: \text{Art}_k \rightarrow \text{Sets}$ by

$$\begin{aligned} \text{Def}_{(S,F)}(A) &:= \{\text{iso. classes of deformations of } (S, F) \text{ over } A\}, \\ \text{Def}_S(A) &:= \{\text{iso. classes of deformations of } S \text{ over } A\}, \\ \text{Def}_F(A) &:= \{\text{iso. classes of deformations of } F \text{ over } A\}. \end{aligned} \tag{3.1}$$

The automorphism groups $\text{Aut}(S, F)$, $\text{Aut}(S)$, and $\text{Aut}(F)$ act on appropriate deformation functors, and this action will be studied in § 4. The reader should be familiar with the definitions of $\text{Aut}(S)$ and $\text{Aut}(F)$, but perhaps not of $\text{Aut}(S, F)$.

DEFINITION 3.3. An *automorphism* of (S, F) is a pair (σ, τ) that consists of:

- (1) an automorphism $\sigma: S \xrightarrow{\sim} S$;
- (2) an isomorphism of sheaves $\tau: \sigma_*F \xrightarrow{\sim} F$.

The group of automorphisms of (S, F) , denoted by $\text{Aut}(S, F)$, fits into the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Aut}(F) \longrightarrow \text{Aut}(S, F) \longrightarrow \text{Aut}(S), \\ (\sigma, \tau) \longmapsto \sigma. \end{aligned} \tag{3.2}$$

These automorphism groups act naturally on their respective functors.

DEFINITION 3.4. Let (S, F) be a given pair. Then we define the *natural action* of

- (1) $\text{Aut}(S, F)$ on $\text{Def}_{(S, F)}$ by making an element $(\sigma, \tau) \in \text{Aut}(S, F)$ act as:

$$(S_A, F_A, i, j) \mapsto (S_A, F_A, \sigma \circ i, \tau \circ \sigma_*(j)).$$

Here $\tau \circ \sigma_*(j)$ is the composition $\sigma_* i_*(F_A \otimes_A k) \xrightarrow{\sigma_*(j)} \sigma_*(F) \xrightarrow{\tau} F$;

- (2) $\text{Aut}(S)$ on Def_S by making an element $\sigma \in \text{Aut}(S)$ acts as $(S_A, i) \mapsto (S_A, \sigma \circ i)$;
- (3) $\text{Aut}(F)$ on Def_F by making an element $\tau \in \text{Aut}(F)$ acts as $(F_A, j) \mapsto (F_A, \tau \circ j)$.

Later we will relate the above deformation functors to the Quot scheme, so it is convenient to introduce the deformation functors arising from the Quot scheme. To avoid irrelevant foundational issues, we only define the deformation functors associated to nodal curves.

DEFINITION 3.5. Let X be a nodal curve, F be a coherent sheaf on X , and $q: \mathcal{O}_X^{\oplus r} \rightarrow F$ be a surjection. A *deformation of the pair* (X, q) over $A \in \text{Art}_k$ is a quadruple (X_A, i, q_A, j) , where $q_A: \mathcal{O}_{X_A}^r \rightarrow F_A$ is a surjection such that (X_A, F_A, i, j) is a deformation of (X, F) in the sense of Definition 3.1. Furthermore, we require that the isomorphism $j: i_*(F_A \otimes_A k) \xrightarrow{\sim} F$ respect quotient maps, in the sense that $q = j \circ i_*(q_A \otimes 1)$.

Given a second deformation (X'_A, i', q'_A, j') of (X, q) over A , an *isomorphism* from (X_A, i, q_A, j) to (X'_A, i', q'_A, j') is defined to be a pair (ϕ, ψ) consisting of

- (1) an isomorphism $\phi: X_A \xrightarrow{\sim} X'_A$ over A ;
- (2) an isomorphism $\psi: \phi_*(F_A) \xrightarrow{\sim} F'_A$ of $\mathcal{O}_{A'}$ -modules such that $\psi \circ \phi_*(q_A) = q'_A$.

A *deformation of q* over $A \in \text{Art}_k$ is defined to be a deformation of (X, q) of the form $(X \otimes_k A, i_{\text{can}}, q_A, j)$, where $(X \otimes_k A, i_{\text{can}})$ is the trivial deformation. An *isomorphism* from one deformation of q to another is defined to be an isomorphism of the associated deformations of (X, q) .

The deformation functors Def_q and $\text{Def}_{(X, q)}$ are defined in the expected manner.

DEFINITION 3.6. We define functors $\text{Def}_q, \text{Def}_{(X, q)}: \text{Art}_k \rightarrow \text{Sets}$ by

$$\begin{aligned} \text{Def}_{(X, q)}(A) &:= \{\text{iso. classes of deformations of } (X, q) \text{ over } A\}, \\ \text{Def}_q(A) &:= \{\text{iso. classes of deformations of } q \text{ over } A\}. \end{aligned} \tag{3.3}$$

To study $\bar{J}_{d, g}$, we also need a slight generalization of $\text{Def}_{(X, q)}$.

DEFINITION 3.7. Suppose that X is a stable curve, F is a coherent sheaf, $q: \mathcal{O}_X^{\oplus r} \rightarrow F$ is a quotient map, and $p: X \hookrightarrow \mathbb{P}^N$ is a 10-canonical embedding. A *deformation of the pair* (p, q) over $A \in \text{Art}_k$ is a quadruple (p_A, i, q_A, j) , where $p_A: X_A \hookrightarrow \mathbb{P}_A^N$ is a closed embedding and (X_A, i, q_A, j) is a deformation of the pair (X, q) . We further require

- (1) the line bundles $\mathcal{O}_{X_A}(1)$ and $\omega_{X_A/A}^{\otimes 10}$ are isomorphic;
- (2) $p_A \otimes 1 = p \circ i$.

Given a second deformation (p'_A, i', q'_A, j') of (p, q) , we define an *isomorphism* from the first deformation to the second to be an isomorphism (ϕ, ψ) of the associated deformations of (X, q) with the property that

$$p_A = p'_A \circ \phi.$$

DEFINITION 3.8. Define the functor $\text{Def}_{(p,q)}: \text{Art}_k \rightarrow \text{Sets}$ by

$$\text{Def}_{(p,q)}(A) := \{\text{iso. classes of deformations of } (p, q) \text{ over } A\}.$$

Note that there are forgetful transformations $\text{Def}_q \rightarrow \text{Def}_F$ and $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,q)} \rightarrow \text{Def}_{(X,F)}$ that are formally smooth once F is sufficiently positive (see Lemma 6.3).

The deformation functors we study are parameterized by complete local k -algebras. There are several different ways in which a complete local k -algebra can parameterize a deformation functor.

We say that a functor $\text{Def}: \text{Art}_k \rightarrow \text{Sets}$ is *pro-representable* if it is isomorphic to the formal spectrum functor

$$\begin{aligned} \text{Spf}(R): \text{Art}_k &\longrightarrow \text{Sets}, \\ A &\longmapsto \text{Hom}_{\text{loc}}(R, A) \end{aligned} \tag{3.4}$$

for some complete local k -algebra R with residue field k . A pair (R, π) consisting of such an algebra R and an isomorphism $\pi: \text{Spf}(R) \xrightarrow{\sim} \text{Def}$ is said to be a *universal deformation ring* for Def . An easy application of Yoneda’s lemma shows that if (R, π) and (R', π') are both universal deformation rings for Def , then there is a canonical isomorphism $R \cong R'$. An exercise in unraveling definitions shows that the completed local ring of an appropriate Quot scheme is a deformation ring for Def_q , and similarly for $\text{Def}_{(X,q)}$.

The functors Def_F and $\text{Def}_{(X,F)}$ are not always pro-representable, but do satisfy the weaker condition of admitting a miniversal deformation ring. Suppose that we are given a pair (R, π) consisting of a complete local k -algebra R and a natural transformation $\pi: \text{Spf}(R) \rightarrow \text{Def}$. We say that (R, π) is a *versal deformation ring* for Def if π is formally smooth. If π has the additional property that it induces an isomorphism on tangent spaces, then we say that (R, π) is a *miniversal* (or semiuniversal) *deformation ring*. One can show that if (R, π) and (R', π') are both miniversal deformation rings for Def , then R is isomorphic to R' , but in contrast to the situation for deformation rings, there is no distinguished isomorphism $R \cong R'$. We now proceed to construct miniversal deformation rings for Def_I and $\text{Def}_{(X,I)}$.

3.2. The miniversal deformation rings

The existence of miniversal deformation rings for Def_I and $\text{Def}_{(X,I)}$ can be deduced from theorems of Schlessinger, but for later computations, we will want an explicit description of these rings. We derive such a description by relating Def_I and $\text{Def}_{(X,I)}$ to the analogous deformation functors associated to the node \mathcal{O}_0 . We begin by fixing some notation for the node.

DEFINITION 3.9. The *standard node* \mathcal{O}_0 is the complete local k -algebra $k[[x, y]]/(xy)$. The *normalization* of the standard node is denoted by $\tilde{\mathcal{O}}_0$.

