Blowing up subvarieties

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Fix a base field k. Consider two Zariski closed subsets $Y \subseteq X \subseteq \mathbb{A}^r$ $(r \geq 1)$ with Y proper. Our aim is to define and understand the *blow up of* X *along* Y (in symbols, this will be $\mathrm{Bl}_Y X$). Blow ups are important, for example they're used to:

- study rational morphisms;
- study singular curves;
- classify algebraic surfaces;
- build compactifications of moduli spaces of points.

The definition we give of blow up is taken from [J].

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1 Motivating example

Let's build the blow up in a particular case to give an idea about this. In next section, we will generalize the construction. Take $k = \mathbb{R}$, $r = 2$, $X = \mathbb{A}^2$ and $Y = \{(0,0)\}$. The blow up of $\mathbb{A}^2$ at $(0,0)$ is the Zariski closed subset:

$$\{((x,y),(u:v)) \in \mathbb{A}^2 \times \mathbb{P}^1 | xv = yu\},$$

together with the projection map $\sigma((x,y),(u:v)) = (x,y)$. We give a picture of this.

2 The construction

Let J be the ideal in $k[x_1, \ldots, x_r]$ corresponding to Y (i.e., $Y = Z(J)$). Let $\{g_0, \ldots, g_s\}$ be a finite set of generators for J, whose existence is guaranteed by Hilbert's Basissatz. Consider the following rational map:

$$\begin{aligned} \varphi \colon \mathbb{A}^r &\dashrightarrow \mathbb{P}^s \text{ s.t.} \\ p &\mapsto (g_0(p) : \ldots : g_s(p)). \end{aligned}$$

Obviously, φ is not defined on Y by our construction. Let Γ_φ be the graph of φ. Explicitly:

$$\Gamma_\varphi = \{(p, \varphi(p)) | p \in X \backslash Y\} \subseteq \mathbb{A}^r \times \mathbb{P}^s.$$

Finally, call σ the restriction of the projection $\mathbb{A}^r \times \mathbb{P}^s \to \mathbb{A}^r$ to the Zariski closure of Γ_φ. Now it's all set to give our definition of blow up.

Definition 1. We'll define $\mathrm{Bl}_Y X$ to be the Zariski closed subset $\overline{\Gamma}_\varphi \subseteq \mathbb{A}^r \times \mathbb{P}^s$ together with the projection map $\sigma \colon \overline{\Gamma}_\varphi \to \mathbb{A}^r$. $E := \sigma^{-1}(Y) \subseteq \mathrm{Bl}_Y X$ is called the *exceptional divisor*.

3 First properties of the blow up

Here we list the first properties of the blow up. For simplicity of notation, define $Z := \mathrm{Bl}_Y X$.

Proposition 1.

(i) *σ is a closed map;*

(ii) *$\sigma(Z) = X$;*

(iii) *$Z \backslash E = \Gamma_\varphi$, or equivalently $Z = \Gamma_\varphi \coprod E$;*

(iv) *$\sigma|_{Z \backslash E}$ is an isomorphism, in particular $Z \backslash E$ and $X \backslash Y$ are birationally equivalent.*

Proof.

(i) This is a consequence of [S, Chapter I, Section 5.2, Theorem 3], which is a pretty (basic) important statement.

(ii) Obviously, $\sigma(\Gamma_\varphi) = X \backslash Y \Rightarrow \overline{\sigma(\Gamma_\varphi)} = X$. But σ is closed, therefore $\overline{\sigma(\Gamma_\varphi)} = \sigma(\overline{\Gamma_\varphi}) = \sigma(Z) = X$.

(iii) $(p, q) \in \Gamma_\varphi \Rightarrow p \in X \backslash Y \Rightarrow (p, q) \notin E$.

