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RESTRICTED INFINITESIMAL DEFORMATIONS OF RESTRICTED SIMPLE LIE ALGEBRAS

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We compute the restricted infinitesimal deformations of the restricted simple Lie algebras over an algebraically closed field of characteristic $p \geq 5$.

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1. Introduction

Simple Lie algebras over an algebraically closed field F of characteristic $p \neq 2, 3$ have recently been classified by Block–Wilson–Premet–Strade (see [1, 22–28, 31–37, 39]). The classification says that for $p \geq 7$ the simple Lie algebras can be of two types: of classical type and of generalized Cartan type.

The algebras of *classical type* are obtained by considering the simple Lie algebras in characteristic zero (classified via Dynkin diagrams), by taking a model over the integers via the Chevalley bases and then reducing modulo the prime p (see [30]).

The algebras of *generalized Cartan type* were constructed by Kostrikin–Shafarevich, Wilson and Kac ([15–20, 45, 46]) and are divided into four families, called Witt–Jacobson, special, Hamiltonian and contact algebras. These four families are the finite-dimensional analogue of the four classes of infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification [2] of Lie pseudogroups.

In characteristic $p = 5$, apart from the above two types of algebras, there is one more family of simple Lie algebras called *Melikian algebras* (introduced in [21]).

In characteristic $p = 2, 3$, there are many exceptional simple Lie algebras (see [37, p. 209]) and the classification seems still far away.

We are interested in a particular class of modular Lie algebras called *restricted*. These can be characterized as those modular Lie algebras such that the p-power of an inner derivation (which in characteristic p is a derivation) is still inner. Important examples of restricted Lie algebras are the ones coming from groups schemes. Indeed there is a one-to-one correspondence between restricted Lie algebras and finite group schemes whose Frobenius vanishes (see [7, Chap. 2]).

The aim of this paper is to compute the *restricted infinitesimal deformations* of the restricted simple Lie algebras in characteristic $p \geq 5$. By standard facts of deformation theory (see for example [10]), restricted infinitesimal deformations of a restricted Lie algebra $\mathfrak{g}$ are parameterized by the second restricted cohomology group $H^2_*(\mathfrak{g}, \mathfrak{g})$ of $\mathfrak{g}$ with values in the adjoint representation (see [12]).

The restricted simple Lie algebras of classical type are known to be rigid as Lie algebras, under the assumption $p \geq 5$ (see [29]). This is equivalent to the vanishing of the second ordinary cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$ for $\mathfrak{g}$ restricted simple of classical type. Therefore, using the so-called Hochschild 6-term exact sequence (see (2.1) below), one can easily deduce the vanishing of $H^2_*(\mathfrak{g}, \mathfrak{g})$, which implies that these algebras are rigid also as restricted Lie algebras. Note that in characteristic zero, the vanishing of $H^2(\mathfrak{g}, \mathfrak{g})$ for $\mathfrak{g}$ simple follows from a classical result, known as second Whitehead's Lemma (see for example [11]). Interestingly, some of the classical simple Lie algebras admit non-trivial deformations in characteristic $p = 2$ or 3 (see [4–6, 8]).

Under the assumption $p \geq 5$, we compute the restricted infinitesimal deformations of the restricted simple Lie algebras not of classical type: the four infinite families $W(n) := W(n, \underline{1})$, $S(n) := S(n, \underline{1})$, $K(n) := K(n, \underline{1})$, $H(n) := H(n, \underline{1})$ and the exceptional restricted Melikian algebra $M := M(1, 1)$ in characteristic $p = 5$. Using the notations about those algebras which we are going to recall in what follows and the squaring operation Sq (see Sec. 2.2), we can state our results as follows.

Theorem 1.1. *The infinitesimal restricted deformations of the restricted Jacobson–Witt algebra $W(n)$ are given by*

$$H^2_*(W(n), W(n)) = H^2(W(n), W(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(D_i) \rangle_F.$$

Theorem 1.2. *The infinitesimal restricted deformations of the restricted special algebra $S(n)$ are given by*

$$H^2_*(S(n), S(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(D_i) \rangle_F.$$

Theorem 1.3. *The infinitesimal restricted deformations of the restricted contact algebra $K(n)$ are given by*

$$H^2_*(K(n), K(n)) = H^2(K(n), K(n)) = \bigoplus_{i=1}^{2m} \langle \mathrm{Sq}(x_i)\rangle_F \oplus \langle \mathrm{Sq}(1)\rangle_F.$$

Theorem 1.4. *The infinitesimal restricted deformations of the restricted Hamiltonian algebra $H(n)$ are given by*

$$H^2_*(H(n), H(n)) = \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(x_i)\rangle_F \oplus \langle \Phi\rangle_F,$$

where the cocycle Φ is defined as

$$\Phi(x^a, x^b) = \sum_{\substack{\underline{0}<\delta\leq a,\widehat{b}\\ |\delta|=3}} \binom{a}{\delta}\binom{b}{\widehat{\delta}} \sigma(\delta)\, \delta!\, x^{a+b-\delta-\widehat{\delta}}.$$

Theorem 1.5. *The restricted infinitesimal deformations of the restricted Melikian algebra M are given by*

$$H^2_*(M, M) = H^2(M, M) = \langle \mathrm{Sq}(1)\rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i)\rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(\widetilde{D_i})\rangle_F.$$

We prove these theorems using the computation of the second ordinary cohomology group $H^2(\mathfrak{g},\mathfrak{g})$ that we performed in [41–43], together with the 6-term exact sequence of Hochschild (see Sec. 2.1) that relates the ordinary and restricted cohomology.

Note that the second restricted cohomology group for the above algebras is freely generated over F by the squaring operators of the elements of negative degree, with one remarkable exception: For the Hamiltonian algebra $H(n)$, there is an exceptional extra-cocycle Φ. Observe moreover that, quite interestingly, for the special algebras $S(n)$ and the Hamiltonian algebras $H(n)$, the restricted cohomology group $H^2_*(\mathfrak{g},\mathfrak{g})$ is a proper subgroup of the ordinary cohomology group $H^2(\mathfrak{g},\mathfrak{g})$. In other words, there are infinitesimal deformations of the algebra that do not admit a restricted structure.

Since the restricted infinitesimal deformations of a restricted Lie algebra $\mathfrak{g}$ correspond to the infinitesimal deformations of the associated finite group scheme G of height one, the above results give the infinitesimal deformations of some simple finite group schemes (see [40, 44] for more details). In order to complete the picture, it would be very interesting to extend the above computations to the minimal p-envelope of all the simple Lie algebras.

The paper is organized as follows. Section 2 contains preliminary results. We quickly review the ordinary and restricted cohomology of Lie algebras and the 6-term Hochschild exact sequence relating the first two ordinary and restricted cohomology groups. Moreover we recall the definition of the squaring operation.