As a subring of the total ring of fractions $\text{Frac}(\mathcal{O}_0)$, the normalization of \mathcal{O}_0 is equal to $\tilde{\mathcal{O}}_0 = \mathcal{O}_0[x/(x+y)]$. It follows that the quotient $\tilde{\mathcal{O}}_0/\mathcal{O}_0$ is a one-dimensional k -vector space spanned by the image of $x/(x+y)$. Recall that $\tilde{\mathcal{O}}_0$ is also isomorphic to the ring $k[[x]] \oplus k[[y]]$, and the inclusion $\mathcal{O}_0 \rightarrow \tilde{\mathcal{O}}_0$ factors as

$$\frac{k[[x, y]]}{(xy)} \longrightarrow k[[x]] \oplus k[[y]] \xrightarrow{\sim} \frac{k[[x, y]]}{(xy)} \left[\frac{x}{x+y} \right],$$

where the first map is given by $h(x, y) \mapsto (h(x, 0), h(0, y))$ and the second map is given by $(f, g) \mapsto (fx + gy)/(x + y)$.

Over \mathcal{O}_0 , there are exactly two rank 1, torsion-free modules up to isomorphism: the free module and a unique module that fails to be locally free. A proof of this statement can be found in [14], where it is deduced from [43, Theorem 3.1]. There are several ways to describe the module that fails to be locally free.

DEFINITION 3.10. The unique rank 1, torsion-free module I_0 over \mathcal{O}_0 that fails to be locally free can be described as any one of the following modules:

- (1) the ideal $(x, y) \subset \mathcal{O}_0$, considered as an \mathcal{O}_0 -module;
- (2) the extension $\tilde{\mathcal{O}}_0 \supset \mathcal{O}_0$, considered as an \mathcal{O}_0 -module;
- (3) the \mathcal{O}_0 -module with presentation $\langle e, f : y \cdot e = x \cdot f = 0 \rangle$.

An isomorphism from the third module to the first module is given by $e \mapsto x, f \mapsto y$, while an isomorphism from the third to the second is given by $e \mapsto x/(x + y), f \mapsto y/(x + y)$. In passing from one model of I_0 to another, we will always implicitly identify the modules via these specific isomorphisms.

3.2.1. *Formal smoothness and reduction to the case of nodes.* If I is a rank 1, torsion-free sheaf on a nodal curve X , then the study of Def_I and $\text{Def}_{(X,I)}$ reduces to the study of Def_{I_0} and $\text{Def}_{(\mathcal{O}_0, I_0)}$. Indeed, say that Σ is the set of nodes where I fails to be locally free. For a given $e \in \Sigma$, let X_e denote the spectrum of the completed local ring $\widehat{\mathcal{O}}_{X,e}$ and I_e the pullback of I to X_e . There are forgetful transformations relating global deformations to local deformations:

$$\begin{aligned} \text{Def}_{(X,I)} &\longrightarrow \prod_{e \in \Sigma} \text{Def}_{(X_e, I_e)}, \\ \text{Def}_I &\longrightarrow \prod_{e \in \Sigma} \text{Def}_{I_e}, \\ \text{Def}_X &\longrightarrow \prod_{e \in \Sigma} \text{Def}_{X_e}. \end{aligned} \tag{3.5}$$

All of these transformations are formally smooth. Indeed, for the last transformation, this is [15, Proposition 1.5]. That result together with [17, A.1–4] shows that the first transformation is formally smooth. Essentially the same argument also shows that the middle transformation is formally smooth, and this is a special case of [17, B.1].

We now construct deformation rings for Def_{I_0} and $\text{Def}_{(\mathcal{O}_0, I_0)}$. We begin by parameterizing deformations of (\mathcal{O}_0, I_0) .

DEFINITION 3.11. Define $S_2 = S_2(\mathcal{O}_0, I_0) := k[[t, u, v]]/(uv - t)$. The deformation $(\mathcal{O}_{S_2}, I_{S_2}, i, j)$ of (\mathcal{O}_0, I_0) over S_2 is defined by setting

- (1) $\mathcal{O}_{S_2} := S_2[[x, y]]/(xy - t)$;
- (2) I_{S_2} equal to the \mathcal{O}_{S_2} -module with presentation

$$I_{S_2} := \langle \tilde{e}, \tilde{f} : y \cdot \tilde{e} = -u \cdot \tilde{f}, x \cdot \tilde{f} = -v \cdot \tilde{e} \rangle; \tag{3.6}$$

- (3) $i : \mathcal{O}_{S_2} \otimes_{S_2} k \xrightarrow{\sim} \mathcal{O}_0$ equal to the isomorphism that is the identity on the variables x and y ;
- (4) $j : i_*(I_{S_2} \otimes_{S_2} k) \xrightarrow{\sim} I_0$ equal to the isomorphism given by rules $\tilde{e} \otimes 1 \mapsto e$ and $\tilde{f} \otimes 1 \mapsto f$.

Deformations of I_0 alone are parameterized similarly.

DEFINITION 3.12. Define $S_1 = S_1(I_0) := k[[u, v]]/(uv)$. The algebraic deformation (I_{S_1}, j) of I_0 over S_1 is defined by setting

- (1) $\mathcal{O}_{S_1} = S_1[[x, y]]/(xy)$;
- (2) I_{S_1} equal to the \mathcal{O}_{S_1} -module with presentation

$$\mathcal{I} := \langle \tilde{e}, \tilde{f} : y \cdot \tilde{e} = -u \cdot \tilde{f}, x \cdot \tilde{f} = -v \cdot \tilde{e} \rangle; \tag{3.7}$$
- (3) $j : i_*(I_{S_1} \otimes_{S_1} k) \xrightarrow{\sim} I_0$ equal to isomorphism given by rules $\tilde{e} \otimes 1 \mapsto e$ and $\tilde{f} \otimes 1 \mapsto f$.

REMARK 3.13. It may be more intuitive to describe the deformations in geometric terms. There is a versal deformation (respectively, trivial deformation) $\mathcal{X} \rightarrow B$ of the node, with base

$$B = \text{Spec } k[u, v, t]/(uv - t) \text{ (respectively, } B = \text{Spec } k[u, v]/(uv))$$

and total space

$$\mathcal{X} = B \times \text{Spec } k[x, y]/(xy - t) \text{ (respectively, } \mathcal{X} = B \times \text{Spec } k[x, y]/(xy)).$$

The module I_{S_2} (respectively, I_{S_1}) is essentially the ‘universal’ ideal $I = (x - u, y - v) \subseteq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ considered as a module as in Definition 3.10(3).

LEMMA 3.14. *The ring S_2 is a miniversal deformation ring for $\text{Def}_{(\mathcal{O}_0, I_0)}$. More precisely, the algebraic deformation $(\mathcal{O}_{S_2}, i, I_{S_2}, j)$ defines a transformation $\text{Spf}(S_2) \rightarrow \text{Def}_{(\mathcal{O}_0, I_0)}$ that realizes S_2 as the miniversal deformation ring for Def_I . Similarly, S_1 is a miniversal deformation ring for Def_{I_0} .*

Proof. The claim concerning the ring S_1 was established in the course of proving [11, Proposition 2.6]. The same argument holds for S_2 , provided that one replaces the standard irreducible, nodal plane cubic used in that proof with a general pencil containing such a curve. □

Given a rank 1, torsion-free sheaf I that fails to be locally free at a set of nodes Σ , there is a simple relation between Def_I and $\prod_{e \in \Sigma} \text{Def}_{I_e}$.

DEFINITION 3.15. Let $\text{Def}_I^{\text{lt}} \subset \text{Def}_I$ be the subfunctor parameterizing deformations that map to the trivial deformation under $\text{Def}_I \rightarrow \prod_{e \in \Sigma} \text{Def}_{I_e}$. Define $\text{Def}_{(X, I)}^{\text{lt}}$ similarly. Elements of these deformation functors (valued in a given ring) are called *locally trivial* deformations (over that ring).

LEMMA 3.16. *Let X be a nodal curve, Σ be a set of nodes, $g : X_{\Sigma} \rightarrow X$ be the map that normalizes the nodes Σ , and $I := g_*(L)$ be the direct image of a line bundle L on X_{Σ} . Then the rule*

$$\begin{aligned} \text{Def}_L(A) &\longrightarrow \text{Def}_I^{\text{lt}}(A), \\ (L_A, j) &\longmapsto ((g \times \text{id})_*(L), (g \times \text{id})_*(j)) \end{aligned} \tag{3.8}$$

for any $A \in \text{Art}_k$ defines an isomorphism $\text{Def}_L \xrightarrow{\cong} \text{Def}_I^{\text{lt}}$.

Proof. The map $\text{Def}_L \rightarrow \text{Def}_I$ defined by Equation (3.8) has the property that the composition $\text{Def}_L \rightarrow \text{Def}_I \rightarrow \prod_{e \in \Sigma} \text{Def}_{I_e}$ is the trivial map, so there is an induced map $\text{Def}_L \rightarrow \text{Def}_I^{\text{lt}}$. Studying the map $\text{Def}_I \rightarrow \prod \text{Def}_{I_e}$ and the associated map on tangent-obstruction theories, one can show using the local-to-global spectral sequence for Ext that Def_I^{lt} is formally smooth with tangent space $T(\text{Def}_I^{\text{lt}}) = H^1(\underline{\text{End}}(I))$. This vector space is just $H^1(X_{\Sigma}, \mathcal{O}_{X_{\Sigma}})$ (see, for example, the proof of Lemma 4.2), which can be identified with

the tangent space to Def_L in such a way that $T(\text{Def}_L) \rightarrow T(\text{Def}_I^{\text{lt.}})$ is the identity. By formal smoothness, it follows that $\text{Def}_L \rightarrow \text{Def}_I^{\text{lt.}}$ is an isomorphism. \square

Let us denote by R_1 the miniversal deformation ring of Def_I and by R_2 the miniversal deformation ring of $\text{Def}_{(X,I)}$ (which exists by, say, [17, §A]). Lemma 3.14 together with the discussion following Equation (3.5) allows us to describe the miniversal deformation rings R_1 and R_2 as follows.

