

Picard group of moduli of hyperelliptic curves

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Abstract The main subject of this work is the difference between the coarse moduli space and the stack of hyperelliptic curves. We compute their Picard groups, giving explicit description of the generators. We get an application to the non-existence of a tautological family over the coarse moduli space.

Keywords Hyperelliptic curves · Coarse moduli scheme · Stack · Picard group

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1 Introduction

Throughout this paper we work over a field k of characteristic different from 2 and we fix an integer $g \geq 2$. A hyperelliptic curve of genus g over an algebraically closed field is a smooth curve of genus g which is a double cover of the projective line \mathbb{P}^1 ramified at $2g + 2$ points. We say that a smooth morphism $f : \mathcal{F} \rightarrow S$ of k -schemes is a family of hyperelliptic curves or of \mathbb{P}^1 if any geometric fiber of f is isomorphic to a hyperelliptic curve or to \mathbb{P}^1 , respectively.

In this article we are interested in comparing the (coarse) moduli space H_g of hyperelliptic curves and the moduli stack \mathcal{H}_g of hyperelliptic curves. The stack \mathcal{H}_g has been studied

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by Arsie and Vistoli (see [2] and also [16] for $g = 2$) who provided a description of it as a quotient stack and computed its Picard group, which turns out to be isomorphic to $\mathbb{Z}/(4g + 2)\mathbb{Z}$ for g even, and to $\mathbb{Z}/2(4g + 2)\mathbb{Z}$ for g odd. After some auxiliary results on H_g in Sect. 2, we compute in Sect. 3 (away from some bad characteristics of the base field k) the class group $\text{Cl}(H_g)$ and compare it with $\text{Pic}(\mathcal{H}_g)$ in Theorem 3.6 and Corollary 3.8. As an application we prove in Theorem 3.12 the non-existence of a tautological family over H_g^0 for g odd, where H_g^0 is the locus of hyperelliptic curves without extra-automorphisms (for g even a tautological family does not exist over any Zariski open subset in H_g). Also we obtain that the Picard group of the normal variety H_g is trivial, see Corollary 3.10.

Further, for $g = 2$ Vistoli proved in [2] that the Picard group $\text{Pic}(\mathcal{H}_2)$ is generated by the first Chern class of the Hodge bundle. In Theorem 4.1 from Sect. 4 we provide an explicit functorial description of a generator of the Picard group of the stack \mathcal{H}_g for arbitrary g . Moreover, in Theorem 4.2 we consider some natural elements of the Picard group (obtained by pushing-forward linear combinations of the relative canonical divisor and the Weierstrass divisor and then taking the determinant) and express them in terms of the generator found above. In particular, we show in Corollary 4.4 that the first Chern class of the Hodge bundle generates the Picard group if and only if 4 does not divide g . Otherwise the Hodge line bundle generates a subgroup of index 2.

Quite recently Cornalba has computed in [3] the Picard group of the stack of stable hyperelliptic curves. The article [3] also contains a very beautiful proof of the first assertion of Theorem 4.1 over \mathbb{C} by a quite different method from the one used in the present paper.

Let us finally mention that a more detailed version of this text can be found at the web in [5].

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2 Auxiliary results on the moduli space of hyperelliptic curves

Recall that the coarse moduli space H_g parameterizing isomorphism classes of hyperelliptic curves is an irreducible variety of dimension $2g - 1$ and can be realized as follows:

$$H_g = (\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta) / PGL_2, \tag{2.1}$$

where the action of PGL_2 is induced from the natural action on \mathbb{P}^1 and Δ is the closed subset in $\text{Sym}^{2g+2}(\mathbb{P}^1)$ consisting of all $(2g + 2)$ -tuples on \mathbb{P}^1 with at least one coincidence. We identify $\text{Sym}^{2g+2}(\mathbb{P}^1)$ with the projective space $\mathbb{B}(2, 2g + 2)$ of degree $2g + 2$ binary forms. Under this identification $\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta$ corresponds to the open subset $\mathbb{B}_{sm}(2, 2g + 2)$ of smooth forms (i.e., whose all roots are distinct) and the action of PGL_2 is defined by the formula $[A] \cdot [f(x)] = [f(A^{-1}x)]$, where $[A]$ is the class in PGL_2 of a (2×2) non-degenerate matrix A . By H_g^0 denote the open subset of H_g corresponding to hyperelliptic curves without extra-automorphisms apart from the hyperelliptic involution. Let $\mathbb{B}_{sm}(2, 2g + 2)^0$ denote the preimage of H_g^0 in $\mathbb{B}_{sm}(2, 2g + 2)$. Equivalently, $\mathbb{B}_{sm}(2, 2g + 2)^0$ consists of points in $\mathbb{B}_{sm}(2, 2g + 2)$ with trivial stabilizers in PGL_2 .

Proposition 2.1 *The locus $H_g^{\text{aut}} = H_g - H_g^0$ has dimension g and hence codimension $g - 1$ in H_g . Moreover, it has a unique irreducible component of maximal dimension corresponding to hyperelliptic curves that have an extra-involution (besides the hyperelliptic one), acting on $2g + 2$ ramification points as a product of $g + 1$ commuting transpositions.*

Proof The automorphism group $\text{Aut}(C)$ of a hyperelliptic curve C always contains the hyperelliptic involution i as a central element. Consider the group $G = \text{Aut}(C)/\langle i \rangle$. There is a canonical inclusion inside the symmetric group $G \subset S_{2g+2}$, since every automorphism of a hyperelliptic curve acts on the ramification divisor. Hence the variety H_g^{aut} decomposes into the strata

$$H_g^{\text{aut}} = \bigcup_{p \leq 2g+2} H_g^{\text{aut},p},$$

where the union is taken over all primes p , $p \leq 2g - 2$ and $H_g^{\text{aut},p}$ is the set of hyperelliptic curves such that there exists an element of order p in the corresponding group G . There is a canonical finite map $H_g^{\text{aut},p\text{-fixed}} \rightarrow H_g^{\text{aut},p}$, where $H_g^{\text{aut},p\text{-fixed}}$ is the coarse moduli space of pairs (C, σ) such that C is a curve from $H_g^{\text{aut},p}$ and σ is an element of order p in the group G associated with C .