Conversely, let $(p, q) \in Z \backslash E$. Assume by contradiction that $(p, q) \notin \Gamma_\varphi$. By previous point, $p \in X$, but $p \notin Y$, otherwise $(p, q) \in E$. So, $p \in X \backslash Y$ and $q \neq \varphi(p)$. Let $U \subseteq Z$ be the open nonempty subset of all such points (p, q). Explicitly:

$$U := \{(p, q) \in Z | p \in X \backslash Y \text{ and } q \neq \varphi(p)\}.$$

Since Γ_φ is dense in Z, $\Gamma_\varphi \cap U \neq \emptyset$, and this can't be.

(iv) The regular map $X \backslash Y \to Z \backslash E = \Gamma_\varphi$ s.t. $p \mapsto (p, \varphi(p))$ is obviously the inverse of $\sigma|_{Z \backslash E}$.

□

What we wanted to remark in previous proposition is that the exceptional divisor is exactly what we add to Γ_φ in order to get the closure Z.

At this point, since Z is a Zariski closed subset of $\mathbb{A}^r \times \mathbb{P}^s$, it's natural to ask who are the polynomials that determine Z. Let's introduce some notation. Let $I \subseteq k[x_1, \ldots, x_r]$ be the ideal such that $X = Z(I)$. Call $\overline{x}_i$ the class of x_i modulo I, $i = 1, \ldots, r$, and define $R := k[x_1, \ldots, x_r]/I$. Lastly, if t is an indeterminate over R, define $\psi\colon k[x_1, \ldots, x_r, Y_0, \ldots, Y_s] \to R[t]$ to be the homomorphism of k-algebras obtained by extending $x_i \mapsto \overline{x}_i$ and $Y_j \mapsto g_j(\overline{x})t$, $i = 1, \ldots, r, j = 0, \ldots, s$. Of course, ψ can be viewed as a morphism of graded algebras in the following way: assign degree zero to the elements of $k[x_1, \ldots, x_r]$, R and assign degree 1 to $Y_0, \ldots, Y_s, t$. Since by definition ψ preserves the degree, we have a morphism of graded algebras. In particular, $\mathrm{Ker}(\psi)$ will be a homogeneous ideal in the variables $Y_0, \ldots, Y_s$ (this is easy to be proved). Therefore it makes sense to consider:

$$Z(\mathrm{Ker}(\psi)) \subseteq \mathbb{A}^r \times \mathbb{P}^s.$$

Then we have the following result.

Proposition 2. $Z(\mathrm{Ker}(\psi)) = Z(= \mathrm{Bl}_Y X = \overline{\Gamma}_\varphi)$.

Proof.
($\subseteq$) We will prove that $Z(\mathrm{Ker}(\psi)) \backslash E \subseteq \Gamma_\varphi$, which implies $Z(\mathrm{Ker}(\psi)) \subseteq Z$. Pick $(p, q) \in Z(\mathrm{Ker}(\psi)) \backslash E$. Hence $p \notin Y$, and therefore we can assume WLOG that $g_0(p) \neq 0$. We observe that trivially:

$$(Y_i g_j - Y_j g_i | i, j = 0, \ldots, s) \subseteq \mathrm{Ker}(\psi).$$

In particular, if $q = (q_0 : \ldots : q_s)$, we have the following equalities:

$$q_i = \frac{q_0}{g_0(p)} g_i(p), \ i = 0, \ldots, s,$$

and in particular $q_0 \neq 0$. Therefore:

$$q = (q_0 : \ldots : q_s) = \left(\frac{q_0}{g_0(p)} g_0(p) : \ldots : \frac{q_0}{g_0(p)} g_s(p) \right) = \varphi(p) \Rightarrow$$

$$(p, q) = (p, \varphi(p)) \in \Gamma_\varphi.$$

($\supseteq$) It's enough to prove that $\Gamma_\varphi \subseteq Z(\mathrm{Ker}(\psi))$. So, pick $(p, \varphi(p)) \in \Gamma_\varphi$. We have to show that given $f \in \mathrm{Ker}(\psi)$, $f(p, \varphi(p)) = 0$. Since $\mathrm{Ker}(\psi)$ is homogeneous in $Y_0, \ldots, Y_s$, for our purpose we can assume f to be homogeneous in $Y_0, \ldots, Y_s$ of degree d. If briefly $x := (x_1, \ldots, x_r)$ and $\overline{x} := (\overline{x}_1, \ldots, \overline{x}_r)$, we have that:

$$0 = \psi(f) = f(\overline{x}, g_0(\overline{x})t, \ldots, g_s(\overline{x})t) = f(\overline{x}, g_0(\overline{x}), \ldots, g_s(\overline{x}))t^d \Rightarrow$$

$$f(\overline{x}, g_0(\overline{x}), \ldots, g_s(\overline{x})) = 0 \Rightarrow f \in I \Rightarrow f(p, g_0(p), \ldots, g_s(p)) = 0,$$

which means that $(p, \varphi(p)) \in Z(\mathrm{Ker}(\psi))$. □

Now that we're familiar with the construction of the blow up, we make some additional observations.

Observation 1. In the construction of the blow up, if we take $\mathbb{P}^r$ instead of $\mathbb{A}^r$, nothing changes at all. We just have to consider homogeneous generators $\{G_0, \ldots, G_s\}$ for J and consider another rational map:

$$\Phi\colon \mathbb{P}^r \to \mathbb{P}^s \text{ s.t.}$$
$$p \mapsto (G_0(p) : \ldots : G_s(p)).$$

Everything else is the same.

Observation 2. The construction of the blow up doesn't depend on the choice of the generators for the ideal J. We mean that, if $J = (g'_0, \ldots, g'_{s'})$ and $\varphi'(p) := (g'_0(p) : \ldots : g'_{s'}(p))$, then:

$$\Gamma_\varphi \cong \Gamma_{\varphi'}.$$

4 A concrete example: blowing up a linear subspace

Let $X := \mathbb{A}^r$ and let $Y \subseteq X$ be a linear subspace of dimension $0 \leq d < r$. WLOG, we can assume that Y is the following linear subspace:

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_{r-d} = 0. \end{cases}$$

So, $Z = \mathrm{Bl}_Y X = Z(\mathrm{Ker}(\psi))$, where:

$$\begin{aligned} \psi\colon k[x_1, \dots, x_r, Y_0, \dots, Y_{r-d-1}] &\to k[x_1, \dots, x_r][t] \text{ s.t.} \\ x_i &\mapsto x_i,\ i = 1, \dots, r, \\ Y_j &\mapsto x_{j+1}t,\ j = 0, \dots, r-d-1, \end{aligned}$$

and then extended (in particular $s = r - d - 1$). Define the following ideal in $k[x_1, \dots, x_r, Y_0, \dots, Y_{r-d-1}]$:

$$H = (Y_j x_{i+1} - Y_i x_{j+1} | i, j = 0, \dots, r-d-1).$$

The problem now is to show that $H = \mathrm{Ker}(\psi)$. The containment $H \subseteq \mathrm{Ker}(\psi)$ is trivial. The other one is more subtle. Here we prove the case $s = 1$.
Take $f \in \mathrm{Ker}(\psi)$ and we can assume f homogeneous in Y_0, Y_1 ($\mathrm{Ker}(\psi)$ is generated by such polynomials). We have that $f(x_1, \dots, x_r; x_1, x_2) = 0$, therefore $(Y_0 x_2 - Y_1 x_1)$ divides $f(x_1, \dots, x_r; Y_0, Y_1)$ as homogeneous polynomials with coefficients in $k[x_1, \dots, x_r]$ and indeterminates Y_0, Y_1, and we're done.
For an inductive proof of the other cases, here's the idea: by adding suitable monomials to f, we can get a new polynomial which depends on $Y_0, \dots, Y_{s-1}$ and whose class mod $\mathrm{Ker}(\psi)$ is the same as f.

References

[J] J. Harris. *Algebraic Geometry, a First Course*, Springer-Verlag, New York, 1992, Graduate Texts in Mathematics, 133.

[S] I. R. Shafarevich. *Basic Algebraic Geometry: Varieties in Projective Space*, second edition. Springer-Verlag, 1994.