COROLLARY 3.17. *Let X be a nodal curve, I be a rank 1, torsion-free sheaf on X , and Σ be the set of nodes where I fails to be locally free. For every $e \in \Sigma$, fix an identification of $(\hat{\mathcal{O}}_{X,e}, I_e)$ with (\mathcal{O}_0, I_0) . Then the forgetful transformations in Equation (3.5) induce inclusions*

$$\begin{aligned} \widehat{\bigotimes_{e \in \Sigma} k[[U_e^-, U_e^-]] / (U_e^- U_e^-)} &\cong \widehat{\bigotimes_{e \in \Sigma} S_1} \hookrightarrow R_1, \\ \widehat{\bigotimes_{e \in \Sigma} k[[U_e^-, U_e^-, T_e]] / (U_e^- U_e^- - T_e)} &\cong \widehat{\bigotimes_{e \in \Sigma} S_2} \hookrightarrow R_2, \end{aligned}$$

and each inclusion realizes the larger ring as a power series ring over the smaller ring.

4. Automorphism groups and their actions

Automorphism groups appeared in the previous section, where we defined group actions on deformation functors (Definition 3.4). Here we study the structure of these groups with the aim of collecting results to use in §5. There we will study the problem of lifting the action of an automorphism group on a deformation functor to an action on a miniversal deformation ring. The existence of a lift follows from a theorem of Rim if the automorphism group is known to be linearly reductive. Thus, the focus of this section is on showing that the automorphism groups of interest are linearly reductive.

We begin by studying automorphisms of the node \mathcal{O}_0 (Definition 3.9) and its unique rank 1, torsion-free module I_0 that fails to be locally free (Definition 3.10). The automorphism group $\text{Aut}(X_0, I_0)$ fits into the exact sequence

$$0 \longrightarrow \text{Aut}(I_0) \longrightarrow \text{Aut}(X_0, I_0) \longrightarrow \text{Aut}(X_0) \longrightarrow 0, \tag{4.1}$$

and the group $\text{Aut}(I_0)$ admits the following explicit description.

LEMMA 4.1. *Consider I_0 as the normalization $\tilde{\mathcal{O}}_0$. Then the natural action of $\tilde{\mathcal{O}}_0^*$ on I_0 induces an isomorphism $\tilde{\mathcal{O}}_0^* \xrightarrow{\sim} \text{Aut}(I_0)$.*

Proof. We claim that every \mathcal{O}_0 -linear map $\phi: I_0 \rightarrow I_0$ is $\tilde{\mathcal{O}}_0$ -linear. It is enough to show that ϕ commutes with multiplication by $x/(x+y)$, and this is clear: for all $s \in I_0$, we have

$$(x+y) \cdot \phi(x/(x+y) \cdot s) = \phi(x \cdot s) = x \cdot \phi(s).$$

Dividing by $x+y$, we obtain the desired equality. Thus, $\text{Aut}(I_0)$ coincides with the group of $\tilde{\mathcal{O}}_0$ -linear automorphisms, which equals $\tilde{\mathcal{O}}_0^*$. \square

The action of $\tilde{\mathcal{O}}_0^*$ can also be described in terms of the presentation from Definition 3.10. A typical element $f \in \tilde{\mathcal{O}}_0^*$ can be uniquely written as $f = \alpha \frac{x}{x+y} + \beta \frac{y}{x+y} + g(x, y)$, with $\alpha, \beta \in k^*$ and $g(x, y) \in (x, y) \subset \mathcal{O}_0$, and this element acts by

$$e \longmapsto (\alpha + g(x, 0))e, \quad f \longmapsto (\beta + g(0, y))f.$$

We now turn to the global picture. Let I be a rank 1, torsion-free sheaf on a nodal curve X . Set Σ equal to the set of nodes where I fails to be locally free. In analogy with Equation (4.1), $\text{Aut}(X, I)$ fits into the following exact sequence:

$$0 \longrightarrow \text{Aut}(I) \longrightarrow \text{Aut}(X, I) \longrightarrow \text{Aut}(X). \tag{4.2}$$

We describe $\text{Aut}(X, I)$ by describing the outermost groups.

Consider first $\text{Aut}(X)$. Without more information, we can only describe the rough features of this group. For X stable (the main case of interest), $\text{Aut}(X)$ is a finite, reduced group scheme [15, Theorem 1.11], and if we additionally assume that X is general and of genus $g \geq 3$, then this group is trivial. However, $\text{Aut}(X)$ can be highly non-trivial for special curves; see [37] for a sharp bound on the cardinality of $\text{Aut}(X)$ in terms of the genus g , and for a description of the curves attaining the bounds.

The group $\text{Aut}(I)$ admits the following explicit description. In the notation from §3.2.1, there is a natural map $\text{Aut}(I) \rightarrow \text{Aut}(I_e)$ for every $e \in \Sigma$, and we use this map to describe $\text{Aut}(I)$.

LEMMA 4.2. *Let X be a nodal curve, I be a rank 1, torsion-free sheaf, Σ be the set of points where I fails to be locally free, and $g: X_\Sigma \rightarrow X$ be the map that normalizes the nodes Σ . Then there is a unique isomorphism $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*) \cong \text{Aut}(I)$ that extends the inclusion of $H^0(X, \mathcal{O}_X^*)$ in $\text{Aut}(I)$ and makes the diagram*

$$\begin{array}{ccc} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*) & \xrightarrow{\cong} & \text{Aut}(I) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{O}}_{X,e}^* & \xrightarrow{\cong} & \text{Aut}(I_e) \end{array}$$

commute for all $e \in \Sigma$. Here $\tilde{\mathcal{O}}_{X,e}$ is the normalization of the completed local ring at e , the horizontal maps are isomorphisms, and the vertical maps are restrictions.

Proof. Given I , we prove the stronger statement that $\underline{\text{End}}(I)$ is canonically isomorphic to $g_*(\mathcal{O}_{X_\Sigma})$. Because I is torsion-free, $\underline{\text{End}}(I)$ injects into $\underline{\text{End}}(I \otimes \text{Frac}(\mathcal{O}_X))$, which equals $\text{Frac}(\mathcal{O}_X)$ as I is rank 1. Thus, $\underline{\text{End}}(I)$ is a finitely generated, commutative \mathcal{O}_X -algebra satisfying $\mathcal{O}_X \subset \underline{\text{End}}(I) \subset \text{Frac}(\mathcal{O}_X)$. Furthermore, an application of the Cayley–Hamilton Theorem shows that a local section of $\underline{\text{End}}(I)$ satisfies a monic equation whose coefficients are local sections of \mathcal{O}_X . We may conclude that $\underline{\text{End}}(I) \subset \nu_*(\mathcal{O}_{\tilde{X}})$, where $\nu: \tilde{X} \rightarrow X$ is the (full) normalization. To complete the proof, it is enough to show that the support of $\nu_*(\mathcal{O}_{\tilde{X}})/\underline{\text{End}}(I)$ is precisely Σ . However, this can be checked on the level of completed stalks, and so we may deduce the claim from Lemma 4.1. The result now follows by taking global sections of $\underline{\text{End}}(I)$ and passing to units. \square

One consequence of the previous two lemmas is that many of the groups appearing in this paper are linearly reductive. Recall that the ground field k may have positive characteristic, and in positive characteristic, linear reductivity is a strong condition to impose. Indeed, while many algebraic groups (for example, $\text{GL}_r, \text{SL}_r, \dots$) are linearly reductive in characteristic 0, Nagata has shown that the only linearly reductive groups in characteristic $p > 0$ are the groups G whose identity component G_0 is a multiplicative torus and whose étale quotient G/G_0 has prime-to- p order. We now list the groups we have shown to satisfy this condition.

COROLLARY 4.3. *Let X be a nodal curve and I be a rank 1, torsion-free sheaf. Then the following groups are reduced and linearly reductive:*

- (1) *the automorphism group $\text{Aut}(I)$;*
- (2) *the quotient group $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$;*
- (3) *the automorphism group $\text{Aut}(X, I)$ when X is stable and does not admit an order $p = \text{char}(k)$ automorphism.*

Proof. Lemma 4.1 shows $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$ is a multiplicative torus, and Lemma 4.2 shows the same is true for $\text{Aut}(I)$. Given this, an inspection of Equation (4.2) proves that $\text{Aut}(X, I)$ is linearly reductive. □

5. Group actions on rings

In this section, we show that, in the cases of interest, the actions on deformation functors from Definition 3.4 lift to unique actions on miniversal deformation rings (Fact 5.4), which we then compute (Theorem 5.10). These results are used in §6, where we show that the action on the miniversal deformation ring can be described using the GIT construction of the compactified Jacobian (Lemmas 6.4 and 6.6). We then use this observation to deduce the main theorem of the paper (Theorem 6.1). Key to this section are the linear reductivity results from the previous section.

