Blow up of affine schemes, fat points and universal property of the blow up

Luca Schaffler

Spring, 2013

In our first talk, we suggested an approach to blow ups from the point of viev of classical algebraic geometry, and this approach is useful to get hands dirty with blow ups. Now we would like to give a more modern approach to the subject using the language of schemes. What we will discuss is a natural generalization to what discussed so far.

1 Blow up of affine schemes: the blow up algebra

Let A be a noetherian ring (commutative, with unit). Take an ideal $I \subseteq A$ and define the *blow up algebra of A along I* as:

$$\mathrm{Bl}_I A := \bigoplus_{d \geq 0} I^d, \text{ where } I^0 := A.$$

The blow up algebra as an obvious structure of graded A-algebra. If t is an indeterminate over A it's easy to see that there is an isomorphism of graded A-algebras:

$$\mathrm{Bl}_I A \cong A[It] = A[it | i \in I].$$

Now, if $I = (g_0, \ldots, g_s)$, $A[It] = A[g_j t | j = 0, \ldots, s]$, which is nothing else than the image of that morphism ψ defined in the first talk (see [S]).
We define the blow up of $X := \mathrm{Spec}(A)$ along the subscheme $V(I)$ to be:

$$\mathrm{Bl}_{V(I)} X := \mathrm{Proj}(\mathrm{Bl}_I A),$$

together with the induced morphism $\sigma \colon \mathrm{Bl}_{V(I)} X \to X$. We want to remark that $\mathrm{Bl}_{V(I)} X$ depends deeply on the scheme structure of $V(I)$, and in next section we'll have a better intuition for this.

It's interesting to see how algebraic properties of A and I reflect on the geometry of the blow up (see [L, Lemma 8.1.2]). In particular we discuss the following.

Proposition 1. *Assume $I = (a)$ with $a \in A$ regular element. Then* $\mathrm{Bl}_{V(I)}X \cong X$*.*

Why do we like this statement? Geometrically, it means that blowing up in codimension 1 doesn't give anything new. Let's prove Proposition 1.

Proof. Since a is regular, $\mathrm{Bl}_{V(a)}A \cong A[at] \cong A[t]$. From [H, Chapter II, Proposition 2.5(b)], we know that:

$$D_+(t) \cong \mathrm{Spec} A[t]_{(t)}.$$

But $D_+(t) = \mathrm{Proj}(A[t])$, because any homogeneous prime ideal $\mathfrak{p} \subseteq A[t]$ which not contain (t) is in $D_+(t)$. Moreover $A[t]_{(t)} = A$, because if $x \in A[t]_{(t)}$, $x = \frac{\alpha t^n}{t^n}$ for some $n \geq 0$ and $\alpha \in A$. We just proved that $\mathrm{Bl}_{V(a)}X \cong X$. $\square$

2 Study of the behavior of the blow up at a fat point

Here we want to work with the complex numbers. The motivation for this section is the following. Let's fix a base field k. We want to understand what's the difference between, say, $\mathrm{Bl}_{V(x,y)}\mathbb{A}^2$ and $\mathrm{Bl}_{V(x,y^2)}\mathbb{A}^2$. We know that $\mathrm{Bl}_{V(x,y)}\mathbb{A}^2$ is regular. What about $\mathrm{Bl}_{V(x,y^2)}\mathbb{A}^2$? Using the computational tools we introduced in [S], we get that:

$$\mathrm{Bl}_{V(x,y^2)}\mathbb{A}^2 = Z(Y_0y^2 - Y_1x) \subseteq \mathbb{A}^2 \times \mathbb{P}^1,$$

where $(Y_0 : Y_1)$ are the homogeneous coordinates. Let's fix this notation once and for all:

$$w := \frac{Y_1}{Y_0},\ z = \frac{Y_0}{Y_1},$$

$$U_0 := \{((x,y),(Y_0 : Y_1)) \in \mathbb{A}^2 \times \mathbb{P}^1 | Y_0 \neq 0\},$$

$$U_1 := \{((x,y),(Y_0 : Y_1)) \in \mathbb{A}^2 \times \mathbb{P}^1 | Y_1 \neq 0\}.$$

So, we observe that in U_0, the equation of our blow up is $y^2 - wx = 0$, which is singular at $(0,0,0)$ (or, in homogeneous coordinates, in $((0,0),(1:0))$).

The second natural question to ask is how $\text{Bl}_{V(x,y^2)}\mathbb{A}^2$ behaves with the separation of tangent directions at the origin: more precisely we want to pick two nonsingular curves through the origin, blow up along $V(x, y^2)$, compute their strict transforms and see if they intersect along the exceptional divisor. We'll go through several examples until we see a pattern.