In each of the remaining five sections, we recall the basic definitions of the five classes of restricted simple Lie algebras of non-classical type and we compute the corresponding infinitesimal deformations.

2. Preliminaries

2.1. *Ordinary and restricted cohomology*

In this section we review, in order to fix notations, the ordinary and restricted cohomology of Lie algebras, following [12, 13].

Let $\mathfrak{g}$ be a Lie algebra over a field F. We denote by $U_{\mathfrak{g}}$ the universal enveloping algebra of $\mathfrak{g}$ and by $I_{\mathfrak{g}}$ its augmentation ideal, that is the kernel of the augmentation map $\epsilon : U_{\mathfrak{g}} \to F$. For a $\mathfrak{g}$-module M (or, equivalently, an unital $U_{\mathfrak{g}}$-module), the (ordinary) cohomology groups $H^n(\mathfrak{g}, M)$ are the right derived functor of the fixed point functor $M \mapsto M^{\mathfrak{g}}$, considered as a functor from the category of $\mathfrak{g}$-modules to the category of abelian groups. They can be computed into two different ways, using a Lie complex or an associative complex.

The Lie complex has n-dimensional cochains $C^n(\mathfrak{g}, M) = \{f : \Lambda^n(\mathfrak{g}) \to M\}$ and differential $\mathrm{d} : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ defined by

$$\begin{aligned} \mathrm{d}f(x_0, \ldots, x_n) = &\sum_{i=0}^{n} (-1)^i x_i \cdot f(x_0, \ldots, \hat{x}_i, \ldots, x_n) \\ &+ \sum_{p<q} (-1)^{p+q} f([x_p, x_q], x_0, \ldots, \hat{x}_p, \ldots, \hat{x}_q, \ldots, x_n), \end{aligned}$$

where the sign $\hat{}$ means that the argument below must be omitted.

The associative complex has n-dimensional cochains $C^n(I_{\mathfrak{g}}, M) = \{g : I_{\mathfrak{g}}^{\otimes n} \to M\}$, and differential $\mathrm{d} : C^n(I_{\mathfrak{g}}, M) \to C^{n+1}(I_{\mathfrak{g}}, M)$ defined by

$$\mathrm{d}g(s_0, \ldots, s_n) = s_0 \cdot g(s_1, \ldots, s_n) + \sum_{i=1}^{n} (-1)^i g(s_0, \ldots, s_{i-1}s_i, \ldots, s_n).$$

Now let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over F. Denote by $U_{\mathfrak{g}}^{[p]} := U_{\mathfrak{g}}/(x^p - x^{[p]})$ the restricted enveloping algebra of $(\mathfrak{g}, [p])$ and with $I_{\mathfrak{g}}^{[p]}$ its augmented ideal. For a restricted $\mathfrak{g}$-module M (or, equivalently, an unital $U_{\mathfrak{g}}^{[p]}$-module), the restricted cohomology groups $H_*^n(\mathfrak{g}, M)$ are the right derived functor of the fixed point functor $M \mapsto M^{\mathfrak{g}}$, considered as a functor from the category of restricted $\mathfrak{g}$-modules to the category of abelian groups. Explicitly, these can be calculated via an associative complex which is obtained from the one described above for ordinary cohomology groups simply by replacing I_g with $I_{\mathfrak{g}}^{[p]}$. Observe, on the other hand, that the ordinary Lie complex does not generalize to restricted cohomology, a fact which makes the computation of the restricted cohomology harder than the ordinary one.

There is a 6-term exact sequence relating the first two ordinary and restricted cohomology groups (see [12]):

$$\begin{aligned} 0 \to H^1_*(\mathfrak{g}, M) \to H^1(\mathfrak{g}, M) &\xrightarrow{D} \mathrm{Hom}_{Fr}(\mathfrak{g}, M^L) \to \\ \to H^2_*(\mathfrak{g}, M) \to H^2(\mathfrak{g}, M) &\xrightarrow{H} \mathrm{Hom}_{Fr}(\mathfrak{g}, H^1(\mathfrak{g}, M)), \end{aligned} \tag{2.1}$$

where $\mathrm{Hom}_{Fr}(V, W)$ denotes the Frobenius-semilinear morphisms between the two F-vector spaces V and W, that is

$$\mathrm{Hom}_{Fr}(V, W) = \{f : V \to W \mid f(\alpha x + \beta y) = \alpha^p f(x) + \beta^p f(y)\},$$

for any $\alpha, \beta \in F, x, y \in V$ and D and H are defined on the Lie cochains $\phi \in H^1(\mathfrak{g}, M)$ and $\psi \in H^2(\mathfrak{g}, M)$ as, respectively (for any $x, y \in \mathfrak{g}$):

$$\begin{cases} D_\phi(x) = x^{p-1} \circ \phi(x) - \phi(x^{[p]}), \\ H_\psi(x) \cdot y = \displaystyle\sum_{j=0}^{p-1} x^j \circ \psi(x, (\mathrm{ad}\, x)^{p-1-j}(y)) - \psi(x^{[p]}, y). \end{cases}$$

In the particular case in which $M = \mathfrak{g}$ is the adjoint representation and the algebra $\mathfrak{g}$ has no center, the above 6-term exact sequence (2.1) becomes

$$\begin{cases} H^1_*(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}), \\ 0 \to H^2_*(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathfrak{g}) \xrightarrow{H} \mathrm{Hom}_{Fr}(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g})), \end{cases} \tag{2.2}$$

and the operator H becomes (for any $\psi \in H^2(\mathfrak{g}, \mathfrak{g})$ and $x, y \in \mathfrak{g}$)

$$H_\psi(x) \cdot y = \sum_{j=0}^{p-1} (\mathrm{ad}\, x)^j \circ \psi(x, (\mathrm{ad}\, x)^{p-1-j}(y)) - \psi(x^{[p]}, y). \tag{2.3}$$

Remark 2.1. The Hochschild 6-term exact sequence (2.1) has been interpreted as the initial sequence of two different spectral sequences relating the ordinary and restricted cohomology:

$$\begin{cases} E_1^{p,q} = \mathrm{Hom}_{Fr}(S^p\mathfrak{g}, H^{q-p}(\mathfrak{g}, M)) \Rightarrow H^{p+q}_*(\mathfrak{g}, M), \quad \text{if } p \neq 2, \quad \text{(see [14])} \\ E_2^{p,q} = \mathrm{Hom}_{Fr}(\Lambda^q\mathfrak{g}, H^p_*(\mathfrak{g}, M)) \Rightarrow H^{p+q}(\mathfrak{g}, M), \quad \text{(see [9])} \end{cases}$$

where $S^p\mathfrak{g}$ and $\Lambda^q\mathfrak{g}$ denote, respectively, the pth symmetric power and the qth alternating power.