We begin by showing that certain actions are trivial.

LEMMA 5.1. *The action of $1 + (x, y)\mathcal{O}_0 \subset \text{Aut}(I_0)$ on Def_{I_0} is trivial.*

Proof. Suppose that we are given $A \in \text{Art}_k$ and a deformation (I_A, j) of I_0 over A . Given $\tau \in 1 + (x, y)\mathcal{O}_0$, we must show that (I_A, j) and $(I_A, \tau^{-1} \circ j)$ are isomorphic deformations. But this is clear: τ lies in \mathcal{O}_0 , and multiplication by $\tau \otimes 1 \in \mathcal{O}_0 \otimes_k A$ defines an isomorphism $(I_A, j) \xrightarrow{\sim} (I_A, \tau^{-1} \circ j)$. □

Essentially the same argument proves the following two lemmas.

LEMMA 5.2. *Let X be a nodal curve and I be a rank 1, torsion-free sheaf on X . Then the subgroup $\mathbb{G}_m \subset \text{Aut}(I)$ of scalar automorphisms acts trivially on Def_I . Under the inclusion (4.2), \mathbb{G}_m also acts trivially on $\text{Def}_{(X, I)}$.*

Proof. We give a proof for Def_I ; the case of $\text{Def}_{(X, I)}$ is similar, and left to the reader. If (I_A, j) is a deformation of I and $\tau \in \mathbb{G}_m \subset \text{Aut}(I)$ is a scalar automorphism, then τ trivially extends to an automorphism $\tilde{\tau}$ of I_A that defines an isomorphism of (I_A, j) with $(I_A, \tau^{-1} \circ j)$. □

LEMMA 5.3. *Let X be a nodal curve and I be a rank 1, torsion-free sheaf. Then $\text{Aut}(I)$ acts trivially on the subfunctor $\text{Def}_I^{\text{lt}} \subset \text{Def}_I$. Under the inclusion (4.2), $\text{Aut}(I)$ also acts trivially on the subfunctor $\text{Def}_{(X, I)}^{\text{lt}} \subset \text{Def}_{(X, I)}$.*

Proof. The lemma is a consequence of Lemmas 3.16 and 4.1. □

We may now invoke a theorem of Rim to show that the actions uniquely lift to actions on miniversal deformation rings.

FACT 5.4 (Rim [40]). Let X be a nodal curve and I a rank 1, torsion-free sheaf. Then:

- (i) there is a unique action of $\text{Aut}(I_0)$ on the miniversal deformation ring S_1 (respectively, S_2) that makes the map $\text{Spf}(S_1) \rightarrow \text{Def}_{I_0}$ (respectively, $\text{Spf}(S_2) \rightarrow \text{Def}_{(\mathcal{O}_0, I_0)}$) equivariant and has the property that the subgroup $1 + (x, y)\mathcal{O}_0 \subset \hat{\mathcal{O}}_0^* = \text{Aut}(I_0)$ acts trivially;
- (ii) there is a unique action of $\text{Aut}(I)$ on the miniversal deformation ring R_1 of Def_I that makes $\text{Spf}(R_1) \rightarrow \text{Def}_I$ equivariant;
- (iii) there is a unique action of $\text{Aut}(X, I)$ on the miniversal deformation ring R_2 of $\text{Def}_{(X, I)}$ that makes $\text{Spf}(R_2) \rightarrow \text{Def}_{(X, I)}$ equivariant, provided that X is stable and it does not admit an order $p = \text{char}(k)$ automorphism.

Proof. This is a special case of [40, p. 225]. Indeed, the functors Def_{I_0} , Def_I , $\text{Def}_{(\mathcal{O}_0, I_0)}$, and $\text{Def}_{(X, I)}$ are all examples of a deformation functor F associated to a ‘homogeneous fibered category in groupoid’ satisfying a finiteness condition. Given an action of a linearly reductive group on such a category, there is an induced action on F , and Rim’s Theorem asserts that there exists a miniversal deformation ring R that admits an action of G making $\text{Spf}(R) \rightarrow F$ equivariant. Furthermore, as an algebra with G -action, R is unique up to a (non-unique) isomorphism.

One may verify that the actions on Def_{I_0} , Def_I , $\text{Def}_{(\mathcal{O}_0, I_0)}$, and $\text{Def}_{(X, I)}$ are defined on the level of groupoids. The claims concerning R_1 and R_2 follow immediately because we have shown that $\text{Aut}(I)$ and $\text{Aut}(X, I)$ are linearly reductive. The group $\text{Aut}(I_0)$ is certainly not linearly reductive, but Lemma 5.1 asserts that this group acts through its linearly reductive quotient $\text{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$. Case (i) then follows as well. \square

The actions described by Fact 5.4 are, of course, unique only up to a non-unique isomorphism. Because of the non-uniqueness, it is not immediate that the group action is functorial. This issue is addressed in the lemma below.

LEMMA 5.5. *Let X be a nodal curve and I be a rank 1, torsion-free sheaf. For every point $e \in X$ where I fails to be locally free, fix an isomorphism between $(\hat{\mathcal{O}}_{X, e}, I \otimes \hat{\mathcal{O}}_{X, e})$ and (\mathcal{O}_0, I_0) . Then the restriction transformations*

$$\text{Def}_I \longrightarrow \prod_{e \in \Sigma} \text{Def}_{I_0} \quad \text{resp.} \quad \text{Def}_{(X, I)} \longrightarrow \prod_{e \in \Sigma} \text{Def}_{(\mathcal{O}_0, I_0)} \tag{5.1}$$

lift to transformations of miniversal deformation rings

$$\text{Spf}(R_1) \longrightarrow \prod_{e \in \Sigma} \text{Spf}(S_1) \quad \text{resp.} \quad \text{Spf}(R_2) \longrightarrow \prod_{e \in \Sigma} \text{Spf}(S_2)$$

that are equivariant with respect to the homomorphism

$$\text{Aut}(I) \longrightarrow \prod_{e \in \Sigma} \text{Aut}(I_0) \tag{5.2}$$

and the actions of $\text{Aut}(I)$ and $\text{Aut}(I_0)$ described in Fact 5.4.

Proof. The only condition that is not immediate is that the natural transformations can be chosen to be equivariant. We give the proof for $\text{Spf}(R_1)$ and leave the task of extending the argument to $\text{Spf}(R_2)$ to the interested reader.

As $\text{Spf}(S_1) \rightarrow \text{Def}_{I_0}$ is formally smooth, there exists a lift $\text{Spf}(R_1) \rightarrow \prod \text{Spf}(S_1)$ of the forgetful transformation $\text{Def}_I \rightarrow \prod \text{Def}_{I_0}$, and such a lift is automatically formally smooth. Writing R_1 as a power series ring over $\hat{\otimes} S_1$, it is easy to see that there exists an action of $\text{Aut}(I)$ on $\text{Spf}(R_1)$ that makes $\text{Spf}(R_1) \rightarrow \prod \text{Spf}(S_1)$ equivariant and has the property that the

induced action on the tangent space $T(\mathrm{Spf}(R_1))$ coincides with the natural action on $T(\mathrm{Def}_I)$. To complete the proof, we must show that this action makes $\mathrm{Spf}(R_1) \rightarrow \mathrm{Def}_I$ equivariant, and hence satisfies the conditions of Fact 5.4.

Consider the composition $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Spf}(S_1) \rightarrow \prod \mathrm{Def}_{I_0}$. This transformation is formally smooth and hence realizes R_1 as a (non-minimal) versal deformation ring for Def_{I_0} . Furthermore, the constructed action of $\mathrm{Aut}(I)$ on R_1 makes $\mathrm{Spf}(R_1) \rightarrow \prod \mathrm{Def}_{I_0}$ equivariant and induces the standard action on $T(R_1) = T(\mathrm{Def}_I)$. A second action on R_1 with this property is the unique action that makes $\mathrm{Spf}(R_1) \rightarrow \mathrm{Def}_I$ equivariant. An inspection of Rim’s *proof* shows that the uniqueness statement in Fact 5.4 still holds if the miniversality hypothesis is weakened to versality, provided the action on the tangent space is specified. In particular, there is an automorphism of R_1 transforming the first action into the second. We can conclude that the map in (5.2) and the action in Fact 5.4 can be chosen so that $\mathrm{Spf}(S_1) \rightarrow \mathrm{Def}_I$ is equivariant. This completes the proof. \square

We now compute the actions described by Fact 5.4. Let us start with the action of $\mathrm{Aut}(I_0)$ on S_1 .

LEMMA 5.6. *In terms of the presentation from Definitions 3.12 and 3.11, define an action of $\mathrm{Aut}(I_0)$ on S_1 and S_2 by making $\tau = a(x/(x+y)) + b(y/(x+y)) + g \in \mathrm{Aut}(I_0)$ act as*

$$u \longmapsto ab^{-1} \cdot u, \quad v \longmapsto a^{-1}b \cdot v, \quad t \longmapsto t.$$

Here $a, b \in k^*$ and $g \in (x, y)\tilde{\mathcal{O}}_0$. Then this action is the unique action described by Fact 5.4(i).

