Compactifications of moduli spaces.

§ Introduction: examples of key concepts.

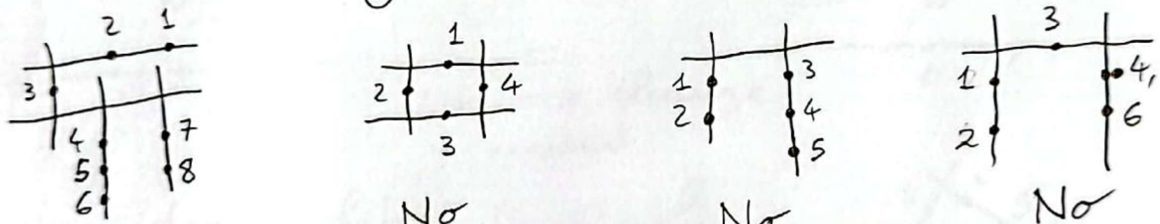
Fundamental concepts for us: stable varieties and stable pairs (in the sense of the MMP), and the procedure of stable replacement.

PART 1.

Def. A stable n-pointed rational curve is a nodal curve X, $p_a(X)=0$, together with n distinct smooth marked pts $p_1,\dots,p_n \in X$ s.t. $\forall\, Y\subseteq X$ irr comp has at least 3 special pts.
A special point is a node of X or a marked point.
The above data are usually summarized as a pair $\left(X, \sum_{i=1}^{n} p_i\right)$, which is an example of stable pair.

↳ divisor, plus it is understood that we keep track of the labeling.

Ex.



Yes No No No

Thm (Knudsen) For $n\geq 3$, $\exists$ proj. fine moduli space $\overline{M}_{0,n}$ paramet. iso. classes of stable n-pointed rat. curves.
(Note: we later review fine/coarse mod space. For now, retain the intuitive picture that points in $\overline{M}_{0,n}$ are in bijection with the above objects.)

Ex. $D_1,\dots,D_6 \subseteq \mathbb{P}^1\times\mathbb{A}^1$ images of the sections $\sigma_i : \mathbb{A}^1 \to \mathbb{P}^1\times\mathbb{A}^1$ below, and let $B\subseteq \mathbb{A}^1$, $0\in B$, be the Zariski open subset such that $\sigma_1,\dots,\sigma_6$ are disjoint over $B\setminus\{0\}$. Let $\mathcal{X} := \mathbb{P}^1\times B \xrightarrow{\pi} B$.