2.2. *Squaring operation*

There is a canonical way to produce 2-cocycles in $H^2(\mathfrak{g}, \mathfrak{g})$ over a field of characteristic $p > 0$, namely the squaring operation (see [10]). Given a derivation γ (inner or not), one defines the squaring of γ to be

$$\mathrm{Sq}(\gamma)(x,y) = \sum_{i=1}^{p-1} \frac{[\gamma^i(x), \gamma^{p-i}(y)]}{i!(p-i)!} \tag{2.4}$$

where γ^i is the i-th iteration of γ. In [10] it is shown that $[\mathrm{Sq}(\gamma)] \in H^2(\mathfrak{g},\mathfrak{g})$ is an obstruction to integrability of the derivation γ, that is to the possibility of finding an automorphism of $\mathfrak{g}$ extending the infinitesimal automorphism given by γ.

3. The Witt–Jacobson Algebra

Let us recall the definition of the restricted Witt–Jacobson algebra, following [38, Sec. 4.2]. Let $A(n) = A(n;\underline{1}) := F[x_1,\ldots,x_n]/(x_1^p,\ldots,x_n^p)$ be the ring of p-truncated polynomials in n variables over a field F of positive characteristic $p \geq 5$.

Definition 3.1. The restricted Witt–Jacobson algebra $W(n) = W(n;\underline{1})$ is the restricted Lie algebra $\mathrm{Der}_F A(n)$ of derivations of $A(n) = F[x_1,\ldots,x_n]/(x_1^p,\ldots,x_n^p)$.

The Witt–Jacobson algebra $W(n)$ is a free $A(n)$-module with basis $\{D_1,\ldots,D_n\}$, where we put $D_j := \frac{\partial}{\partial x_j}$. Therefore $\dim_F(W(n)) = np^n$ with a basis over F given by $\{x^a D_j \mid 1 \leq j \leq n,\ 0 \leq a_i \leq p-1\}$. It has a natural grading obtained by assigning to the element $x^a D_j$ the degree $|a| - 1 := \sum_{i=1}^n a_i - 1$. In particular the elements of negative degree are $W(n)_{-1} = \langle D_1,\ldots,D_n\rangle_F$. The $[p]$-map is defined on the elements of the base by

$$(x^a D_j)^{[p]} = \begin{cases} x^a D_j, & \text{if } x^a D_j = x_j D_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is a classical result of Celousov (see [3] or [38, Sec. 4.8]) that every derivation of $W(n)$ is inner or in other words that

$$H^1_*(W(n),W(n)) = H^1(W(n),W(n)) = 0. \tag{3.1}$$

Therefore, from the Hochschild exact sequence (2.2), we deduce that $H^2_*(W(n), W(n)) = H^2(W(n),W(n))$. The Theorem 1.1 follows from [41, Theorem 1.1]:

$$H^2(W(n),W(n)) = \bigoplus_{i=1}^n \langle \mathrm{Sq}(D_i)\rangle_F.$$

4. The Special Algebra

Let us recall the definition of the restricted special algebra, following [38, Sec. 4.3]. Fix an integer $n \geq 3$ and a field F of characteristic $p \geq 5$. Consider the following map, called divergence:

$$\operatorname{div}: \begin{cases} W(n) \to A(n), \\ \sum_{i=1}^{n} f_i D_i \mapsto \sum_{i=1}^{n} D_i(f_i). \end{cases}$$

The kernel of the divergence map $S'(n) = S'(n;\underline{1}) = \{E \in W(n) \mid \operatorname{div}(E) = 0\}$ is a graded subalgebra of $W(n)$ of dimension $(n-1)p^n + 1$.

Definition 4.1. The restricted special algebra $S(n) = S(n;\underline{1})$ is the derived subalgebra of $S'(n)$:

$$S(n) := S'(n)^{(1)} = [S'(n), S'(n)].$$

It turns out that there is an exact sequence

$$0 \to S(n) \to S'(n) \to \bigoplus_{i=1}^{n} \langle x^{\tau-(p-1)\epsilon_i} D_i \rangle_F \to 0,$$

where $\tau := (p-1, \ldots, p-1)$ and ϵ_i is the n-tuple having 1 at the ith place and 0 outside. Therefore $S(n)$ has F-dimension $(n-1)(p^n-1)$. A set of generators (but not linearly independent!) of $S(n)$ is given by the elements $\{D_{ij}(f) \mid f \in A(n),\ 1 \leq i < j \leq n\}$, where the maps D_{ij} are defined by:

$$D_{ij} : \begin{cases} A(n) \to W(n), \\ f \mapsto D_j(f)D_i - D_i(f)D_j. \end{cases}$$

In particular, the elements of negative degree are $S(n)_{-1} = \langle D_1, \ldots, D_n \rangle_F$. The $[p]$-map on the above generators is given by

$$D_{ij}(x^a)^{[p]} = \begin{cases} D_{ij}(x^a), & \text{if } x^a = x_i x_j, \\ 0, & \text{otherwise.} \end{cases}$$

The first cohomology group of the adjoint representation is equal to (see [3] or [38, Sec. 4.8]):

$$H^1_*(S(n), S(n)) = H^1(S(n), S(n)) = \bigoplus_{i=1}^{n} \langle \operatorname{ad}(x^{\tau-(p-1)\epsilon_i} D_i) \rangle_F \bigoplus \langle \operatorname{ad}(x_1 D_1) \rangle_F.$$

From this result, we can deduce a criterion saying when a derivation of $S(n)$ is inner. First we introduce the following notation. Observe that, expressing any element $E \in S(n)$ as F-linear combination of the generators $D_{ij}(x^\tau)$, the coefficients of the terms of minimal degree $D_{ij}(x_j) = D_i$ and of maximal degree $D_{ij}(x^\tau) = x^{\tau-\epsilon_j} D_i - x^{\tau-\epsilon_i} D_j$ are well-defined, that is they are the same for any such expression of E. We call the above coefficients E_{D_i} and $E_{D_{ij}(x^\tau)}$, respectively.

Lemma 4.2. *A derivation $\gamma : S(n) \to S(n)$ is inner if and only it satisfies the following conditions*:

(i) *For every $1 \le i \le n$, there exists $j \ne i$ such that $\gamma(x_i^{p-1}D_j)_{D_{ij}(x^\tau)} = 0$.*
(ii) $\sum_{k=1}^n \gamma(D_k)_{D_k} = 0$.