Proof. We give a proof for the case of S_1 ; the case of S_2 is similar, and left to the reader. The rule above is easily seen to define an action of $\mathrm{Aut}(I_0)$ on S_1 with the property that $1 + (x, u)\mathcal{O}_0$ acts trivially, so we need only show that this action makes $\mathrm{Spf}(S_1) \rightarrow \mathrm{Def}_{I_0}$ into an equivariant map. In fact, it is enough to verify this for the subgroup of $\mathrm{Aut}(I_0)$ that consists of elements of the form $\tau := a(x/(x+y)) + b(y/(x+y))$ because this subgroup maps isomorphically onto $\mathrm{Aut}(I_0)/(1 + (x, y)\mathcal{O}_0)$.

Given such a τ , what is the pullback of the miniversal deformation (I_{S_1}, i) under τ ? It is the module with presentation

$$\langle \tilde{e}', \tilde{f}' : y \cdot \tilde{e}' = -a^{-1}bu \cdot \tilde{f}', x \cdot \tilde{f}' = -ab^{-1}v \cdot \tilde{e}' \rangle, \tag{5.3}$$

together with the identification j sending $\tilde{e}' \mapsto e, \tilde{f}' \mapsto f$. One isomorphism between this deformation and the deformation $(I_{S_1}, \tau^{-1} \circ j)$ is

$$\tilde{e}' \longmapsto b^{-1}\tilde{e}, \quad \tilde{f}' \longmapsto a^{-1}\tilde{f}.$$

This completes the proof. \square

We now turn our attention to the action of $\mathrm{Aut}(I)$ on R_1 . It is convenient to introduce some combinatorial language.

DEFINITION 5.7. Let $e \in \Sigma$ be a node that lies on the intersection of the irreducible components v and w . Write \overleftarrow{e} for the pair (v, w) and \overrightarrow{e} for the pair (w, v) . Define $s, t: \{\overleftarrow{e}, \overrightarrow{e}\} \rightarrow \{v, w\}$ to be projection onto the first component and onto the second component, respectively.

This notation is intended to be suggestive of graph theory. We may consider v and w as being vertices of the dual graph Γ_X that are connected by an edge corresponding to e . The pairs \overleftarrow{e} and \overrightarrow{e} should be thought of as orientations of this edge, and the maps s and t are the ‘source’ and ‘target’ maps sending an oriented edge to its source vertex and its target vertex,

respectively. The relation with graph theory is developed more systematically by the authors in [12].

The group $\text{Aut}(I)$ can also be described using similar notation.

DEFINITION 5.8. Let X be a nodal curve, I be a rank 1, torsion-free sheaf, Σ be the set of nodes where I fails to be locally free, and V be the set of irreducible components of X . Define T_Σ to be the subgroup

$$T_\Sigma \subset \prod_{v \in V} \mathbb{G}_m$$

that consists of sequences (λ_v) with the property that $\lambda_{v_1} = \lambda_{v_2}$ for every two components v_1 and v_2 whose intersection contains some node *not* in Σ .

REMARK 5.9. The torus T_Σ is isomorphic to $\text{Aut}(I) = H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*)$ (Lemma 4.2). Indeed, the element $\lambda = (\lambda_v) \in T_\Sigma$ corresponds to the regular function $f \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}^*)$ that is equal to the constant λ_v on the component v . It is convenient to have the following explicit isomorphism of $\text{Aut}(I)$ with a split torus. Let Γ_X be the dual graph of X and let $\Gamma = \Gamma_X(\Sigma)$ be the dual graph of a curve obtained from X by smoothing the nodes not in Σ . There is a map of vertices $c : V(\Gamma_X) \rightarrow V(\Gamma)$ ([12, § 2.1]) and it is easy to check there is an isomorphism

$$\phi : T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m \xrightarrow{\sim} T_\Sigma = \text{Aut}(I) \subseteq \prod_{w \in V(\Gamma_X)} \mathbb{G}_m$$

defined as follows. Given $(g_v) \in \prod_{v \in V(\Gamma)} \mathbb{G}_m$, set $\phi((g_v))_w = g_{c(w)}$ for each $w \in V(\Gamma_X)$.

We use the description of $\text{Aut}(I)$ as T_Σ to describe the action of $\text{Aut}(I)$ on R_1 and on R_2 .

THEOREM 5.10. Let X be a nodal curve, I be a rank 1, torsion-free sheaf, Σ be the set of nodes where I fails to be locally free, and $g_\Sigma := h^1(X_\Sigma, \mathcal{O}_{X_\Sigma})$ the arithmetic genus of X_Σ . Then the following conditions are satisfied.

(i) Define an action of $T_\Sigma = \text{Aut}(I)$ on

$$R_1(\Sigma) := k[[\{U_{\bar{e}}, U_{\bar{e}} : e \in \Sigma\}; W_1, \dots, W_{g_\Sigma}]] / (U_{\bar{e}} U_{\bar{e}} : e \in \Sigma)$$

by making $\lambda \in T_\Sigma$ act as

$$U_{\bar{e}} \mapsto \lambda_{s(\bar{e})} \cdot U_{\bar{e}} \cdot \lambda_{t(\bar{e})}^{-1}, \quad U_{\bar{e}} \mapsto \lambda_{s(\bar{e})} \cdot U_{\bar{e}} \cdot \lambda_{t(\bar{e})}^{-1}, \quad W_i \mapsto W_i. \tag{5.4}$$

Then there exists an isomorphism $R_1 \cong R_1(\Sigma)$ that identifies the above action of T_Σ on $R_1(\Sigma)$ with the action of $\text{Aut}(I)$ on R_1 from Fact 5.4.

(ii) Suppose that $\text{Aut}(X)$ is trivial, and define an action of $T_\Sigma = \text{Aut}(X, I)$ on

$$R_2(\Sigma) := k[[\{U_{\bar{e}}, U_{\bar{e}}, T_e : e \in \Sigma\}; W_1, \dots, W_m]] / (U_{\bar{e}} U_{\bar{e}} - T_e : e \in \Sigma)$$

for some $m \in \mathbb{Z}_{\geq 0}$ by making $\lambda \in T_\Sigma$ act as in (5.4) and as $T_e \mapsto T_e$. Then there exists an isomorphism $R_2 \cong R_2(\Sigma)$ that identifies the above action of $T(\Sigma)$ on $R_2(\Sigma)$ with the action of $\text{Aut}(X, I)$ on R_2 from Fact 5.4.

REMARK 5.11. Let $\Gamma = \Gamma_X(\Sigma)$ be the dual graph of any curve obtained from X by smoothing the nodes not in Σ . Then one can check that, in the notation of the theorem above, $g_\Sigma = g(X) - b_1(\Gamma)$. It is also easy to see that the action of T_Γ on R_I and $R_{(X, I)}$ defined in Theorem A agrees with the action of T_Σ defined above.

Proof. This is a consequence of results already proved in his section. We only prove the statement about R_1 and leave the task of extending the proof to R_2 to the interested reader.

Suggestively set

$$S(\Sigma) := k[[U_{\bar{e}}, U_{\bar{e}} : e \in \Sigma]] / (U_{\bar{e}} U_{\bar{e}} : e \in \Sigma). \tag{5.5}$$

This is a miniversal deformation ring for $\prod \text{Def}_{I_0}$, where the product runs over the elements of Σ . If we fix an isomorphism between $(\mathcal{O}_{X,e}, I \otimes \widehat{\mathcal{O}}_{X,e})$ and (\mathcal{O}_0, I_0) for every node $e \in \Sigma$, then, by Corollary 3.17 and Lemma 5.5, there exists an equivariant map $S(\Sigma) \hookrightarrow R_1$ realizing R_1 as a power series ring over $S(\Sigma)$. To complete the proof, we need to show that there exists an expression of R_1 as a power series ring generated by variables invariant under the group action.

Thus, consider the map from the cotangent space of $\text{Spf}(R_1)$ to the cotangent space of Def_I^{lt} . This is an equivariant map, and the action of $\text{Aut}(I)$ on the target space is trivial (Lemma 5.3). Because $\text{Aut}(I)$ is linearly reductive, we can find invariant elements $\bar{W}_1, \dots, \bar{W}_{g_\Sigma} \in R_1$ whose images in the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ map isomorphically onto the cotangent space of Def_I^{lt} .

Letting W_1, \dots, W_{g_Σ} denote indeterminates, define a map

$$\phi: S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]] \longrightarrow R_1$$

by sending W_i to \bar{W}_i . The target and source of ϕ are isomorphic, and the induced map on tangent spaces is an isomorphism, hence ϕ itself must be an isomorphism.

Furthermore, if we make T_Σ act on $S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]]$ by making the group act trivially on the indeterminates, then ϕ is equivariant. The ring $S(\Sigma)[[W_1, \dots, W_{g_\Sigma}]]$, together with this group action, is nothing other than $R_1(\Sigma)$, so the proof is complete. \square

Observe that the theorem computes the action of $\text{Aut}(X, I)$ on R_2 when X is automorphism-free. It would be interesting to compute the action when X is stable, but possibly admits non-trivial automorphisms. Indeed, such a result (combined with a suitable extension of Theorem 6.1) would allow us to remove the hypothesis that X does not have an automorphism from Theorem A. When X does not admit an automorphism of order $p = \text{char}(k)$, Fact 5.4 states that there is a unique action of $\text{Aut}(X, I)$, so the problem is to modify the action described in Theorem 5.10 to incorporate $\text{Aut}(X)$. The case where X admits an order $p = \text{char}(k)$ automorphism is more challenging, for then we can no longer cite Rim’s work to assert that $\text{Aut}(X, I)$ acts on R_2 or to assert that such an action, if it exists, is unique. Simply knowing if R_2 still admits a unique action of $\text{Aut}(X, I)$ would be interesting. More generally, it would be interesting to know if Rim’s Theorem remains true if the assumption that the group G acting is linearly reductive is weakened.