Since $\sigma \in G$ is uniquely determined by any automorphism of \mathbb{P}^1 preserving the ramification divisor, we see that, in fact, $H_g^{\text{aut},p\text{-fixed}}$ is the coarse moduli space of pairs (D, τ) such that D is a reduced effective divisor of degree $2g + 2$ on \mathbb{P}^1 and τ is an automorphism of \mathbb{P}^1 of order p that satisfies $\tau(D) = D$.

Consider the natural quotient map

$$\pi : \mathbb{P}^1 = \mathbb{P}_1^1 \xrightarrow{p:1} \mathbb{P}_2^1 = \mathbb{P}^1 / \langle \tau \rangle.$$

Since p is prime, it is well-known that π has a cyclic ramification of order p at two points $x_1, x_2 \in \mathbb{P}_1^1$ and τ is uniquely determined by the points x_1 and x_2 . There are three possibilities for the divisor $D \subset \mathbb{P}_1^1$:

- (0) D contains no points among x_1 and x_2 ,
- (1) D contains only one point among x_1 and x_2 ,
- (2) D contains both points x_1 and x_2 .

Hence we get one more stratification:

$$H_g^{\text{aut},p\text{-fixed}} = \bigcup_{l=0,1,2} H_g^{\text{aut},p\text{-fixed},l},$$

where $H_g^{\text{aut},p\text{-fixed},l}$ is the coarse moduli space of pairs (R, E) such that R and E are non-intersecting reduced effective divisors on \mathbb{P}_2^1 of degrees 2 and $(2g + 2 - l)/p$, respectively (in particular, we require that $2g + 2 - l$ is divisible by p). Thus we get the equality

$$\dim H_g^{\text{aut},p\text{-fixed},l} = 2 + \frac{2g + 2 - l}{p} - 3 = \frac{2g + 2 - l}{p} - 1.$$

Notice that the case $p = 2$ and $l = 1$ is impossible because of the divisibility condition. Further, if $p \geq 3$, then

$$\frac{2g + 2 - l}{p} - 1 \leq \frac{2g + 2}{3} - 1 \leq g - 1$$

and for $p = 2, l = 2$ we have

$$\frac{2g + 2 - 2}{2} - 1 = g - 1.$$

So, we get the inequality

$$\dim \left(H_g^{\text{aut}, 2\text{-fixed}, 2} \cup \bigcup_{3 \leq p} H_g^{\text{aut}, p\text{-fixed}} \right) = \max_{(p,l) \neq (2,0)} \{ \dim(H_g^{\text{aut}, p\text{-fixed}, l}) \} \leq g - 1.$$

Suppose that $p = 2$ and $l = 0$; then $\dim(H_g^{\text{aut}, 2\text{-fixed}, 0}) = g$. We claim that in this case the curve C has an element $\tilde{\sigma}$ of order two in the automorphism group $\text{Aut}(C)$ itself (not only in G). Indeed, consider the composition

$$\varphi : C \xrightarrow{2:1} \mathbb{P}_1^1 \xrightarrow{2:1} \mathbb{P}_2^1.$$

This is a Galois map of degree 4 with Galois group H generated in $\text{Aut}(C)$ by any preimage $\tilde{\sigma} \in \text{Aut}(C)$ of $\sigma \in G$ and i . It is easily seen that the ramification of φ consists only of pairs of double points. If $H \cong \mathbb{Z}/4\mathbb{Z}$, then the inertia group at all ramification points of φ would be the same, namely $\langle i \rangle \subset H$. This would mean that the map $\pi : \mathbb{P}_1^1 = C/\langle i \rangle \rightarrow \mathbb{P}_2^1$ is unramified. This contradiction shows that $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\tilde{\sigma} \in \text{Aut}(C)$ has order two.

Conversely, if $\text{Aut}(C)$ has an element $\sigma \neq i$ of order two, then the corresponding number l equals zero. Indeed, otherwise the inertia group of φ at any point from $D \cap \{x_1, x_2\}$ would be isomorphic to $\mathbb{Z}/4\mathbb{Z}$, hence H would be isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and would have only one element of order two.

Note that $H_g^{\text{aut}, 2\text{-fixed}, 0}$ is irreducible and, moreover, it follows from the explicit geometric description of the ramification of the covering $\varphi : C \rightarrow \mathbb{P}_2^1$ that $\sigma \in G \subset S_{2g+2}$ must be equal to the product of $g + 1$ commuting transpositions. This completes the proof.

Remark 2.2 It is possible to give a purely combinatorial proof of a weaker version of this proposition (see [5, Proposition 4.3’]).

Remark 2.3 It follows from Proposition 2.1 that for $g \geq 3$ the smooth locus of the normal variety H_g is equal to H_g^0 (see [5, Proposition 4.5]).

Remark 2.4 It is interesting to compare the above results with the analogous ones for the coarse moduli space M_g of smooth curves of genus $g \geq 3$. The locus M_g^{aut} of curves with non-trivial automorphisms is a closed subset of dimension $2g - 1$, and it has a unique irreducible component of maximal dimension corresponding to hyperelliptic curves. Moreover, the smooth locus of M_g is equal to M_g^0 if $g \geq 4$, while the smooth locus of M_3^{smooth} is equal to $M_3^0 \cup H_3^0$ (see [9, 13–15]).

The following result is needed for the sequel.

Lemma 2.5 *Let D be the unique irreducible divisor on $\mathbb{B}_{sm}(2, 6)$ from $\mathbb{B}_{sm}(2, 6) - \mathbb{B}_{sm}(2, 6)^0$ (see Proposition 2.1) and let \bar{D} be its closure in $\mathbb{B}(2, 6)$. Then \bar{D} is an irreducible hypersurface in $\mathbb{B}(2, 6) = \mathbb{P}^6$ of degree 15.*

Proof Consider the natural map $\pi : (\mathbb{P}^1)^6 \xrightarrow{/S_6} \text{Sym}^6(\mathbb{P}^1)$. Suppose that an element $\sigma \in S_6$ is conjugate to the permutation (12)(34)(56). Denote by D_σ the divisor in $(\mathbb{P}^1)^6$ consisting of all points $(P_1, \dots, P_6) \in (\mathbb{P}^1)^6$ such that there exists a non-trivial element $A \in PGL_2$ that satisfies $A(P_1, \dots, P_6) = (\sigma(P_1), \dots, \sigma(P_6))$.