Proof. We first prove that the two conditions are necessary. Consider an inner derivation $\mathrm{ad}(D)$, with $D \in S(n) \subset W(n)$. To prove condition (i), write D as linear combination of the base elements $x^a D_h \in W(n)$. Consider the element of $W(n)$ given by (for $i \ne j$)

$$\mathrm{ad}(x^a D_h)(x_i^{p-1}D_j) = [x^a D_h, x_i^{p-1}D_j] = x^a D_h(x_i^{p-1})D_j - x_i^{p-1}D_j(x^a)D_h.$$

Clearly, the two elements at the end cannot be equal to $x^{\tau-\epsilon_j}D_i$ and therefore $\mathrm{ad}(D)(x_i^{p-1}D_j)_{D_{ij}(x^\tau)} = 0$. To prove condition (ii), write $D = \sum_{i=1}^n a_i x_i D_i + E$ with $E_{x_i D_i} = 0$ for every i. Clearly $D \in S(n)$ if and only if $E \in S(n)$ and $\sum_{i=1}^n a_i = 0$. We compute

$$\sum_{k=1}^n \mathrm{ad}(D)(D_k)_{D_k} = -\sum_{k=1}^n a_k = 0.$$

However, the two conditions are also sufficient since

$$\begin{cases} \displaystyle\sum_{k=1}^n \mathrm{ad}(x_1D_1)(D_k)_{D_k} = \mathrm{ad}(x_1D_1)(D_1)_{D_1} = -1, \\ \mathrm{ad}(x^{\tau-(p-1)\epsilon_i}D_i)(x_i^{p-1}D_j) = -x^{\tau-\epsilon_i}D_j + x^{\tau-\epsilon_j}D_i = -D_{ij}(x^\tau). \end{cases}$$ □

In [41, Theorem 1.2], we prove that the second ordinary cohomology group of the adjoint representation of $S(n)$ is

$$H^2(S(n), S(n)) = \bigoplus_{i=1}^n \langle \mathrm{Sq}(D_i)\rangle_F \bigoplus \langle \Theta\rangle_F, \tag{4.1}$$

where Θ is defined by $\Theta(D_i, D_j) = D_{ij}(x^\tau)$ and extended by 0 elsewhere. Using this result and the Hochschild exact sequence (2.2), we can compute the second restricted cohomology group.

Proof of Theorem 1.2. The cocycle Θ does not belong to $H_*^2(S(n), S(n))$. Indeed, using that $D_i^{[p]} = 0$, we compute (for $i \ne j$)

$$H_\Theta(D_i) \cdot x_i^{p-1}D_j = \sum_{k=0}^{p-1} D_i^{p-1-k}\Theta(D_i, D_i^k(x_i^{p-1}D_j)) = -\Theta(D_i, D_j) = -D_{ij}(x^\tau).$$

Therefore, according to Lemma 4.2(i), the derivation $H_\Theta(D_i)$ is not inner and hence $\Theta \notin H_*^2(S(n), S(n))$ by the Hochschild exact sequence (2.2).

On the other hand, we are going to prove that $\mathrm{Sq}(D_h) \in H_*^2(S(n), S(n))$ (for any h) by showing that for any $D_{rs}(x^a) \in S(n)$ the derivation $H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a))$ satisfies the two conditions of Lemma 4.2.

Suppose first, by contradiction, that the first condition of Lemma 4.2 is not satisfied for certain indices $i \neq j$, that is $[H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot x_i^{p-1} D_j]_{D_{ij}(x^\tau)} \neq 0$. Then the index i must be equal to r or s and therefore, since we can choose the index $j \neq i$, we can assume without loss of generality that $(i,j) = (r,s)$. However, from the definition of the operator H, it is straightforward to see that

$$H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot x_r^{p-1} D_s \in \langle x^{pa-\epsilon_r-p\epsilon_s-p\epsilon_h} D_s, x^{pa-(p+1)\epsilon_s-p\epsilon_h} D_r \rangle_F,$$

and this contradicts the hypothesis since the multi-index $pa - \epsilon_r - p\epsilon_s - p\epsilon_h$ cannot be equal to the multi-index $\tau - \epsilon_r$.

Suppose next, again by contradiction, that the second condition of Lemma 4.2 is not satisfied, that is $\sum_k [H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot D_k]_{D_k} \neq 0$. Then the element $D_{rs}(x^a)$ must have degree 1 and the formula (2.3) for H simplifies as

$$H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot D_k = \mathrm{Sq}(D_h)(D_{rs}(x^a), \mathrm{ad}\, D_{rs}(x^a)^{p-1} \cdot D_k).$$

From this formula it is straightforward to see that

$$H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot D_k \in \langle D_{rs}(x^{pa-p\epsilon_h-(p-1)(\epsilon_r+\epsilon_s)-\epsilon_k}) \rangle_F.$$

Therefore if $[H_{\mathrm{Sq}(D_h)}(D_{rs}(x^a)) \cdot D_k]_{D_k} \neq 0$, we must have that $x^a = x_r x_s x_h$ and that $k = r$ or s. Now we distinguish two cases, according to whether h is equal to one of the two indices r and s, or not. If $h \neq r, s$, using the formulas

$$\begin{cases} \mathrm{ad}\, D_{rs}(x_r x_s x_h)^{p-1} \cdot D_r = D_{rs}(x_s x_h^{p-1}), \\ \mathrm{ad}\, D_{rs}(x_r x_s x_h)^{p-1} \cdot D_s = -D_{rs}(x_r x_h^{p-1}), \end{cases}$$

we get a contradiction with the non-vanishing hypothesis because of the following

$$\begin{cases} H_{\mathrm{Sq}(D_h)}(D_{rs}(x_r x_s x_h)) \cdot D_r = \mathrm{Sq}(D_h)(D_{rs}(x_r x_s x_h), D_{rs}(x_s x_h^{p-1})) \\ \qquad\qquad = [D_{rs}(x_r x_s), D_{rs}(x_s)] = -D_r, \\ H_{\mathrm{Sq}(D_h)}(D_{rs}(x_r x_s x_h)) \cdot D_s = \mathrm{Sq}(D_h)(D_{rs}(x_r x_s x_h), -D_{rs}(x_r x_h^{p-1})) \\ \qquad\qquad = -[D_{rs}(x_r x_s), D_{rs}(x_r)] = D_s. \end{cases}$$

On the other hand, if $h = r \neq s$, one can prove by induction on $1 \leq t \leq p-1$ that

$$\begin{cases} \mathrm{ad}\, D_{hs}(x_h^2 x_s)^t \cdot D_h = \prod_{u=1}^{t} (u-3) \cdot D_{hs}(x_h^t x_s), \\ \mathrm{ad}\, D_{hs}(x_h^2 x_s)^t \cdot D_s = -t!\, D_{hs}(x_h^{t+1}). \end{cases}$$

Therefore both the above expressions vanish for $t = p-1$, which contradicts our assumption. $\square$