6. Luna slice argument

We now prove that the invariant subrings in Theorem 5.10 are isomorphic to the completed local rings of the compactified Jacobians. The main result is the following.

THEOREM 6.1. *Let X be a nodal curve and I be a rank 1, torsion-free sheaf.*

(i) *Let $\bar{J}(X)$ be a compactified Jacobian of X and assume that I is polystable with respect to the associated stability condition. Then the action from Fact 5.4 of $\text{Aut}(I)$ on the deformation ring R_1 parameterizing deformations of I satisfies*

$$\widehat{\mathcal{O}}_{\bar{J}(X), [I]} \cong R_1^{\text{Aut}(I)}.$$

(ii) *Assume that X is stable and does not admit an order p automorphism, and I is slope polystable with respect to the dualizing sheaf ω_X . Then the action of $\text{Aut}(X, I)$ on the deformation ring R_2 satisfies*

$$\widehat{\mathcal{O}}_{\bar{J}_{a,g}, [(X, I)]} \cong R_2^{\text{Aut}(X, I)}.$$

In the theorem, the isomorphisms between the complete local rings are non-canonical, but this is necessarily so as the rings R_1 and R_2 are themselves only defined up to non-canonical isomorphism.

REMARK 6.2. Observe that Theorem 6.1, together with Theorem 5.10, establishes Theorem A (see also Remarks 5.9 and 5.11). An elementary argument in GIT shows that the ring $\widehat{B(\Gamma)}^{Tr}$ defined in Theorem A has dimension $b_1(\Gamma) + \#E(\Gamma)$. Since $\bar{J}_{d,g}$ has dimension $4g - 3$, it follows that $m = 4g - 3 - b_1(\Gamma) - \#E(\Gamma)$ in Theorem 5.10.

The proof of Theorem 6.1 is given at the end of the section, where it is deduced from the following sequence of lemmas.

LEMMA 6.3. *Let X be a nodal curve, I be a rank 1, torsion-free sheaf, and $q: \mathcal{O}_X^{\oplus r} \rightarrow I$ be a surjection. If $H^1(X, I) = 0$, then the forgetful morphism $\text{Def}_q \rightarrow \text{Def}_I$ is formally smooth. Assume further that X is stable and $p: X \hookrightarrow \mathbb{P}^N$ is a 10-canonical embedding. Then $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,I)}$ is formally smooth.*

Proof. We prove the statement about $\text{Def}_q \rightarrow \text{Def}_I$ and leave the proof for $\text{Def}_{(p,q)} \rightarrow \text{Def}_{(X,I)}$ to the interested reader. Given a surjection $B \twoheadrightarrow A$ of artin local k -algebras, a deformation (I_B, j) of I over B , and a deformation (q_A, j) of q such that the associated deformation of I is isomorphic to $(I_B \otimes_B A, j \otimes 1)$, we must show that there exists a deformation (q_B, j) extending (q_A, j) and inducing (I_B, j) . A filtering argument shows that the vanishing $H^1(X, I) = 0$ implies that $H^0(X_B, I_B) \rightarrow H^0(X_A, I_A)$ is surjective. Now suppose $s_1, \dots, s_r \in H^0(I_A)$ is the image of the standard basis for $H^0(X_A, \mathcal{O}_{X_A}^{\oplus r})$. If we lift these elements to $\tilde{s}_1, \dots, \tilde{s}_r \in H^0(X_B, I_B)$ and define $q_B: \mathcal{O}_{X_B}^{\oplus r} \rightarrow I_B$ to be the map that sends the i th standard basis element to \tilde{s}_i , then (q_B, j) has the desired properties. \square

We now relate R_1 and R_2 to the appropriate Quot schemes.

LEMMA 6.4. *Let X be a nodal curve, I be a rank 1, torsion-free sheaf, and $q: \mathcal{O}_X^{\oplus r} \rightarrow I$ be a quotient map corresponding to a point $\tilde{x} \in \text{Quot}(\mathcal{O}_X^{\oplus r})$. Assume:*

- (1) $H^1(X, I) = 0$;
- (2) $q: H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I)$ is an isomorphism.

(i) *If Z is a slice through \tilde{x} in some invariant affine open neighborhood $\tilde{x} \in U \subseteq \text{Quot}(\mathcal{O}_X^{\oplus r})$, then the completed local ring $\widehat{\mathcal{O}}_{Z, \tilde{x}}$ of Z at \tilde{x} is a miniversal deformation ring for Def_I .*

(ii) *Assume additionally that X is stable. Let $p: X \hookrightarrow \mathbb{P}^N$ be a 10-canonical embedding with (p, q) corresponding to the point \tilde{y} of the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r})$ (as in the proof of Fact 2.11). If Z is a slice through \tilde{y} in some invariant affine open neighborhood $\tilde{y} \in V \subseteq \text{Quot}(\mathcal{O}_{X_g}^{\oplus r})$, then the completed local ring of $\widehat{\mathcal{O}}_{Z, \tilde{y}}$ of Z at \tilde{y} is a miniversal deformation ring for $\text{Def}_{(X,I)}$.*

Proof. We prove the statement relating $\text{Quot}(\mathcal{O}_X^{\oplus r})$ to Def_I and leave the task of extending the argument to $\text{Def}_{(X,I)}$ to the interested reader. The necessary changes are primarily notational (for example, the action of SL_r must be replaced with that of $\text{SL}_r \times \text{SL}_{N+1}$).

Temporarily set F equal to the functor pro-represented by $\widehat{\mathcal{O}}_{Z,x}$. There is a natural forgetful map $\text{Def}_q \rightarrow \text{Def}_I$, and our goal is to show that the restriction of this map to F is formally smooth and an isomorphism on tangent spaces. We do this by proving that $F(A) \rightarrow \text{Def}_I(A)$ is injective for $A = k[\epsilon]$ and has the same image as $\text{Def}_q(A) \rightarrow \text{Def}_I(A)$ for all $A \in \text{Art}_k$. Because $\text{Def}_q \rightarrow \text{Def}_I$ is formally smooth (Lemma 6.3), the lemma will then follow.

The desired facts are proved by studying the action of the Lie algebra of SL_r on deformations. Set \mathfrak{sl}_r equal to the deformation functor pro-represented by the completed local ring of SL_r at the identity and \mathfrak{h} equal to the deformation functor associated to the stabilizer $H := \text{Stab}(\tilde{x}) \subset SL_r$. There is a natural map $\mathfrak{sl}_r/\mathfrak{h} \rightarrow \text{Def}_q$ given by the derivative of the orbit map. Concretely, this is defined by the rule $g \mapsto g \cdot v_{\text{triv}}$, where v_{triv} is the trivial deformation (over an unspecified artin local algebra). Because U admits a slice, there exists a morphism $\text{Def}_q \rightarrow \mathfrak{sl}_r/\mathfrak{h}$ that is a contraction onto the orbit in the sense that the derivative of the orbit map defines a section. Furthermore, this morphism has the property that the preimage of the trivial element $0 \in \mathfrak{sl}_r/\mathfrak{h}(A)$ is $F(A) \subset \text{Def}_q(A)$. The construction of the morphism is immediate: the scheme $Z \times_H SL_r$ admits a global contraction morphism given by projection onto the second factor, and the desired infinitesimal contraction is obtained by choosing a local inverse of $Z \times_H SL_r \rightarrow \text{Quot}(\mathcal{O}_X^{\oplus r})$.

We can use the contraction morphism to deduce the second claim, that $\text{Def}_q(A) \rightarrow \text{Def}_I(A)$ and $F(A) \rightarrow \text{Def}_I(A)$ have the same image. Indeed, if $v \in \text{Def}_q(A)$ maps to an element of $\mathfrak{sl}_r/\mathfrak{h}(A)$ represented by $g \in \mathfrak{sl}_r(A)$, then $g^{-1} \cdot v$ lies in $F(A)$. Because both v and $g^{-1} \cdot v$ map to the same element of $\text{Def}_I(A)$, we have proved the claim.

We also need to verify that $F(k[\epsilon]) \rightarrow \text{Def}_I(k[\epsilon])$ is injective. This too can be proved using the contraction map, but we must first relate the kernel of $F(k[\epsilon]) \rightarrow \text{Def}_I(k[\epsilon])$ to the contraction. Specifically, we claim the kernel equals the image of the orbit map. It is immediate that the image is contained in the kernel, but the reverse inclusion requires more justification. Thus, suppose $(q_1: \mathcal{O}_{X_1}^{\oplus r} \rightarrow I_1, j)$ is a first-order deformation with the property that (I_1, j) is the trivial deformation. Because q induces an isomorphism on global sections, we can chose bases and use the identification j to represent $q_1: H^0(X, \mathcal{O}_{X_1}^{\oplus r}) \rightarrow H^0(X, I_1)$ by a matrix g that reduces to the identity modulo ϵ . The matrix g may not lie in $\mathfrak{sl}_r(k[\epsilon])$, but if we set $\delta := \det(g)$, then the product $\delta \cdot g^{-1}$ does. One may check that $\delta \cdot g^{-1}$ maps to the deformation represented by $(q_1: \mathcal{O}_{X_1}^{\oplus r} \rightarrow I_1, j)$, establishing the reverse inclusion.