It follows from Proposition 2.1 that $\pi^{-1}(\overline{D}) = \bigcup_{\sigma} D_{\sigma}$, where the union is taken over the 15 elements of S_6 conjugated to (12)(34)(56).

Let us compute the class of D_{σ} in the Picard group $\text{Pic}((\mathbb{P}^1)^6) \cong (\mathbb{Z})^6$. Without loss of generality we may suppose that $\sigma = (12)(34)(56)$. Take a line $l = \{P_1\} \times \dots \times \{P_5\} \times \mathbb{P}^1$ in $(\mathbb{P}^1)^6$ for general points $P_i \in \mathbb{P}^1$. It is well-known that there exists a unique non-trivial element $A \in PGL_2$ exchanging P_1 with P_2 and P_3 with P_4 . In particular, A has order two. Hence the point $P_6 = A(P_5)$ is uniquely determined and the intersection $l \cap D_{(12)(34)(56)}$ consists of one point. It is easy to prove that this intersection is actually transversal. By the symmetry of $D_{(12)(34)(56)}$, the same is true for all other “coordinate” lines in $(\mathbb{P}^1)^6$ and the class of $\overline{D}_{(12)(34)(56)}$ in $\text{Pic}((\mathbb{P}^1)^6)$ is equal to $(1, 1, 1, 1, 1, 1)$. This completes the proof of Lemma 2.5.

3 Comparison between Picard groups of moduli space and stack of hyperelliptic curves

Recall that \mathcal{H}_g is a category such that objects in \mathcal{H}_g are families $\pi : \mathcal{F} \rightarrow S$ of hyperelliptic curves of genus g and morphisms in \mathcal{H}_g are Cartesian diagrams between such families. Associating the base to a family, we obtain that \mathcal{H}_g is a category fibered in groupoids over the category of k -schemes. By \mathcal{H}_g^0 denote the full fibered subcategory of \mathcal{H}_g such that the objects in \mathcal{H}_g^0 are families of hyperelliptic curves whose geometric fibers have no extra-automorphisms.

Let us cite from [2] two fundamental facts about the fibered category \mathcal{H}_g . We keep notations from the previous section.

Theorem 3.1 (Arsie–Vistoli)

- (i) *The fibered category \mathcal{H}_g is a Deligne–Mumford algebraic stack and can be realized as $\mathcal{H}_g = [\mathbb{A}_{sm}(2, 2g + 2)/(GL_2/\mu_{g+1})]$ with the usual action given by $[A] \cdot f(x) = f(A^{-1} \cdot x)$, where μ_n denotes the group scheme of n -th power roots of unity for n prime to $\text{char}(k)$.*
- (ii) *Suppose that $\text{char}(k)$ does not divide $2g + 2$. Then the Picard group $\text{Pic}(\mathcal{H}_g)$ is the quotient of $\text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}(2, 2g + 2)) = (GL_2/\mu_{g+1})^* = \mathbb{Z}$ of order equal to $4g + 2$ if g is even and $2(4g + 2)$ if g is odd (where $G^* = \text{Hom}(G, \mathbb{G}_m)$ for an algebraic group G).*

In addition, there is a well-known explicit description of $(GL_2/\mu_{2g+2})^*$.

Lemma 3.2

- (i) *If g is even, then there is an isomorphism of algebraic groups $GL_2/\mu_{g+1} \rightarrow GL_2$ given by $[A] \mapsto \det(A)^{\frac{g}{2}} A$ and $(GL_2/\mu_{g+1})^* = \mathbb{Z}$ is generated by \det^{g+1} .*
- (ii) *If g is odd, then there is an isomorphism of algebraic groups $GL_2/\mu_{g+1} \rightarrow \mathbb{G}_m \times PGL_2$ given by $[A] \mapsto \left(\det(A)^{\frac{g+1}{2}}, [A] \right)$ and $(GL_2/\mu_{g+1})^* = \mathbb{Z}$ is generated by $\det^{\frac{g+1}{2}}$.*

Now we compare the stack \mathcal{H}_g and its coarse moduli scheme H_g . In particular, we compare the Picard group of \mathcal{H}_g^0 and the Picard group of H_g^0 . With this aim it is natural to introduce a new stack, which is “intermediate” between \mathcal{H}_g and H_g .

Definition 3.3 Let \mathcal{D}_{2g+2} be a category such that the objects in \mathcal{D}_{2g+2} are families $p : \mathcal{C} \rightarrow S$ of \mathbb{P}^1 together with an effective Cartier divisor $D \subset \mathcal{C}$ finite and étale over S of degree $2g + 2$ and and the morphisms in \mathcal{D}_{2g+2} are natural Cartesian diagrams.

Associating the base to each family, we obtain that \mathcal{D}_{2g+2} is a category fibered in groupoids over the category of k -schemes. We say that a divisor $D \subset \mathbb{P}^1$ has no automorphisms if there is no non-trivial element $f \in \text{Aut}(\mathbb{P}^1)$ such that $f(D) = D$. By \mathcal{D}_{2g+2}^0 denote the full fibered subcategory of \mathcal{D}_{2g+2} such that the objects in \mathcal{D}_{2g+2}^0 are families whose all geometric fibers have no automorphisms.

The following result is analogous to Theorem 3.1, as well as its proof.

Proposition 3.4 *The fibered category \mathcal{D}_{2g+2} is a Deligne–Mumford algebraic stack and can be realized as $[\mathbb{B}_{sm}(2, 2g + 2)/PGL_2]$ with the usual action given by $[A] \cdot [f(x)] = [f(A^{-1} \cdot x)]$. Moreover, there is a natural isomorphism of stacks $[\mathbb{B}_{sm}(2, 2g + 2)/PGL_2] \cong [\mathbb{A}_{sm}(2, 2g + 2)/(GL_2/\mu_{2g+2})]$, where we consider the same action of GL_2 on $\mathbb{A}_{sm}(2, 2g + 2)$ as in Theorem 3.1.*