5. The Contact Algebra

Let us recall the definition of the restricted contact algebra. Fix an odd integer $n = 2m+1 \geq 3$ and a field F of characteristic $p \geq 5$. For any $j \in \{1, \ldots, 2m\}$, we define the sign $\sigma(j)$ and the conjugate j' of j as follows:

$$\sigma(j) = \begin{cases} 1, & \text{if } 1 \leq j \leq m, \\ -1, & \text{if } m < j \leq 2m, \end{cases} \quad \text{and} \quad j' = \begin{cases} j+m, & \text{if } 1 \leq j \leq m, \\ j-m, & \text{if } m < j \leq 2m. \end{cases}$$

Consider the operator $D_H : A(n) \to W(n)$ defined as

$$D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_{j'} = \sum_{i=1}^{m} [D_i(f) D_{i+m} - D_{i+m}(f) D_i],$$

where, as usual, $D_i := \frac{\partial}{\partial x_i} \in W(n)$. We denote by $K'(n) = K'(n, \underline{1})$ the graded Lie algebra over F whose underlying F-vector space is $A(n)$, endowed with the grading defined by $\deg(x^a) = |a| + a_n - 2 = \sum_{i=1}^{2m} a_i + 2a_n - 2$ and with the Lie bracket defined by

$$[x^a, x^b] = D_H(x^a)(x^b) + [a_n \deg(x^b) - b_n \deg(x^a)] x^{a+b-\epsilon_n}.$$

Definition 5.1. The contact algebra is the derived subalgebra of $K'(n)$:

$$K(n) = K(n; \underline{1}) = K'(n)^{(1)} = [K'(n), K'(n)].$$

Indeed it turns out that

$$K(n) = \begin{cases} K'(n), & \text{if } p \nmid (m+2), \\ K'(n)_{\neq \tau}, & \text{if } p \mid (m+2), \end{cases}$$

where $K'(n)_{\neq\tau}$ is the sub-vector space of $K'(n)$ generated over F by the monomials x^a such that $a \neq \tau := (p-1, \ldots, p-1)$. Note that the elements of negative degree are $K(n)_{-2} = \langle 1 \rangle_F$ and $K(n)_{-1} = \langle x_1, \ldots, x_n \rangle_F$. The $[p]$-map on the base $\{x^a\}$ is given by

$$(x^a)^{[p]} = \begin{cases} x^a, & \text{if } x^a = x_i x_{i'} \text{ or } x^a = x_n, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that our notations for the contact algebra differ slightly from the ones of [38, Sec. 4.5], since we have dropped the operator D_K used there. Therefore, our elements x^a correspond to their elements $D_K(x^a)$.

The first cohomology group of the adjoint representation is equal to (see [3] or [38, Sec. 4.8]):

$$H^1_*(K(n), K(n)) = H^1(K(n), K(n)) = \begin{cases} 0, & \text{if } p \nmid (m+2), \\ \langle \operatorname{ad} x^\tau \rangle_F, & \text{if } p \mid (m+2). \end{cases} \tag{5.1}$$

From this result, we can deduce a criterion saying when a derivation of $K(n)$ is inner in the case $p|(m+2)$ (in the other case, every derivation is inner). As usual, for any two elements $E, x^a \in K(n)$, we indicate with E_{x^a} the coefficient of E with respect to the base element x^a.

Lemma 5.2. *Suppose that p divides $m+2$. Then a derivation $\gamma : K(n) \to K(n)$ is inner if and only if $\gamma(1)_{x^{\tau-\epsilon_n}} = 0$.*

Proof. We first prove that the condition is necessary. Consider an inner derivation $\mathrm{ad}(x^a)$, with $x^a \in K(n)$. Then, from the computation $\mathrm{ad}(x^a)(1) = [x^a, 1] = -2a_n x^{a-\epsilon_n}$, we deduce that $\mathrm{ad}(x^a)(1)_{x^{\tau-\epsilon_n}} = 0$ since $x^\tau \notin K(n)$ by the hypothesis $p|(m+2)$. However, the condition is also sufficient since $\mathrm{ad}(x^\tau)(1) = 2x^{\tau-\epsilon_n}$. $\square$

In [42, Theorem 1.1], we prove that the second cohomology group of the adjoint representation is

$$H^2(K(n), K(n)) = \bigoplus_{i=1}^{2m} \langle \mathrm{Sq}(x_i)\rangle_F \oplus \langle \mathrm{Sq}(1)\rangle_F. \tag{5.2}$$

Using this result and the exact sequence (2.2), we compute the second restricted cohomology group.

Proof of Theorem 1.3. The theorem is clearly true in the case when p does not divide $m+2$, because in this case the first cohomology group vanishes. In the case where $p \,|\, (m+2)$, we are going to show that for any $x^a \in K(n)$ the derivations $H_{\mathrm{Sq}(1)}(x^a)$ and $H_{\mathrm{Sq}(x_i)}(x^a)$ satisfy the condition of Lemma 5.2.

Consider first the cocycle $\mathrm{Sq}(1) = 2\mathrm{Sq}(D_n)$. From the definition (2.3) of the operator H, it is straightforward to check that

$$H_{\mathrm{Sq}(1)}(x^a) \cdot 1 \in \langle x^{pa-2p\epsilon_n}\rangle_F.$$

Therefore the condition of Lemma 5.2 is satisfied since the multi-index $pa - 2p\epsilon_n$ cannot be equal to $\tau - \epsilon_n$.

Consider next the cocycle $\mathrm{Sq}(x_i)$. From the definition (2.3) of H together with the fact that $\mathrm{ad}(x_i) = \sigma(i) D_{i'} + x_i D_n$, it is straightforward to check that

$$H_{\mathrm{Sq}(x_i)}(x^a) \cdot 1 \in \left\langle \sum_{r,s\in\mathbb{Z}} x^{pa + r\epsilon_i - (p-r)\epsilon_{i'} - s\epsilon_n} \right\rangle_F.$$

Since $n = 2m+1 \neq 3$ by the hypothesis $p \,|\, (m+2)$ and $p \geq 5$, the multi-index $pa + r\epsilon_i - (p-r)\epsilon_{i'} - s\epsilon_n$ cannot be equal to $\tau - \epsilon_n$ and therefore the condition of Lemma 5.2 is satisfied. $\square$

6. The Hamiltonian Algebra

Let us recall the definition of the restricted Hamiltonian algebra. Fix an even integer $n = 2m \geq 2$ and a field F of characteristic $p \geq 5$.