We now prove injectivity by showing directly that the image of the orbit map has trivial intersection with $F(k[\epsilon])$. Given v in this intersection, the image in $\mathfrak{sl}_r/\mathfrak{h}(k[\epsilon])$ under the contraction morphism is zero because v lies in $F(k[\epsilon])$. But, as v also lies in the image of the orbit map, the image under the composition $\text{Def}_q(k[\epsilon]) \rightarrow \mathfrak{sl}_r/\mathfrak{h}(k[\epsilon]) \rightarrow \text{Def}_q(k[\epsilon])$ of the contraction map with the orbit map is v . Thus, $v = 0$, and the proof is complete. \square

The following definition and lemma relate the stabilizer of a point of the Quot scheme to an automorphism group.

DEFINITION 6.5. Let I be a rank 1, torsion-free sheaf on a nodal curve X and $q: \mathcal{O}_X^{\oplus r} \twoheadrightarrow I$ be a quotient map corresponding to a point \tilde{x} belonging to some Quot scheme $\text{Quot}(\mathcal{O}_X^{\oplus r})$. Assume

- (1) I is generated by global sections;
- (2) $q: H^0(X, \mathcal{O}_X^{\oplus r}) \rightarrow H^0(X, I)$ is an isomorphism.

(i) If $\text{Stab}(\tilde{x}) \subset SL_r$ is the stabilizer under the natural action on $\text{Quot}(\mathcal{O}_X^{\oplus r})$, then the natural homomorphism

$$\text{Stab}(\tilde{x}) \longrightarrow \text{Aut}(I)$$

is defined by sending $g \in \text{Stab}(\tilde{x})$ into the unique automorphism $\alpha(g): I \rightarrow I$ with the property that $\alpha(g) \circ q = q \circ g^{-1}$ (which exists since I is generated by the image of $H^0(X, \mathcal{O}_X^{\oplus r})$).

Assume additionally that X is stable. Let $p: X \hookrightarrow \mathbb{P}^N$ be a 10-canonical embedding with (p, q) corresponding to the point \tilde{y} of the relative Quot scheme $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r})$ (as in the proof of Fact 2.11).

(ii) If $\text{Stab}(\tilde{y}) \subset \text{SL}_r \times \text{SL}_{N+1}$ is the stabilizer under the natural action on $\text{Quot}(\mathcal{O}_{X_g}^{\oplus r})$, then the natural homomorphism

$$\text{Stab}(\tilde{y}) \longrightarrow \text{Aut}(X, I)$$

is defined by sending $g = (g_1, g_2) \in \text{Stab}(\tilde{y})$ into the unique element $\alpha(g) = (\alpha_1(g), \alpha_2(g)) \in \text{Aut}(X, I)$ such that $p \circ \alpha_1(g) = g_2 \circ p$ and $\alpha_2(g) \circ \alpha_1(g)_*(q) = q \circ g_1^{-1}$.

LEMMA 6.6. *Same notation as in Definition 6.5.*

(i) *The natural homomorphism $\text{Stab}(\tilde{x}) \rightarrow \text{Aut}(I)$ is injective with image equal to the subgroup $\text{Aut}_1(I) \subset \text{Aut}(I)$ consisting of the automorphisms $\tau \in \text{Aut}(I)$ with the property that the induced automorphism $H^0(X, I) \rightarrow H^0(X, I)$ has determinant +1.*

(ii) *The natural homomorphism $\text{Stab}(\tilde{y}) \rightarrow \text{Aut}(X, I)$ is injective with image equal to the subgroup $\text{Aut}_1(X, I) \subset \text{Aut}(X, I)$ consisting of the automorphisms $(\sigma, \tau) \in \text{Aut}(X, I)$ with the property that the induced automorphism $H^0(X, I) \xrightarrow{\text{can}} H^0(X, \sigma_*(I)) \xrightarrow{\tau} H^0(X, I)$ has determinant +1.*

Proof. As in the last proof, we only prove the statement for $\text{Stab}(\tilde{x})$ and leave the case of $\text{Stab}(\tilde{y})$ to the interested reader. Set $s_1, \dots, s_r \in H^0(X, I)$ equal to the image of the standard basis for $H^0(X, \mathcal{O}_X^{\oplus r})$. We first show injectivity. Given $g \in \text{Stab}(\tilde{x})$, write $(a_{i,j}) := g^{-1}$. Then $\alpha(g)$ satisfies

$$\alpha(g)(s_i) = a_{i,1}s_1 + \dots + a_{i,r}s_r. \tag{6.1}$$

If $\alpha(g)$ is the identity, then we must have $\alpha(g)(s_i) = s_i$ for all i . But the elements s_1, \dots, s_r form a basis, so this is only possible if $g = \text{id}_r$, showing injectivity. Similarly, given an $\alpha \in \text{Aut}(I)$ that induces a determinant +1 automorphism of $H^0(X, I)$, define scalars $a_{i,j}$ as in Equation (6.1). Then $g := (a_{i,j})^{-1} \in \text{SL}_r$ is an element of $\text{Aut}_1(I)$ with $\alpha(g) = \alpha$. This completes the proof. \square

The last lemma we need asserts that the formation of the relevant group quotients commutes with completion.

LEMMA 6.7. *Let Z be an affine algebraic scheme, $\tilde{x} \in Z$ be a point, and H be an algebraic group acting on Z that fixes \tilde{x} . Assume that H is linearly reductive. Then the formation of H -invariants commutes with completion, that is, if we call x the image of \tilde{x} in Z/H , then we have*

$$\widehat{\mathcal{O}}_{Z,\tilde{x}}^H \cong \widehat{\mathcal{O}}_{Z/H,x}.$$

Proof. This is an exercise in linear reductivity. The quotient map induces a local homomorphism $\widehat{\mathcal{O}}_{Z/H,x} \rightarrow \widehat{\mathcal{O}}_{Z,\tilde{x}}$. Because \tilde{x} is a fixed point, H acts continuously on $\widehat{\mathcal{O}}_{Z,\tilde{x}}$, and passing to invariants, we may replace the target of this map with $\widehat{\mathcal{O}}_{Z,\tilde{x}}^H$. Our goal is to show that the resulting map is an isomorphism.

For injectivity, say $r \in \widehat{\mathcal{O}}_{Z/H,x}$ lies in the kernel. By picking a sequence $\{r_i\}_{i=1}^\infty$ in $\mathcal{O}_{Z/H,x}$ converging to r and studying the valuation of r_i , one can show that $r = 0$. Surjectivity requires more work.

Given $r \in \widehat{\mathcal{O}}_{Z,\tilde{x}}^H$, consider the reduction map $\widehat{\mathcal{O}}_{Z,\tilde{x}}^H \rightarrow \widehat{\mathcal{O}}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$. The element r maps to an H -invariant element \bar{r} in the target, which is canonically isomorphic to $\mathcal{O}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$. Fixing an equivariant splitting of $\mathcal{O}_{Z,\tilde{x}} \rightarrow \mathcal{O}_{Z,\tilde{x}}/\mathfrak{m}_{\tilde{x}}^{i+1}$ (which exists by linear reductivity), we can lift \bar{r} to an invariant element r_i of $\mathcal{O}_{Z,\tilde{x}}$. The collection of all these elements defines a sequence $\{r_i\}_{i=1}^\infty$ whose limit is r . Furthermore, every term in the sequence lies in $\widehat{\mathcal{O}}_{Z/H,x}$; thus the limit must lie in this ring as well. This completes the proof. \square

Proof of Theorem 6.1. The proof is an application of the Luna Slice Theorem, together with the previous lemmas. As usual, we only give the proof for a compactified Jacobian of a fixed nodal curve and leave the task of extending the argument to the universal compactified Jacobian to the interested reader (replacing Lemma 6.4(i) with Lemma 6.4(ii) and Lemma 6.6(i) with Lemma 6.6(ii) in the argument that follows).

According to Corollary 2.10, we can assume that $\bar{J}(X) \cong U // \mathrm{SL}_r = U^{\mathrm{ss}} / \mathrm{SL}_r$, where U is the open subset of the Quot scheme $\mathrm{Quot}(\mathcal{O}_X^{\oplus r})$ defined in *loc. cit.*

Take now any lift of $[I] \in \bar{J}(X)$ to a point $\tilde{x} \in U \subseteq \mathrm{Quot}(\mathcal{O}_X^{\oplus r})$, corresponding to a quotient map $q: \mathcal{O}_X^{\oplus r} \rightarrow I$, and observe that the orbit of \tilde{x} is closed in the semistable locus U^{ss} since I is polystable. Lemma 6.6(i) identifies $\mathrm{Stab}(\tilde{x})$ with the subgroup $\mathrm{Aut}_1(I) \subset \mathrm{Aut}(I)$ which is a (multiplicative) torus (since $\mathrm{Aut}(I)$ is a torus by Lemma 4.2 and any subgroup of a torus is a torus), hence linearly reductive. Therefore, we can apply the Luna Slice Theorem (see § 2.1) in order to get a slice Z of U at \tilde{x} .