- $x = y$ and $x = -y$ are not separated;
- $x = 0$ and $y = ax$ are separated (so, let's stick to the tangent direction $x = 0$ but let's rise the multiplicity of intersection);
- $x = 0$ and $x = y^2$ are separated (let's rise again the multiplicity);
- $x = 0$ and $x = y^3$ are not separated;

We can state now a general result. (Note: in the propositions that follow, if we take a curve C we are implicitly assuming that C is different from $x = 0$.)

Proposition 2. *Fix a positive integer e and an irreducible nonsingular curve C in $\mathbb{A}^2$ such that the multiplicity of intersection of C with the line $l : x = 0$ at $(0, 0)$ is less or equal than e. Then, if we consider $\text{Bl}_{V(x,y^e)}\mathbb{A}^2$, $\widehat{C} \cap \widehat{l} \cap E = \emptyset$.*

Proof. Let $f(x, y)$ be the polynomial whose zero set is C. Since C is nonsingular and $(0, 0) \in C$, then the cone of C is in the form cx for some $c \neq 0$. Since $\partial_x f(x, y)|_{(0,0)} = c$, we can use the implicit function theorem and say that C is locally described at $(0, 0)$ by $x = g(y)$. If we expand g at zero we get:

$$x = \alpha y^a + \text{higher degree terms},$$

where a is the multiplicity of intersection of C with l. Therefore we can assume that $C : x = \alpha y^a$.

Now if we look at U_1, the preimage of C has equations:

$$\begin{cases} x = \alpha y^a \\ zy^e - x = 0 \Rightarrow zy^e - \alpha y^a = 0, \end{cases}$$

and hence $\widehat{C}$ must satisfy:

$$\begin{cases} x = \alpha y^a \\ zy^{e-a} - \alpha = 0. \end{cases}$$

The preimage of the line l is described by:

$$\begin{cases} x = 0 \\ zy^e - x = 0 \Rightarrow zy^e = 0, \end{cases}$$

and the strict transform $\widehat{l}$ satisfies:

$$\begin{cases} x = 0 \\ z = 0, \end{cases}$$

Now if $\widehat{l}$ and $\widehat{C}$ intersect along E, (0,0,0) must be the intersection point. But it's obvious that $(0,0,0) \notin \widehat{C}$ because $\alpha \neq 0$. □

Working with similar ideas, we can generalize previous result to the case in which $x = 0$ can be any other irreducible nonsingular curve through $(0,0)$ with tangent direction $x = 0$.

Proposition 3. *Fix a positive integer e. Take two irreducible nonsingular curves C_1, C_2 which pass through the origin with tangent direction $l : x = 0$ and $e \geq m_{(0,0)}(C_1 \cap l) \neq m_{(0,0)}(C_2 \cap l) \leq e$. Then, if we consider $\mathrm{Bl}_{V(x,y^e)}\mathbb{A}^2$, $\widehat{C}_1 \cap \widehat{C}_2 \cap E = \emptyset$.*

Proof. Doing a reduction similar to the one we did in previous proposition, we can assume WLOG that:

$$C_1 : x = \alpha y^a, \; C_2 : x = \beta y^b.$$

Let's check the statement in U_1. The preimage of C_1 has equations:

$$\begin{cases} x = \alpha y^a \\ zy^e - x = 0 \Rightarrow zy^e - \alpha y^a = 0, \end{cases}$$

therefore $\widehat{C}_1$ has equations:

$$\begin{cases} x = \alpha y^a \\ zy^{e-a} = \alpha. \end{cases}$$

Similarly, the strict transform of C_2 has equations:

$$\begin{cases} x = \beta y^b \\ zy^{e-b} = \beta. \end{cases}$$

Now at least one between $e - a, e - b \neq 0$, say $e - b \neq 0$. So, if we have any intersection point along E, this should have $x = y = 0 \Rightarrow \beta = 0$, and this can't be. □

Obviously, up to a suitable translation and rotation, we have the same results for the blow up at any point in $\mathbb{A}^2$ and with any chosen tangent direction at that point.

The “innatural” hypothesis $m_{(0,0)}(C_1 \cap l) \neq m_{(0,0)}(C_2 \cap l)$ is necessary, otherwise the thesis is false in some cases. For instance take:

$$C_1 : x = y^2,\ C_2 : x = y^2 + y^3,$$

and blow up along $V(x, y^2)$. They both intersect $x = 0$ at $(0, 0)$ with multeplicity 2, but if we blow up the strict transforms intersect in $(0, 0, 1) \in E$ in the chart U_1.