Proof Consider the auxiliary functor $\tilde{\mathcal{D}}_{2g+2}$ that associates with a k -scheme S the set of collection

$$\tilde{\mathcal{D}}_{2g+2}(S) = \left\{ \left(\mathcal{C} \xrightarrow{p} S, D, \phi : \mathcal{C} \cong \mathbb{P}_S^1 \right) \right\},$$

where the family $\mathcal{C} \rightarrow S$ and the divisor D are as in Definition 3.3 and ϕ is an isomorphism over S between the family $\mathcal{C} \rightarrow S$ and the trivial family $\mathbb{P}_S^1 := S \times_k \mathbb{P}^1$. Clearly, $\tilde{\mathcal{D}}_{2g+2} \cong \text{Hom}(-, \mathbb{B}_{sm}(2, 2g + 2))$. The group sheaf $\underline{\text{Aut}}(\mathbb{P}^1) = PGL_2$ acts on $\tilde{\mathcal{D}}_{2g+2}$ by composing with the isomorphism ϕ and it is easy to check that the corresponding action of PGL_2 on $\mathbb{B}_{sm}(2, 2g + 2)$ is the one given in the statement of Proposition 3.4. Finally, descent theory implies that the forgetful morphism $\tilde{\mathcal{D}}_{2g+2} \rightarrow \mathcal{D}_{2g+2}$ is a principle bundle over \mathcal{D}_{2g+2} with the group PGL_2 . Thus we get the description of \mathcal{D}_{2g+2} as a quotient stack $[\mathbb{B}_{sm}(2, 2g + 2)/PGL_2]$.

To prove the second part of the proposition observe that, applying Lemma 3.2(ii) with $g + 1$ replaced by $2g + 2$, one deduces an isomorphism $GL_2/\mu_{2g+2} \cong \mathbb{G}_m \times PGL_2$. It is easily shown that the corresponding action of $\mathbb{G}_m \times PGL_2$ on $\mathbb{A}_{sm}(2, 2g + 2)$ is given by $(\alpha, [A]) \cdot f(x) = \alpha^{-1} \cdot (\det A)^{g+1} f(A^{-1} \cdot x)$. Hence the quotient stack of $\mathbb{A}_{sm}(2, 2g + 2)$ by $GL_2/\mu_{2g+2} \cong \mathbb{G}_m \times PGL_2$ can be taken in two steps: first, we take the quotient over the subgroup $\mathbb{G}_m/\mu_{2g+2} \cong \mathbb{G}_m$, which is isomorphic to $\mathbb{B}_{sm}(2, 2g + 2)$ since the action is free, and then we take the quotient over $GL_2/\mathbb{G}_m \cong PGL_2$ with the usual action.

From these explicit descriptions we get a diagram

$$\begin{array}{ccc} \mathcal{H}_g & \xrightarrow{\Psi} & \mathcal{D}_{2g+2} \\ & \searrow \Phi_{\mathcal{H}} & \swarrow \Phi_{\mathcal{D}} \\ & & H_g \end{array}$$

where H_g is the coarse moduli space for both stacks and the morphism $\Psi : \mathcal{H}_g \rightarrow \mathcal{D}_{2g+2}$ corresponds to the fact that every family $\pi : \mathcal{F} \rightarrow S$ of hyperelliptic curves is a double cover of a family $p : \mathcal{C} \rightarrow S$ of \mathbb{P}^1 such that the ramification divisor $W \subset \mathcal{F}$ and the branch divisor $D \subset \mathcal{C}$ are both finite and étale over S of degree $2g + 2$ (see [10]).

The following result is well-known.

Corollary 3.5 *There is an isomorphism $\mathcal{D}_{2g+2}^0 \cong H_g^0$, i.e., H_g^0 is a fine moduli scheme for \mathcal{D}_{2g+2}^0 .*

Proof By Proposition 3.4, $\mathcal{D}_{2g+2}^0 \cong [\mathbb{B}_{sm}(2, 2g + 2)^0/PGL_2]$. By definition, the action of PGL_2 on $\mathbb{B}_{sm}(2, 2g + 2)^0$ is free and a standard fact is that that the line bundle $\mathcal{O}(1)$ on $\mathbb{B}(2, 2g + 2)$ admits a canonical PGL_2 -linearization. Therefore, $[\mathbb{B}_{sm}(2, 2g + 2)^0/PGL_2] = \mathbb{B}_{sm}(2, 2g + 2)^0/PGL_2 = H_g^0$.

Now we compute the Picard group of the stack \mathcal{D}_{2g+2} and relate it to the Picard group of \mathcal{H}_g .

Theorem 3.6 *The natural map $\text{Pic}(\mathcal{D}_{2g+2}) \rightarrow \text{Pic}(\mathcal{H}_g)$ is injective, being an isomorphism for g even and an inclusion of index 2 for g odd. Therefore if $\text{char}(k)$ does not divide $2g + 2$ then $\text{Pic}(\mathcal{D}_{2g+2}) = \mathbb{Z}/(4g + 2)\mathbb{Z}$.*

Proof Theorem 3.1(i) and the second description from Proposition 3.4 show that there exists a Cartesian diagram

$$\begin{array}{ccc} \mathcal{H}_g & \rightarrow & [\mathbb{A}(2, 2g + 2)/(GL_2/\mu_{g+1})] \\ \downarrow & & \downarrow \\ \mathcal{D}_{2g+2} & \rightarrow & [\mathbb{A}(2, 2g + 2)/(GL_2/\mu_{2g+2})], \end{array}$$

where each horizontal arrow is an open embedding. Hence the statement in question reduces to the study of the map $\text{Pic}^{GL_2/\mu_{2g+2}}(\mathbb{A}(2, 2g + 2)) = (GL_2/\mu_{2g+2})^* \rightarrow \text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}(2, 2g + 2)) = (GL_2/\mu_{g+1})^*$. The needed results follow immediately from Lemma 3.2 and Theorem 3.1(ii).

Remark 3.7 It can easily be checked that the map $\text{Pic}^{PGL_2}(\mathbb{B}(2, 2g + 2)) = \langle \mathcal{O}(1) \rangle \rightarrow \text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}(2, 2g + 2))$ takes $\mathcal{O}(1)$ to the character defined by \det^{g+1} .