We introduce some notations that will be used in this section. As before, for any $j \in \{1, \ldots, 2m\}$, we define the sign $\sigma(j)$ and the conjugate j' of j as follows:

$$\sigma(j) = \begin{cases} 1, & \text{if } 1 \leq j \leq m, \\ -1, & \text{if } m < j \leq 2m, \end{cases} \quad \text{and} \quad j' = \begin{cases} j + m, & \text{if } 1 \leq j \leq m, \\ j - m, & \text{if } m < j \leq 2m. \end{cases}$$

Given two n-tuples of natural numbers $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we say that $a \leq b$ if $a_i \leq b_i$ for every i. We define the degree of $a \in \mathbb{N}^n$ as $|a| = \sum_{i=1}^n a_i$ and the factorial as $a! = \prod_{i=1}^n a_i!$. For two multi-indices $a, b \in \mathbb{N}^n$ such that $b \leq a$, we set $\binom{a}{b} := \prod_{i=1}^n \binom{a_i}{b_i} = \frac{a!}{b!(a-b)!}$. Moreover, we define the sign of $a \in \mathbb{N}^{2m}$ as $\sigma(a) = \prod \sigma(i)^{a_i}$ and the conjugate of a as the multi-index $\hat{a}$ such that $\hat{a}_i = a_{i'}$ for every $1 \leq i \leq 2m$. We set $\tau := (p-1, \ldots, p-1)$ (as usual) and $\underline{0} := (0, \ldots, 0)$.

We denote by $\widetilde{H(n)} = \widetilde{H(n; \underline{1})}$ the graded F-Lie algebra whose underlying vector space is $A(n)$, endowed with the grading defined by $\deg(x^a) = |a| - 2$ and with the Lie bracket defined by

$$[x^a, x^b] = D_H(x^a)(x^b),$$

where $D_H : A(n) \to W(n)$ is defined (as before) by

$$D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_{j'} = \sum_{i=1}^{m} \left[D_i(f) D_{i+m} - D_{i+m}(f) D_i \right].$$

We denote by $H'(n) = H'(n, \underline{1})$ the quotient of $\widetilde{H(n)}$ by the central element 1.

Definition 6.1. The restricted Hamiltonian algebra is the derived subalgebra of $H'(n)$:

$$H(n) = H(n; \underline{1}) = H'(n)^{(1)} = [H'(n), H'(n)].$$

Observe that $H(n)$ has F-dimension $p^n - 2$, with a base given by the elements $\{x^a\}$ such that $x^a \neq 1$ and $x^a \neq x^\tau$. The elements of negative degree are $H(n)_{-1} = \langle x_1, \ldots, x_n \rangle_F$. On the elements of the base, the $[p]$-map is given by

$$(x^a)^{[p]} = \begin{cases} x^a, & \text{if } x^a = x_i x_{i'}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that our notations for the Hamiltonian algebra differ slightly from the ones of [38, Sec. 4.4], since for simplicity we have dropped the operator D_H used there. Therefore, our elements x^a correspond to their elements $D_H(x^a)$.

The first cohomology group of the adjoint representation is given by (see [3] or [38, Sec. 4.8]):

$$H^1_*(H(n), H(n)) = H^1(H(n), H(n)) = \langle \operatorname{ad} x^\tau \rangle_F \bigoplus_{i=1}^{n} \langle x_i^{p-1} D_{i'} \rangle_F \oplus \langle \underline{\deg} \rangle_F, \quad (6.1)$$

where $x_i^{p-1}D_{i'}$ is the derivation which sends $x^a \in H(n)$ into $x_i^{p-1}D_{i'}(x^a) \in H(n)$ and $\underline{\deg}$ is the operator degree defined by $\underline{\deg}(x^a) = \deg(x^a)\, x^a$.

From this result, we can deduce a criterion saying when a derivation of $H(n)$ is inner. As usual, given two elements $E, x^a \in H(n)$, we denote by E_{x^a} the coefficient of E with respect to the base element x^a.

Lemma 6.2. *A derivation $\gamma : H(n) \to H(n)$ is inner if and only if it satisfies the three following conditions*:

(i) *There exists an index i such that $\gamma(x_i)_{x^{\tau-\epsilon_{i'}}} = 0$.*
(ii) *For any index i, it holds that $\gamma(x^{\tau-(p-1)\epsilon_i})_{x^{\tau-\epsilon_{i'}}} = 0$.*
(iii) *There exists an index i such that $\gamma(x_i)_{x_i} + \gamma(x_{i'})_{x_{i'}} = 0$.*

Proof. We first prove that the three conditions are necessary. Consider an inner derivation $\mathrm{ad}(D)$, with $D \in H(n)$. Write D as linear combination of the base elements x^a. The element $\mathrm{ad}(x^a)(x_i) = [x^a, x_i] = -\sigma(i)a_{i'}x^{a-\epsilon_{i'}}$ cannot belong to $\langle x^{\tau-\epsilon_{i'}}\rangle_F$ since $x^\tau \notin H(n)$. Therefore the condition (i) is verified. Consider now the element

$$[x^a, x^{\tau-(p-1)\epsilon_i}] = -\sigma(i)a_i x^{a+\tau-p\epsilon_i-\epsilon_{i'}} + \sum_{j\neq i,i'} \sigma(j)[a_{j'} - a_j]x^{a+\tau-(p-1)\epsilon_i-\epsilon_j-\epsilon_{j'}}.$$

We have that $x^{a+\tau-p\epsilon_i-\epsilon_{i'}} \notin \langle x^{\tau-\epsilon_{i'}}\rangle_F$ since $a_i \leq p-1$ and $x^{a+\tau-(p-1)\epsilon_i-\epsilon_j-\epsilon_{j'}} \notin \langle x^{\tau-\epsilon_{i'}}\rangle_F$ since $a_{i'} \geq 0$. Therefore condition (ii) is verified. Finally, to prove condition (iii), we write $D = \sum_{i=1}^m a_i x_i x_{i'} + E$ with $E_{x_i x_{i'}} = 0$ for every i. For any index i, we have that

$$D(x_i)_{x_i} + D(x_{i'})_{x_{i'}} = -\sigma(i)a_i + \sigma(i)a_i = 0,$$

and therefore also condition (iii) is verified. However, the three conditions are also sufficient since

$$\begin{cases} \mathrm{ad}(x^\tau)(x_i) = [x^\tau, x_i] = \sigma(i)x^{\tau-\epsilon_{i'}}, \\ (x_i^{p-1}D_{i'})(x^{\tau-(p-1)\epsilon_i}) = -x^{\tau-\epsilon_{i'}}, \\ \underline{\deg}(x_i)_{x_i} + \underline{\deg}(x_{i'})_{x_{i'}} = -1-1 = -2. \end{cases}$$