Work in progress: properties of the blow up of $k[x, y]$ along (x^2, y^2). We observe just that the whole exceptional divisor is singular.

Observation 1. What about the blow up of $k[x, y]$ along $(x, y)^2$? Well, $\mathrm{Bl}_{V((x,y)^2)}\mathbb{A}^2 \cong \mathrm{Bl}_{V(x,y)}\mathbb{A}^2$. Actually, we can state a more general result: for any ring A, any ideal $I \subseteq A$ and any $n \geq 1$, we have that:

$$\mathrm{Bl}_I A \cong \mathrm{Bl}_{I^n} A.$$

To see this it is equivalent to prove the following. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded A-algebra and, for any integer $e \geq 1$, let $S^{(e)}$ be the graded A-algebra with grading:

$$S_d^{(e)} := S_{ed} \text{ for } d \geq 0.$$

Then $\mathrm{Proj}(S^{(e)}) \cong \mathrm{Proj}(S)$.

3 Universal property of the blow up

Now we want to state the universal property that the blow up satisfies, but we want to state this not just for noetherian affine schemes: we want to deal with a general noetherian scheme. More precisely, if earlier we blew up $\mathrm{Spec}(A)$ with A noetherian ring along a closed subscheme $V(I)$, now we want to blow up a noetherian scheme X along a closed subscheme Y. Giving a closed subscheme Y is equivalent to give a quasi-coherent sheaf of ideals $\mathscr{I}$ over X, and since X is noetherian, it happens that $\mathscr{I}$ is coherent. Conversely a coherent sheaf of ideals on X define a closed subscheme $Y \subseteq X$. So we want to blow up X along a coherent sheaf of ideals $\mathscr{I}$. We need to refresh the following definition.

Definition 1. Let $f : Z \to X$ be a morphism of schemes and let $\mathscr{I}$ be a sheaf of ideals on X. Define the *inverse image ideal sheaf* $f^{-1}\mathscr{I} \cdot \mathscr{O}_Z$ *on* Z as follows. Obviously $f^{-1}\mathscr{I}$ is a sheaf of ideals in $f^{-1}\mathscr{O}_X$. $f = (f, f^{\#})$ with

$f^{\#}\colon \mathscr{O}_X \to f_*\mathscr{O}_Z$ which correspond bijectively to a morphism $f^{\flat}\colon f^{-1}\mathscr{O}_X \to \mathscr{O}_Z$. Hence we can take $f^{-1}\mathscr{I}\cdot\mathscr{O}_Z$ to be the the sheaf of ideals in $\mathscr{O}_Z$ generated by $f^{-1}\mathscr{I}$. The subscheme of Z that this new ideal sheaf define can be identified with the fiber product $Z\times_X Y$, if $Y\subseteq X$ is the closed subscheme of X defined by $\mathscr{I}$.

Next theorem/definition will inclose the definition of blow up together with its universal property.

Theorem/Definition 1. *Given a noetherian scheme X and a coherent sheaf of ideals $\mathscr{I}$, there exists another scheme $\widetilde{X}$ together with a morphism $\sigma\colon \widetilde{X}\to X$ such that:*

- $\sigma^{-1}\mathscr{I}\cdot\mathscr{O}_{\widetilde{X}}\in \mathrm{Pic}(\widetilde{X})$;
- *given any other morphism of schemes $f\colon Z\to X$ such that $f^{-1}\mathscr{I}\cdot\mathscr{O}_Z$ is invertible, then there exists a unique morphism $g\colon Z\to\widetilde{X}$ such that the diagram:*

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & \widetilde{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle\sigma} \\ & & X \end{array}$$

commutes.

Such a scheme $\widetilde{X}$ is unique up to isomorphism and is called the blow up of X along $\mathscr{I}$.

The uniqueness part is the usual consequence of the universal property. About the construction of $\widetilde{X}$ see [H, Chapter II, page 160, Construction]. In the affine case it turns out that $\mathrm{Bl}_{V(I)}\mathrm{Spec}(A)$ satisfy the stated universal property, and this is the only interesting case in which one has to prove previous theorem.

References

[S] L. Schaffler. *Blowing up subvarieties*, notes for the VIGRE group in Algebraic Geometry, Spring 2013.

[H] R. Hartshorne. *Algebraic Geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, 52.

[L] Qing Liu. *Algebraic Geometry and Arithmetic Curves*, Oxford mathematics, 2002, Oxford Graduate Texts in Mathematics, 6.