Corollary 3.8 *Suppose that $\text{char}(k)$ does not divide $2g + 2$ and neither is equal to 5 if $g = 2$. Then we have*

$$\text{Cl}(H_g) = \text{Cl}(H_g^0) = \text{Pic}(H_g^0) = \text{Pic}(\mathcal{D}_{2g+2}^0) = \begin{cases} \mathbb{Z}/(4g + 2)\mathbb{Z} & \text{if } g \geq 3, \\ \mathbb{Z}/5\mathbb{Z} & \text{if } g = 2. \end{cases}$$

Moreover the natural map $\text{Pic}(H_g^0) \rightarrow \text{Pic}(\mathcal{H}_g^0)$ is injective, being an isomorphism for g even and an inclusion of index 2 for g odd.

Proof By Theorem 3.1, $\mathcal{H}_g^0 \cong [\mathbb{A}_{sm}(2, 2g + 2)^0/(GL_2/\mu_{g+1})]$. Since H_g^0 is smooth, $\text{Cl}(H_g^0) = \text{Pic}(H_g^0)$. By Corollary 3.5, $\text{Pic}(H_g^0) = \text{Pic}(\mathcal{D}_{2g+2}^0)$. By Proposition 2.1, we have $\text{Pic}(\mathcal{H}_g) = \text{Pic}(\mathcal{H}_g^0)$, $\text{Pic}(\mathcal{D}_{2g+2}) = \text{Pic}(\mathcal{D}_{2g+2}^0)$, and $\text{Cl}(H_g) = \text{Cl}(H_g^0)$ when $g - 1 \geq 2$. Thus the needed statement for $g \geq 3$ follows from Theorem 3.1(ii) and Theorem 3.6. For $g = 2$ the Cartesian diagram

$$\begin{array}{ccc} \mathcal{H}_2^0 & \rightarrow & \mathcal{H}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_2^0 & \rightarrow & \mathcal{D}_2 \end{array}$$

shows that the natural map $\text{Pic}(\mathcal{D}_2^0) \rightarrow \text{Pic}(\mathcal{H}_2^0)$ is an isomorphism. Further, it follows from Lemma 2.5, Theorem 3.1(ii), and Remark 3.7 that $\text{Pic}(\mathcal{H}_2^0) = \mathbb{Z}/5\mathbb{Z}$.

On the other hand, by the result of Igusa (see [8]), for $\text{char}(k) \neq 5$ we have $H_2 = \mathbb{A}^3/\mu_5$, where $\zeta \in \mu_5$ acts on \mathbb{A}^3 by formula $(x_1, x_2, x_3) \mapsto (\zeta x_1, \zeta^2 x_2, \zeta^3 x_3)$. Being the image of the origin, the unique singular point of H_2 corresponds to the curve $C_0 := \{y^2 = x^6 - x\}$. Therefore,

$$\text{Cl}(H_2) = \text{Cl}(H_2 - [C_0]) = \text{Pic}(H_2 - [C_0]) = \text{Pic}^{\mathbb{Z}/5\mathbb{Z}}(\mathbb{A}_k^3 - \{0\}) \cong \mathbb{Z}/5\mathbb{Z},$$

and the natural surjective morphism $\text{Cl}(H_2) \rightarrow \text{Cl}(H_2^0) = \text{Pic}(H_2^0) = \text{Pic}(\mathcal{H}_2^0)$ is actually an isomorphism.

Remark 3.9 It is possible to compute the Picard group $\text{Pic}(H_g^0)$ directly, without using the stack description from Corollary 3.5 and the results of Theorem 3.1 (see [5, Theorem 4.7]).

Corollary 3.10 *Suppose that $\text{char}(k)$ does not divide $(2g + 1)(2g + 2)$. Then $\text{Pic}(H_g) = 0$.*

Proof Since H_g is a normal variety, the map $\text{Pic}(H_g) \rightarrow \text{Cl}(H_g)$ is injective. Put $N_2 = 5$ and $N_g = 4g + 2$ if $g \geq 3$. By Corollary 3.8, $\text{Cl}(H_g) = \text{Pic}(\mathcal{D}_{2g+2}^0)$ is a cyclic group of order N_g generated by the image of the character \det^{g+1} under the natural map $\text{Pic}^{GL_2/\mu_{2g+2}}(\mathbb{A}(2, 2g + 2)) = (GL_2/\mu_{2g+2})^* \rightarrow \text{Pic}^{GL_2/\mu_{2g+2}}(\mathbb{A}_{sm}(2, 2g + 2)^0) = \text{Pic}(\mathcal{D}_{2g+2}^0)$. Hence $\text{Pic}(H_g)$ is contained inside the subgroup of $\text{Pic}(\mathcal{D}_{2g+2}^0)$ generated by the images of characters $\chi \in (GL_2/\mu_{2g+2})^*$ such that the restriction of χ to the GL_2 -stabilizer of any point in $\mathbb{A}_{sm}(2, 2g + 2)$ is equal to a multiple of the character $\det^{N_g(g+1)}$.

Since $\text{char}(k)$ does not divide $(2g + 1)(2g + 2)$, the binary forms $f_1 := X^{2g+1}Y - Y^{2g+2}$ and $f_2 := X^{2g+2} - Y^{2g+2}$ belong to $\mathbb{A}_{sm}(2, 2g + 2)$. The GL_2 -subgroups $\text{diag}\{\mu_{2g+1}, 1\}$ and $\text{diag}\{\mu_{2g+2}, 1\}$ stabilize f_1 and f_2 , respectively. This concludes the proof.

Remark 3.11 For the moduli spaces of smooth curves of genus $g \geq 3$ over \mathbb{C} we have

$$\mathbb{Z} = \text{Pic}(M_g) \hookrightarrow \text{Cl}(M_g) = \text{Cl}(M_g^0) = \text{Pic}(M_g^0) = \text{Pic}(\mathcal{M}_g^0) = \text{Pic}(\mathcal{M}_g) = \mathbb{Z}.$$

Still the index of the first group inside the second one remains unknown (see [1, Sect. 4]).

Now let us give an application of the comparison between Picard groups. Recall that a tautological family of hyperelliptic curves exists over a non-empty Zariski open subset in H_g if and only if g is odd (see [7, Exercise 2.3], where “universal” should be replaced by “tautological”). For g odd we get the following non-existence result.

Theorem 3.12 *For g odd there does not exist a tautological family over H_g^0 (and henceforth over all H_g).*

Proof A tautological family over H_g^0 would define a section of the modular map $H_g^0 \rightarrow \mathcal{H}_g^0$. Consequently there would be a splitting of the map $\text{Pic}(H_g^0) \hookrightarrow \text{Pic}(\mathcal{H}_g^0)$, which is impossible because of the explicit description of this map in Corollary 3.8.