□

In [41, Theorem 1.2], we prove that the second cohomology group of the adjoint representation is

$$H^2(H(n), H(n)) = \begin{cases} \bigoplus_{i=1}^{n} \langle \mathrm{Sq}(x_i)\rangle_F \bigoplus_{\substack{i<j \\ j\neq i'}} \langle \Pi_{ij}\rangle_F \bigoplus_{i=1}^{m} \langle \Pi_i\rangle_F \oplus \langle \Phi\rangle_F, & \text{if } n \geq 4, \\ \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(x_i)\rangle_F \oplus \langle \Phi\rangle_F, & \text{if } n = 2, \end{cases} \tag{6.2}$$

where the above cocycles are defined (and vanish elsewhere) by

$$\begin{cases} \Pi_{ij}(x^a, x^b) = x_{i'}^{p-1} x_{j'}^{p-1}[D_i(x^a)D_j(x^b) - D_i(x^b)D_j(x^a)], & \text{for } j \neq i, i', \\ \Pi_i(x_i x^a, x_{i'} x^b) = x^{a+b+(p-1)(\epsilon_i+\epsilon_{i'})}, & \\ \qquad \text{if } a+b \neq \tau - (p-1)(\epsilon_i + \epsilon_{i'}), & \\ \Pi_i(x_k, x^{\tau-(p-1)(\epsilon_i+\epsilon_{i'})}) = -\sigma(k)x^{\tau-\epsilon_{k'}}, & \text{for any } 1 \leq k \leq n, \\ \Phi(x^a, x^b) = \displaystyle\sum_{\substack{\underline{0} \leq \delta \leq a, \widehat{b} \\ |\delta|=3}} \binom{a}{\delta}\binom{b}{\widehat{\delta}} \sigma(\delta)\delta! x^{a+b-\delta-\widehat{\delta}}. & \end{cases}$$

Using the above result (6.2) and the Hochschild exact sequence (2.2), we can compute the second restricted cohomology group.

Proof of Theorem 1.4. For $n \geq 4$, the cocycles Π_{ij} and Π_i do not belong to $H_*^2(H(n), H(n))$. To unify the notation we set (only during this proof) $\Pi_{ii'} = \Pi_i$ for $1 \leq i \leq m$. Then, using that $x_r^{[p]} = 0$, we compute (for $i < j$):

$$\begin{cases} H_{\Pi_{ij}}(x_i) \cdot x^{\tau-(p-1)\epsilon_{j'}} = \Pi_{ij}(x_i, (\operatorname{ad} x_i)^{p-1}(x^{\tau-(p-1)\epsilon_{j'}})) = \sigma(i)x^{\tau-\epsilon_j}, \\ H_{\Pi_{ij}}(x_j) \cdot x^{\tau-(p-1)\epsilon_{i'}} = \Pi_{ij}(x_j, (\operatorname{ad} x_j)^{p-1}(x^{\tau-(p-1)\epsilon_{i'}})) = -\sigma(j)x^{\tau-\epsilon_i}, \\ H_{\Pi_{ij}}(x_k) \cdot x^{\tau-(p-1)\epsilon_h} = 0, \quad \text{if } k \neq i, j \quad \text{or} \quad h \neq i', j'. \end{cases}$$

Therefore according to Lemma 6.2(ii), no linear combination of Π_{ij} can be in the kernel of the map H and therefore in $H_*^2(H(n), H(n))$.

Next we prove that $\mathrm{Sq}(x_h) = \sigma(h)\mathrm{Sq}(D_{h'}) \in H_*^2(H(n), H(n))$ by showing that for any $x^a \in H(n)$ the derivation $H_{\mathrm{Sq}(x_h)}(x^a)$ satisfies the conditions of Lemma 6.2. Suppose, by contradiction, that the condition (ii) of Lemma 6.2 is not satisfied for some index i. Then for degree reasons we must have

$$p - 2 = \deg(x_i^{p-1} D_{i'}) = \deg(\mathrm{Sq}(x_h)) + p\deg(x^a) = -p + p\deg(x^a),$$

which is impossible since $p \neq 2$. Suppose now, by contradiction, that the condition (iii) of Lemma 6.2 is not satisfied and in particular that

$$[H_{\mathrm{Sq}(x_h)}(x^a) \cdot x_h]_{x_h} + [H_{\mathrm{Sq}(x_h)}(x^a) \cdot x_{h'}]_{x_{h'}} \neq 0. \tag{*}$$

From the definition of H, it is straightforward to see that

$$\begin{cases} H_{\mathrm{Sq}(x_h)}(x^a) \cdot x_h \in \langle x^{pa-2p\epsilon_{h'}-(p-1)\epsilon_h} \rangle_F, \\ H_{\mathrm{Sq}(x_h)}(x^a) \cdot x_{h'} \in \langle x^{pa-(2p-1)\epsilon_{h'}-p\epsilon_h} \rangle_F. \end{cases} \tag{**}$$

From the hypothesis (*), it follows that $x^a = x_{h'}^2 x_h$ and therefore the formula (2.3) simplifies (for any index k) as follows

$$H_{\mathrm{Sq}(x_h)}(x_{h'}^2 x_h) \cdot x_k = \mathrm{Sq}(x_h)(x_{h'}^2 x_h, (\operatorname{ad} x_{h'}^2 x_h)^{p-1} x_k).$$

By induction on $1 \leq r \leq p-1$, one can verify that

$$\begin{cases}(\operatorname{ad} x_{h'}^2 x_h)^r x_{h'} = \sigma(h')^r(-1)(-2)\cdots(-r)x_{h'}^{r+1},\\ (\operatorname{ad} x_{h'}^2 x_h)^r x_h = \sigma(h')^r 2(2-1)\cdots(2-(r-1))x_{h'}^r x_h.\end{cases}$$

In particular we have that $(\operatorname{ad} x_{h'}^2 x_h)^{p-1} x_{h'} = (\operatorname{ad} x_{h'}^2 x_h)^{p-1} x_h = 0$ and, substituting in the above expression, we get $H_{\mathrm{Sq}(x_h)}(x_{h'}^2 x_h)\cdot x_{h'} = H_{\mathrm{Sq}(x_h)}(x_{h'}^2 x_h)\cdot x_h$ contradicting the hypothesis (*). Finally, by using the first equation of (**) and the fact that the multi-index $pa - 2p\epsilon_{h'} - (p-1)\epsilon_h$ cannot be equal to $\tau - \epsilon_{h'}$, the condition (i) of Lemma 6.2 is satisfied for $i = h$.