Lemma 6.4(i) identifies the ring $\widehat{\mathcal{O}}_{Z,\tilde{x}}$ with the miniversal deformation ring R_1 of Def_I . Moreover, an exercise in unwinding the definitions shows that the natural transformation $\pi: \mathrm{Spf} \widehat{\mathcal{O}}_{Z,\tilde{x}} \rightarrow \mathrm{Def}_I$ is equivariant with respect to the natural homomorphism $\mathrm{Stab}(\tilde{x}) \hookrightarrow \mathrm{Aut}(I)$ and the actions of $\mathrm{Stab}(\tilde{x})$ on $\mathrm{Spf} \widehat{\mathcal{O}}_{Z,\tilde{x}}$ and of $\mathrm{Aut}(I)$ on Def_I . Therefore, Fact 5.4 implies that the natural identification $\widehat{\mathcal{O}}_{Z,\tilde{x}} \cong R_1$ is $\mathrm{Stab}(\tilde{x}) \cong \mathrm{Aut}_1(I)$ -equivariant.

Now, applying Equation (2.1) together with Lemma 6.7, we obtain

$$\widehat{\mathcal{O}}_{\bar{J}(X),[I]} \cong \widehat{\mathcal{O}}_{Z,\tilde{x}}^{\mathrm{Stab}(\tilde{x})} \cong R_1^{\mathrm{Aut}_1(I)}. \tag{6.2}$$

Now observe that the subgroup $\mathbb{G}_m \subset \mathrm{Aut}(I)$ of scalar automorphism acts trivially on R_1 , as it follows by the explicit description of the action of $\mathrm{Aut}(I)$ on R_1 given in Theorem 5.10. Thus, the natural action of $\mathrm{Aut}(I)$ on R_1 factors through the quotient $\mathrm{Aut}(I)/\mathbb{G}_m$. Because the natural map $\mathrm{Aut}_1(I) \rightarrow \mathrm{Aut}(I)/\mathbb{G}_m$ is surjective, we obtain that

$$R_1^{\mathrm{Aut}_1(I)} \cong R_1^{\mathrm{Aut}(I)}. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain the conclusion. □

In the introduction, we asked if Theorem 6.1 remains valid when X is allowed to have an automorphism of order p . The condition on the automorphism group was only used to apply the Luna Slice Theorem, which applies to actions of linearly reductive groups. It is probably unreasonable to expect an analog of the Slice Theorem to hold for actions of an arbitrary reductive group (see [25]), but we only need an analog for actions of $\mathrm{Aut}(X, I)$. This group is an extension of the finite (reduced) group $\mathrm{Aut}(X)$ by the multiplicative torus $\mathrm{Aut}(I)$, and it is known that the Slice Theorem holds for both the action of a torus (it is linearly reductive) and for the action of a finite group (see, for example, [19, Proposition 2.2]). Perhaps there is a Slice Theorem for actions of an extension of a torus by an arbitrary finite group?

Finally, we can prove Theorem B from the introduction.

Proof of Theorem B. Given Theorem A, this result follows from [12]. To establish Parts (i) and (ii) of Theorem B, it is enough to fix a point $[I] \in \bar{J}(X)$ with I polystable and prove the analogous statement about the completed local ring $\widehat{\mathcal{O}}_{\bar{J}(X),[I]}$. For the remainder of the proof, we will work exclusively with $\widehat{\mathcal{O}}_{\bar{J}(X),[I]}$.

Theorem A identifies $\widehat{\mathcal{O}}_{\bar{J}(X),[I]}$ with the T_Γ -invariant subring of $R_I = \widehat{A}(\Gamma)[[W_1, \dots, W_{g_\Sigma}]]$, where $g_\Sigma := g(X) - b_1(\Gamma)$. The ring $\widehat{A}(\Gamma)$ is the completion of the ring $A(\Gamma)$ defined in [12, § 6] at the maximal ideal $\tilde{\mathfrak{m}} := (U_e^-, U_e^- : e \in \Sigma)$ and the action of T_Γ on $\widehat{A}(\Gamma)$ is induced by

the action on $A(\Gamma)$ defined in *loc. cit.* By [12, Theorem 6.1], the invariant subring of $A(\Gamma)$ is the cographic toric face ring $R(\Gamma)$ (from [12, Definition 1.4]). Thus, applying Lemma 6.7, we obtain

$$\widehat{\mathcal{O}}_{\bar{J}(X),[I]} \cong \widehat{R}(\Gamma)[[W_1, \dots, W_{g_X}]], \tag{6.4}$$

where $\widehat{R}(\Gamma)$ is the completion of $R(\Gamma)$ at the ideal $\mathfrak{m} := \widetilde{\mathfrak{m}} \cap R(\Gamma)$ (which appears in [12, Proposition 4.6]).

We now prove Part (i) of the theorem. In [12], it is proved that $R(\Gamma)$ is Gorenstein and has slc singularities [12, Theorem 5.7], and these properties persist after passing to a completion and adding power series variables.

To establish Part (ii), it is enough to show that the multiplicity $e_{\mathfrak{m}}(R(\Gamma))$ is equal to 1 if and only if every element of Σ corresponds to a separating edge of the dual graph Γ_X of X . The formula for $e_{\mathfrak{m}}(R(\Gamma))$ given in [12, Theorem 5.7(vii)] shows that if $e_{\mathfrak{m}}(R(\Gamma)) = 1$, then $\Gamma \setminus E(\Gamma)_{\text{sep}}$ has a unique totally cyclic orientation and this can only happen if $\Gamma \setminus E(\Gamma)_{\text{sep}}$ is a disjoint union of points, that is, if Γ is a tree. As Γ is obtained from Γ_X by contracting the edges not in Σ , Part (ii) follows. \square

7. Examples

In this section, we present some examples to further elucidate the connections between the results in this paper and those of [12].

7.1. Integral curves

Suppose that X_0 is an integral nodal curve of arithmetic genus g and with m nodes. From Definition 2.2, it follows that any compactified Jacobian of X is equal to the (fine) moduli space $\bar{J}^d(X)$ of rank 1, torsion-free sheaves on X of degree d (for some $d \in \mathbb{Z}$).

Consider now a point $[I] \in \bar{J}^d(X)$ such that I is not locally free at all the nodes of X (note that I is stable). Then Theorem A(i) gives that

$$\widehat{\mathcal{O}}_{\bar{J}^d(X),[I]} \cong \widehat{\bigotimes}_{i=1}^m \frac{k[[X_i, Y_i]]}{X_i Y_i} \widehat{\bigotimes} k[[T_1, \dots, T_{g-m}]]$$

We recover the well-known fact (see [11, Proposition 2.7]) that $\bar{J}^d(X_0)$ is isomorphic, formal (indeed étale) locally at I , to the product of m nodes and a smooth factor.

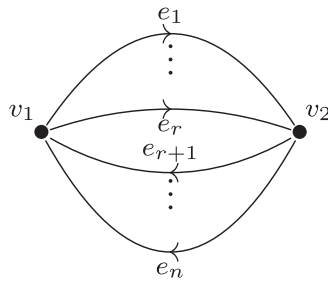
7.2. Two irreducible components

Let X_n be a nodal curve consisting of two smooth irreducible components C_1 and C_2 of genera, respectively, g_1 and g_2 , intersecting in $n \geq 2$ nodes, so that the arithmetic genus of X_n is $g = g_1 + g_2 + n - 1$ (the case $n = 1$ is easy: the curve X_1 is of compact type, hence all compactified Jacobians are smooth and isomorphic to the generalized Jacobian). The dual graph Γ_{X_n} of X_n is depicted in Figure 1 below together with an orientation of it.

Let $\bar{J}(X_n)$ be a compactified Jacobian of X_n and suppose that there exists a polystable sheaf $[I] \in \bar{J}(X_n)$ that fails to be locally free at the n nodes of the curve. Therefore, Equation (6.4) gives

$$\widehat{\mathcal{O}}_{\bar{J}^d(X),[I]} \cong \widehat{R}(\Gamma_{X_n})[[W_1, \dots, W_{g_1+g_2}]].$$

Using this presentation of the complete local ring and the results of [12], we can prove the following properties:

FIGURE 1. The orientation ϕ_r on Γ_{X_n} .

(1) Étale locally at $[I]$, $\bar{J}(X_n)$ has $\sum_{r=1}^{n-1} \binom{n}{r}$ irreducible components, which are in bijection by [12, Theorem 5.7(i)] with the totally cyclic orientations of Γ_n , all of which look like the orientation ϕ_r (for $1 \leq r \leq n-1$) depicted in the figure above.

(2) The dimension of the Zariski tangent space $T_{[I]}\bar{J}(X_n)$ (that is, the embedded dimension of $\bar{J}(X_n)$ at $[I]$) is equal to $g_1 + g_2 + 2\binom{n}{2}$, as it follows from the fact (proved in [12, Theorem 5.7(vi)]) that the embedded dimension of $R(\Gamma_{X_n})$ at the maximal ideal \mathfrak{m} is equal to the number of oriented circuits of Γ_{X_n} , which is $2\binom{n}{2}$.

(3) Finally, it can be proved using [12, Theorem 5.7(vii)] that the multiplicity of $\bar{J}(X_n)$ at $[I]$ is

$$\text{mult}_{[I]}\bar{J} = \sum_{r=1}^{n-1} \binom{n}{r} \binom{n-2}{r-1}.$$

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