Remark 3.13 It is possible to give a direct proof of the last statement: first, compute directly the Picard group of the universal family of \mathbb{P}^1 over H_g^0 , and then find explicitly the class of the universal divisor in this group and check that it is not divisible by two if g is odd, see [5, Proposition 6.13].

Remark 3.14 For $g = 1$ there exists a tautological family over H_1^0 (in this case $\text{Pic}(H_1^0) = 0$), see [12, p. 58].

4 Explicit generators of the Picard group

In this section we give an explicit construction for a generator of the Picard group $\text{Pic}(\mathcal{H}_g)$ in terms of Mumford’s functorial description of the Picard group of a stack (see [4, 12]).

Let $\pi : \mathcal{F} \rightarrow S$ be a family of hyperelliptic curves. In the discussion after Proposition 3.4 we introduced a family $p : \mathcal{C} \rightarrow S$ of \mathbb{P}^1 and two Cartier divisors $W \subset \mathcal{F}$ (so called Weierstrass divisor) and $D \subset \mathcal{C}$. By classical theory of double covers, there exists a line bundle \mathcal{L} on \mathcal{C} such that $(\mathcal{L}^{-1})^{\otimes 2} = \mathcal{O}_{\mathcal{C}}(D)$. This line bundle satisfies two relations:

$$f^*(\mathcal{L}^{-1}) = \mathcal{O}_{\mathcal{F}}(W), \tag{4.1}$$

$$f_*(\mathcal{O}_{\mathcal{F}}) = \mathcal{O}_{\mathcal{C}} \oplus \mathcal{L}. \tag{4.2}$$

Moreover, Hurwitz formula tells that

$$\omega_{\mathcal{F}/S} = f^*(\omega_{\mathcal{C}/S}) \otimes \mathcal{O}_{\mathcal{F}}(W). \tag{4.3}$$

One can check that the fibered category \mathcal{H}_g is equivalent to the fibered category \mathcal{H}'_g such that the objects in \mathcal{H}'_g are collections $(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}})$, where $p : \mathcal{C} \rightarrow S$ is a family of \mathbb{P}^1 , $\mathcal{L} \in \text{Pic}(\mathcal{C})$ and the morphisms in \mathcal{H}'_g are natural Cartesian diagrams (see [2, Sect. 2]).

Theorem 4.1 *Let \mathcal{G} be the element in $\text{Pic}(\mathcal{H}_g)$ such that for any family of hyperelliptic curves $\pi : \mathcal{F} \rightarrow S$ with the Weierstrass divisor W the line bundle $\mathcal{G}(\pi)$ on S is defined by the formula*

$$\mathcal{G}(\pi) = \begin{cases} \pi_* \left(\omega_{\mathcal{F}/S}^{-(g+1)} ((g-1)W) \right) & \text{if } g \text{ is even,} \\ \pi_* \left(\omega_{\mathcal{F}/S}^{-\frac{g+1}{2}} \left(\frac{g-1}{2} W \right) \right) & \text{if } g \text{ is odd.} \end{cases}$$

Then \mathcal{G} generates the Picard group $\text{Pic}(\mathcal{H}_g)$ and equals to the image of the character χ_0 under the natural map $\text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}(2, 2g+2)) = (GL_2/\mu_{g+1})^* \rightarrow \text{Pic}(\mathcal{H}_g)$, where $\chi_0 = \det^{g+1}$ for g even and $\chi_0 = \det^{\frac{g+1}{2}}$ for g odd.

Proof By Theorem 3.1 and Lemma 3.2, $\text{Pic}(\mathcal{H}_g) = \text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}_{sm}(2, 2g+2))$ is a cyclic group generated by the trivial line bundle $\mathbb{A}_{sm}(2, 2g+2) \times k$, on which GL_2/μ_{g+1} acts by the character χ_0 .

Following the proof of Theorem 3.1 from [2], consider the auxiliary functor $\tilde{\mathcal{H}}_g$ that associates with a k -schemes S the set of collections

$$\tilde{\mathcal{H}}_g(S) = \{(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}}, \phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1)))\},$$

where $p : \mathcal{C} \rightarrow S$, $\mathcal{L} \in \text{Pic}(\mathcal{C})$ are as above and the isomorphism ϕ consists of an isomorphisms of S -schemes $\phi_0 : \mathcal{C} \cong \mathbb{P}^1_S$ plus an isomorphism of invertible sheaves $\phi_1 : \mathcal{L} \cong \phi_0^* \mathcal{O}_{\mathbb{P}^1_S}(-g-1)$.

In [2], it was proved that $\tilde{\mathcal{H}}_g \cong \text{Hom}(-, \mathbb{A}_{sm}(2, 2g+2))$ and that the forgetful morphism $\tilde{\mathcal{H}}_g \rightarrow \mathcal{H}'_g \cong \mathcal{H}_g$ is a principal bundle over \mathcal{H}_g with the group GL_2/μ_{g+1} .

Consider the following commutative diagram of GL_2/μ_{g+1} -equivariant maps:

$$\begin{array}{ccc} \tilde{\mathcal{H}}_g \times k & \xrightarrow{\cong} & \mathbb{A}_{sm}(2, 2g+2) \times k \\ \downarrow & & \downarrow \\ \tilde{\mathcal{H}}_g & \xrightarrow{\cong} & \mathbb{A}_{sm}(2, 2g+2). \end{array}$$

The functor $\tilde{\mathcal{H}}_g \times k$ associates with a k -scheme S the set of collections

$$(\tilde{\mathcal{H}}_g \times k)(S) = \left\{ \left(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}}, \phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1)), \mathcal{M} \right) \right\}$$

where $\mathcal{M} = \mathcal{O}_S$ is the trivial line bundle with the action of the group $(GL_2/\mu_{g+1})(S)$ given by the character χ_0 .