Finally, we prove that $\Phi \in H_*^2(H(n), H(n))$ by showing that for any $x^a \in H(n)$ the derivation $H_\Phi(x^a)$ satisfies the conditions of Lemma 6.2. Suppose, by contradiction, that the condition (ii) of Lemma 6.2 is not satisfied for some index i. Then for degree reasons, we must have

$$p - 2 = \deg(x_i^{p-1} D_{i'}) = \deg(\Phi) + p\deg(x^a) = -4 + p\deg(x^a),$$

a contradiction. Analogously, if the condition (iii) of the Lemma is not satisfied by $H_\Phi(x^a)$ for some index i, then we get a contradiction by looking at the degree

$$0 = \deg(\Phi) + p\deg(x^a) = -4 + p\deg(x^a).$$

Finally, suppose that the condition (i) of Lemma 6.2 is not satisfied for some index i, that is $[H_\Phi(x^a)\cdot x_i]_{x^{\tau-\epsilon_{i'}}} \neq 0$. Then, by looking at the degree, we get that

$$\deg(x^a) = 2m - 2\frac{m-1}{p} > 0.$$

In particular we have that $p \mid (m-1)$, from which we deduce that either $m = 1$ or $m \geq p+1 \geq 6$. Suppose first that $m \neq 1$. Then, from the formula (2.3) and using that $(x^a)^{[p]} = 0$, we deduce that

$$H_\Phi(x^a)\cdot x_i \in \sum_{|\delta|=3}\sum_{k=0}^{p-1}\langle(\operatorname{ad} x^a)^{p-1-k}(x^{(k+1)a-(k-1)\epsilon_i - k\epsilon_{i'} - \delta - \widehat{\delta}})\rangle_F.$$

Fix a multi-index δ appearing in the above summation and choose an index $j \neq i, i'$ such $\delta_j = \delta_{j'} = 0$ (this is possible since $|\delta| = 3$ and $n = 2m \geq 12$). Then the jth coefficient of every monomial appearing in the expression

$$\sum_{k=0}^{p-1}(\operatorname{ad} x^a)^{p-1-k}(x^{(k+1)a-(k-1)\epsilon_i - k\epsilon_{i'} - \delta - \widehat{\delta}})$$

is $p\,a_j \neq p-1$. Therefore the monomial $x^{\tau-\epsilon_{i'}}$ cannot appear in the above expression and, repeating the same argument for every multi-index δ as before, we get that $[H_\Phi(x^a)\cdot x_i]_{x^{\tau-\epsilon_{i'}}} = 0$, a contradiction. In the remaining case $m = 1$, we have that

$$H_\Phi(x^a)\cdot x_i \in \sum_{|\delta|=3}\langle x^{pa-(p-2)\epsilon_i-(p-1)\epsilon_{i'}-\delta-\hat{\delta}}\rangle_F.$$

From this and the hypothesis $[H_\Phi(x^a)\cdot x_i]_{x^{\tau-\epsilon_{i'}}} \neq 0$, we deduce that $x^a = x_i^2 x_{i'}^2$. Using the straightforward formulas

$$\begin{cases} (\operatorname{ad} x_i^2 x_{i'}^2)^k (x_i) = [-2\sigma(i)]^k x_i^{k+1} x_{i'}^k, \\ \Phi(x_i^2 x_{i'}^2, x_i^{k+1} x_{i'}^k) = 2\sigma(i)(k+1)k x_i^k x_{i'}^{k-1}, \\ (\operatorname{ad} x_i^2 x_{i'}^2)^{p-1-k} (x_i^k x_{i'}^{k-1}) = [-2\sigma(i)]^{p-1-k} x_i^{p-1} x_{i'}^{p-2}, \end{cases}$$

we get that $H_\Phi(x^a)\cdot x_i = 2\sigma(i)\sum_{k=0}^{p-1}[(k+1)k]x^{\tau-\epsilon_{i'}} = 0$, since $\sum_{k=0}^{p-1} k \equiv 0 \mod p$ for $p \geq 3$ and $\sum_{k=0}^{p-1} k^2 \equiv 0 \mod p$ for $p \geq 5$. This contradiction finishes the proof. □

7. The Melikian Algebra

Let us recall the definition of the restricted Melikian algebra, following [37, Sec. 4.3].

Let F be a field of characteristic $p = 5$. Consider $W(2) = \mathrm{Der}_F A(2) = \mathrm{Der}_F F[x_1, x_2]/(x_1^p, x_2^p)$, the restricted Witt–Jacobson Lie algebra of rank 2. Let $\widetilde{W(2)}$ be a copy of $W(2)$ and for an element $D \in W(2)$ we indicate with $\widetilde{D}$ the corresponding element inside $\widetilde{W(2)}$. The restricted Melikian algebra $M := M(1,1)$ is defined as

$$M = A(2) \oplus W(2) \oplus \widetilde{W(2)},$$

with Lie bracket defined by the following rules (for all $D, E \in W(2)$ and $f, g \in A(2)$):

$$\begin{cases} [D, \widetilde{E}] := \widetilde{[D, E]} + 2\operatorname{div}(D)\widetilde{E}, \\ [D, f] := D(f) - 2\operatorname{div}(D) f, \\ [f_1\widetilde{D_1} + f_2\widetilde{D_2}, g_1\widetilde{D_1} + g_2\widetilde{D_2}] := f_1 g_2 - f_2 g_1, \\ [f, \widetilde{E}] := fE, \\ [f, g] := 2\left(g D_2(f) - f D_2(g)\right)\widetilde{D_1} + 2\left(f D_1(g) - g D_1(f)\right)\widetilde{D_2}, \end{cases}$$

where $\operatorname{div}(f_1 D_1 + f_2 D_2) := D_1(f_1) + D_2(f_2) \in A(2)$. The Melikian algebra M has a $\mathbb{Z}$-grading given by (for all $D, E \in W(2)$ and $f \in A(2)$):

$$\begin{cases} \deg_M(D) := 3\deg(D), \\ \deg_M(\widetilde{E}) := 3\deg(E) + 2, \\ \deg_M(f) := 3\deg(f) - 2. \end{cases}$$

In particular the elements of negative degree are

$$M_{-3} = \langle D_1, D_2\rangle_F, \quad M_{-2} = \langle 1\rangle_F, \quad M_{-1} = \langle \widetilde{D_1}, \widetilde{D_2}\rangle_F.$$

The $[p]$-map is defined on an element $X \in M$ by

$$X^{[p]} = \begin{cases} X, & \text{if } X = x_1 D_1 \text{ or } x_2 D_2, \\ 0, & \text{otherwise.} \end{cases}$$

It is known that every derivation of M is inner (see [37, Sec. 4.3]), that is

$$H^1_*(M, M) = H^1(M, M) = 0. \tag{7.1}$$

Therefore, from the Hochschild exact sequence (2.2), we deduce that $H^2_*(M, M) = H^2(M, M)$. The Theorem 1.5 follows from [43, Theorem 1.1]:

$$H^2(M, M) = \langle \mathrm{Sq}(1) \rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i) \rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(\widetilde{D_i}) \rangle_F.$$

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