Put $\mathbb{P}_S^1 = \mathbb{P}(V_S)$, where V is a two-dimensional vector space over the ground field k . From the Euler exact sequence for the trivial family $p_S : \mathbb{P}_S^1 \rightarrow S$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_S^1} \rightarrow p_S^*(V_S^*)(1) \rightarrow \omega_{\mathbb{P}_S^1/S}^{-1} \rightarrow 0$$

one deduces the $(GL_2/\mu_{g+1})(S)$ -equivariant isomorphism

$$p_S^*((\det V_S)^{-1}) \otimes \mathcal{O}_{\mathbb{P}_S^1}(2) \cong \omega_{\mathbb{P}_S^1/S}^{-1}, \tag{4.4}$$

where we consider the canonical actions of $(GL_2/\mu_{g+1})(S)$ on \mathbb{P}_S^1 and on the invertible sheaves involved. Using projection formula, the fact that $(p_S)_*(\mathcal{O}_{\mathbb{P}_S^1}) = \mathcal{O}_S$, and the $(GL_2/\mu_{g+1})(S)$ -equivariant identity $(\det V_S)^{-(g+1)} = \mathcal{M}$ for g even, and $(\det V_S)^{-\frac{g+1}{2}} = \mathcal{M}$ for g odd, we get $(GL_2/\mu_{g+1})(S)$ -equivariant isomorphisms

$$\begin{cases} \mathcal{M} \cong (p_S)_* \left(\omega_{\mathbb{P}_S^1/S}^{-(g+1)} \otimes \mathcal{O}_{\mathbb{P}_S^1}(-(2g+2)) \right) & \text{if } g \text{ is even,} \\ \mathcal{M} \cong (p_S)_* \left(\omega_{\mathbb{P}_S^1/S}^{-\frac{g+1}{2}} \otimes \mathcal{O}_{\mathbb{P}_S^1}(-(g+1)) \right) & \text{if } g \text{ is odd.} \end{cases}$$

Let us remark that $\phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}(-g-1))$ induces a canonical isomorphism $\omega_{\mathcal{C}/S} \cong \omega_{\mathbb{P}_S^1/S}$ defined by the isomorphism ϕ_0 . Hence the quotient line bundle $\mathcal{G}' = [(\tilde{\mathcal{H}}_g \times k) / (GL_2/\mu_{g+1})]$ over \mathcal{H}'_g is isomorphic to

$$\mathcal{G}'(p) = \begin{cases} \left\{ \mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}}, p_* \left(\omega_{\mathcal{C}/S}^{-(g+1)} \otimes \mathcal{L}^2 \right) \right\} & \text{if } g \text{ is even,} \\ \left\{ \mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}}, p_* \left(\omega_{\mathcal{C}/S}^{-\frac{g+1}{2}} \otimes \mathcal{L} \right) \right\} & \text{if } g \text{ is odd.} \end{cases}$$

Now we express the preceding line bundles as push-forwards with respect to the map f of line bundles on the hyperelliptic family $\pi : \mathcal{F} \rightarrow S$. Using formulas (4.1) and (4.3), we get

$$\begin{cases} f^* \left(\omega_{\mathcal{C}/S}^{-(g+1)} \otimes \mathcal{L}^2 \right) = \omega_{\mathcal{F}/S}^{-(g+1)} ((g-1)W) & \text{if } g \text{ is even,} \\ f^* \left(\omega_{\mathcal{C}/S}^{-\frac{g+1}{2}} \otimes \mathcal{L} \right) = \omega_{\mathcal{F}/S}^{-\frac{g+1}{2}} \left(\frac{g-1}{2} W \right) & \text{if } g \text{ is odd.} \end{cases}$$

To conclude the proof it remains to note that the line bundles $\omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{\otimes (-2)}$ and $\omega_{\mathcal{C}/S}^{\frac{g+1}{2}} \otimes \mathcal{L}^{-1}$ for g odd are trivial on each geometric fiber of p and $p_* = \pi_* f^*$ for them.

Now we express some natural elements of $\text{Pic}(\mathcal{H}_g)$ in terms of the generator found above. Recall that given a family $\pi : \mathcal{F} \rightarrow S$ of hyperelliptic curves, there are two natural line bundles on \mathcal{F} : the relative canonical line bundle $\omega_{\mathcal{F}/S}$ and the line bundle $\mathcal{O}_{\mathcal{F}}(W)$ associated with the Weierstrass divisor $W = W_{\mathcal{F}/S}$. Consider their multiple $\omega_{\mathcal{F}/S}^a \otimes \mathcal{O}_{\mathcal{F}}(bW)$ for any integers a and b . Note that it restricts to any geometric fiber F of the family π as

$$\omega_{\mathcal{F}/S}^a \otimes \mathcal{O}_{\mathcal{F}}(bW)|_F = aK_F + bW_F = [a(g-1) + b(g+1)]g_2^1 = [(a+b)g + b - a]g_2^1.$$

Since $h^0(F, \mathcal{O}_F(kg_2^1)) = k + 1$ for any non-negative integer k , the push-forward $\pi_* \left(\omega_{\mathcal{F}/S}^a \otimes \mathcal{O}_{\mathcal{F}}(bW) \right)$ is a vector bundle of rank $m(a, b) + 1$ on the base S if $m(a, b) \geq 0$, where $m(a, b) := (a + b)g + b - a$. Let $\mathcal{T}_{a,b}$ be an element in $\text{Pic}(\mathcal{H}_g)$ defined by the formula

$$\mathcal{T}_{a,b}(\pi) = \det \left(\pi_* \left(\omega_{\mathcal{F}/S}^a \otimes \mathcal{O}_{\mathcal{F}}(bW) \right) \right) \in \text{Pic}(S).$$

Proposition 4.2 *If $0 \leq m(a, b) < g + 1$, then*

$$\mathcal{T}_{a,b} = \begin{cases} \mathcal{G}^{-\frac{(a+b)(m(a,b)+1)}{2}} & \text{if } g \text{ is even,} \\ \mathcal{G}^{-(a+b)(m(a,b)+1)} & \text{if } g \text{ is odd,} \end{cases}$$

and if $m(a, b) \geq g + 1$, then

$$\mathcal{T}_{a,b} = \begin{cases} \mathcal{G}^{\frac{(a+b-1)(g-m(a,b))}{2}} & \text{if } g \text{ is even,} \\ \mathcal{G}^{(a+b-1)(g-m(a,b))} & \text{if } g \text{ is odd.} \end{cases}$$

Proof It follows from formulas (4.1), (4.2), and (4.3) that $\mathcal{T}_{a,b} \in \text{Pic}(\mathcal{H}_g)$ corresponds to an element $\mathcal{T}'_{a,b} \in \text{Pic}(\mathcal{H}'_g)$ that associates with an object $\left(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{i} \mathcal{O}_{\mathcal{C}} \right)$ from $\mathcal{H}'_g(S)$ the line bundle

$$\mathcal{T}'_{a,b}(p) = \det p_* \left(\omega_{\mathcal{C}/S}^a \otimes \mathcal{L}^{-(a+b)} \right) \otimes \det p_* \left(\omega_{\mathcal{C}/S}^a \otimes \mathcal{L}^{-(a+b)+1} \right) \in \text{Pic}(S).$$

Now we compute the pull-back $\tilde{\mathcal{T}}_{a,b}$ of $\mathcal{T}'_{a,b}$ to $\text{Pic}(\tilde{\mathcal{H}}_g)$. Using the isomorphism $\phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}^1_S = \mathbb{P}(V_S), \mathcal{O}(-g - 1))$ and the Euler formula (4.4), we obtain

$$\begin{aligned} \omega_{\mathcal{C}/S}^a \otimes \mathcal{L}^{-(a+b)} &\cong p_S^* \left((\det V_S)^a \right) \otimes \mathcal{O}_{\mathbb{P}^1_S}(-2a) \otimes \mathcal{O}_{\mathbb{P}^1_S}((a + b)(g + 1)) = \\ &= p_S^* \left((\det V_S)^a \right) \otimes \mathcal{O}_{\mathbb{P}^1_S}(m(a, b)) \end{aligned}$$

and, analogously,

$$\omega_{\mathcal{C}/S}^a \otimes \mathcal{L}^{-(a+b)+1} \cong p_S^* \left((\det V_S)^a \right) \otimes \mathcal{O}_{\mathbb{P}^1_S}(m(a, b) - (g + 1)).$$

Using the relation $\det(\text{Sym}^n(V_S)) = (\det V_S)^{\frac{n(n+1)}{2}}$, we get

$$\begin{aligned} \det(p_S)_* \left(p_S^* \left((\det V_S)^a \right) \otimes \mathcal{O}_{\mathbb{P}^1_S}(m(a, b)) \right) &= \det \left((\det V_S)^a \otimes \text{Sym}^{m(a,b)}(V_S) \right) = \\ &= (\det V_S)^{(a+b)(g+1)\frac{m(a,b)+1}{2}}, \end{aligned}$$

and, analogously,

$$\det(p_S)_* \left(p_S^* \left((\det V_S)^a \right) \otimes \mathcal{O}_{\mathbb{P}^1_S}(m(a, b) - (g + 1)) \right) = (\det V_S)^{(a+b-1)(g+1)\frac{m(a,b)-g}{2}}$$

if $m(a, b) \geq g + 1$ and, otherwise, the latter push-forward is zero. We conclude using the relation $(\det V_S)^{-(g+1)} = \mathcal{M}$, where \mathcal{M} is the pull-back from $\text{Pic}(\mathcal{H}_g)$ to $\text{Pic}(\tilde{\mathcal{H}}_g)$ of the generator \mathcal{G} (see the proof of Theorem 4.1).

Remark 4.3 It is possible to prove the relations from Proposition 4.2 directly for a given family of hyperelliptic curves $\pi : \mathcal{F} \rightarrow S$ without using this stack computation, see [5, Theorem 5.8].

Among the elements $\mathcal{T}_{a,b}$ one is of particular interest, namely the Hodge line bundle $\pi_* \left(\omega_{\mathcal{F}/S} \right)$, which equals $\mathcal{T}_{1,0}$. The following result was proved for $g = 2$ by Vistoli in [16].

Corollary 4.4

(i) *The Hodge line bundle is equal to*

$$\det \pi_*(\omega_{\mathcal{F}/S}) = \begin{cases} \mathcal{G}^{g/2} & \text{if } g \text{ is even,} \\ \mathcal{G}^g & \text{if } g \text{ is odd.} \end{cases}$$

(ii) *Suppose that char(k) does not divide 2g + 2. Then the Hodge line bundle generates the Picard group Pic(H_g) if g is not divisible by 4. Otherwise, it generates a subgroup of index 2 in Pic(H_g).*

Remark 4.5 The Hodge line bundle generates the Picard group of M_g over C (see [6] and [1]).

Let us remark that for g even the generator G of the Picard group Pic(H_g) equals to T_{g/2,1-g/2} = det (π* (ω_{F/S}^{g/2} ⊗ O_F((1 - g/2)W))). It follows from the discussion before Proposition 4.2 that the restriction of the line bundle ω_{F/S}^{g/2} ⊗ O_F((1 - g/2)W) to any geometric fiber of the family π is equal to g₂¹. Note that, being non-unique, a line bundle on F with this property exists for any family π : F → S only if g is even (see [11] and [5, Theorem 3.5]).

Finally, let us provide a functorial description for a generator of the Picard group Pic(D_{2g+2}).

Proposition 4.6 *The image of the line bundle O(1) under the natural map Pic(BB(2, 2g + 2)) → Pic(D_{2g+2}) generates the Picard group Pic(D_{2g+2}) and associates with an object (C \xrightarrow{p} S, D) from D_{2g+2}(S) the line bundle p* (ω_{C/S}^{-(g+1)}(-D)) ∈ Pic(S).*

Proof We compute the image under the natural map Pic(D_{2g+2}) → Pic(H_g) of the element described in the proposition. Suppose that a family p : C → S corresponds to a family π : F → S of hyperelliptic curves. It follows from formulas (4.1) and (4.3) that

$$f^* (ω_{C/S}^{g+1}(D)) \cong ω_{F/S}^{g+1}(-(g + 1)W) \otimes O_{\mathcal{F}}(2W) = ω_{F/S}^{g+1}(-(g - 1)W).$$

Combining Theorem 3.6, Remark 3.7, Theorem 4.1 and the fact that the line bundle ω_{C/S}^{g+1}(D) is trivial on any geometric fiber of p, we get the desired statement.

Remark 4.7 It is possible to prove Proposition 4.6 for H_g⁰ instead of D_{2g+2} directly, without using the stack description, Theorems 3.6, 4.1, see [5, Theorem 6.3 and Remark 6.8].

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