$\sigma_1(t) = ([1:0], t)$

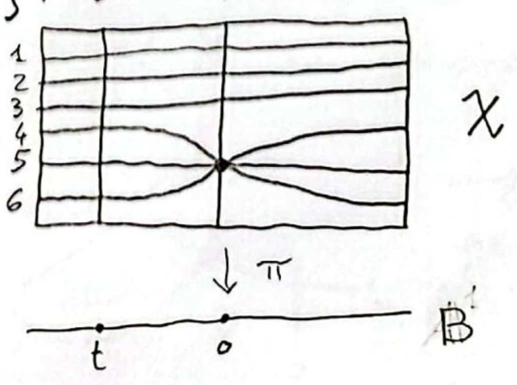
$\sigma_2(t) = ([0:1], t)$

$\sigma_3(t) = ([1:1], t)$

$\sigma_4(t) = ([1:-1], t)$

$\sigma_5(t) = ([1:-1+t], t)$

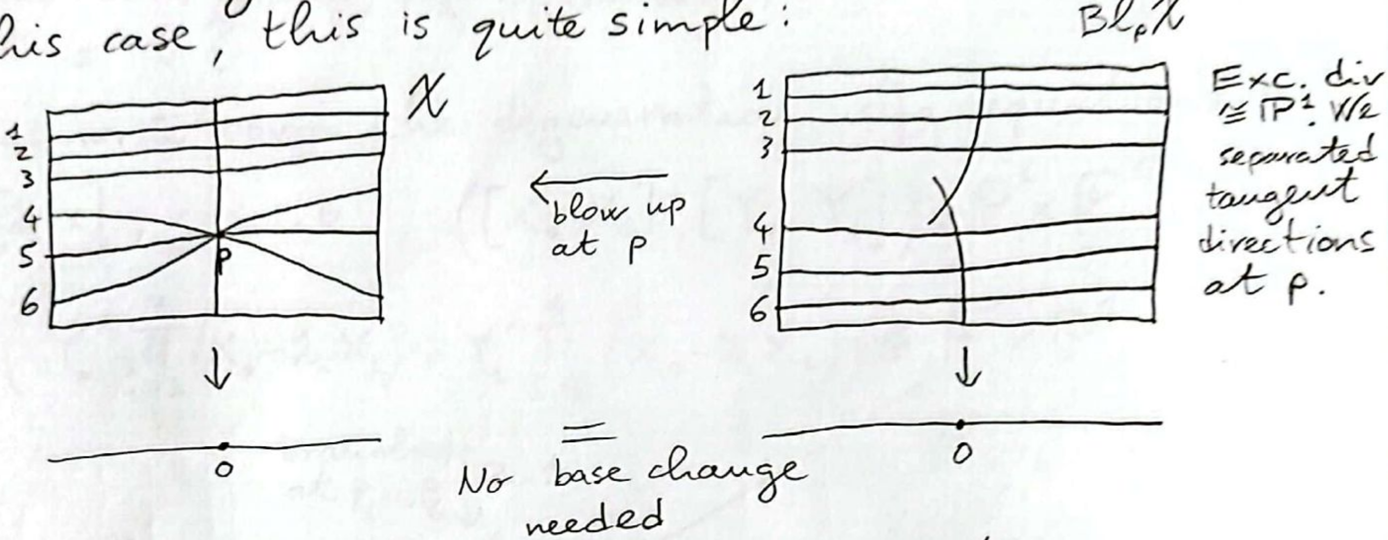
$\sigma_6(t) = ([1:-1+2t], t)$



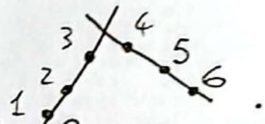
For $t \neq 0$, $\left(\pi^{-1}(t), \sum_{i=1}^{6} \sigma_i(t)\right)$ is stable, but unstable for $t=0$.
However, $\overline{M}_{0,6}$ is compact, so the limit $t \to 0$ exists and corresponds to a stable n-pointed rat. curve.
To find it, we run the <u>stable replacement</u> of the central fiber of $\left(\mathcal{X}, \sum_{i=1}^{6} D_i\right) \to B$. This consists of a combination of:

(1) Birat. modif. of the central fiber;
(2) Base changes.

In this case, this is quite simple:



The new central fiber is stable:



<u>Rmk</u>. One thing that made this particularly simple, is the fact that the family $\mathcal{X}$ is smooth. We will soon see a case where this does not happen.

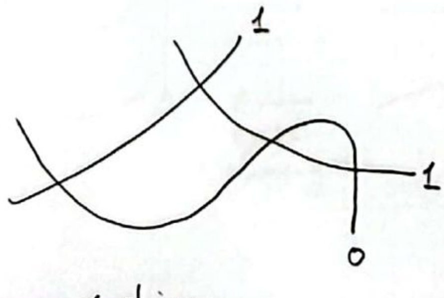
<u>PART 2</u>.

<u>Def</u>. A <u>stable curve</u> of genus $g \geq 2$ is a nodal curve X, $p_a(X) = g$, s.t. $\forall$ smooth rat. irr. comp. $Y \subseteq X$, Y intersects the other components of X in more than 2 points.

<u>Ex</u>.

2 $\dim_{\mathbb{R}}$

or

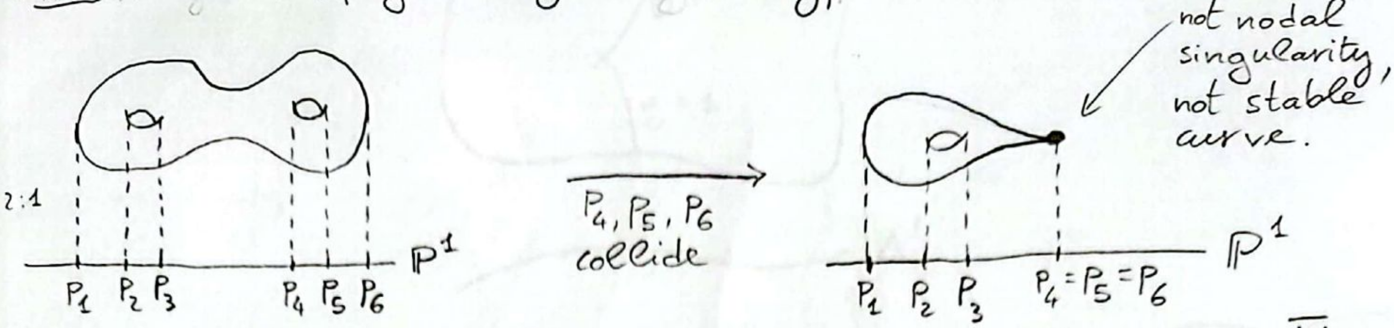


1 $\dim_{\mathbb{C}}$

This has genus 4. (We'll learn how to compute this.)

Thm (Deligne–Mumford) $\forall g \geq 2$, $\exists$ proj. coarse mod. space $\overline{M}_g$ paramet. iso classes of stable curves of genus g.

Ex. Curves of genus $g=2$ are hyperelliptic.



On the other hand, $\overline{M}_2$ is compact, so a limit in $\overline{M}_2$ exists. What is it?

Let us write down the degeneration using equations.

$P_i = [\lambda_i : 1]$, $i = 1, \dots, 6$, $([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$.

$$V\left(Y_0^2 \prod_{i=1}^{3} (X_0 - \lambda_i X_1) + Y_1^2 \prod_{i=4}^{6} (X_0 - \lambda_i X_1)\right) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$$

$$\Big\downarrow \text{branched at } P_1, \dots, P_6 \quad 2:1$$

$$\mathbb{P}^1_{[X_0 : X_1]}$$

To degenerate it, we introduce a parameter $t \in \Delta$

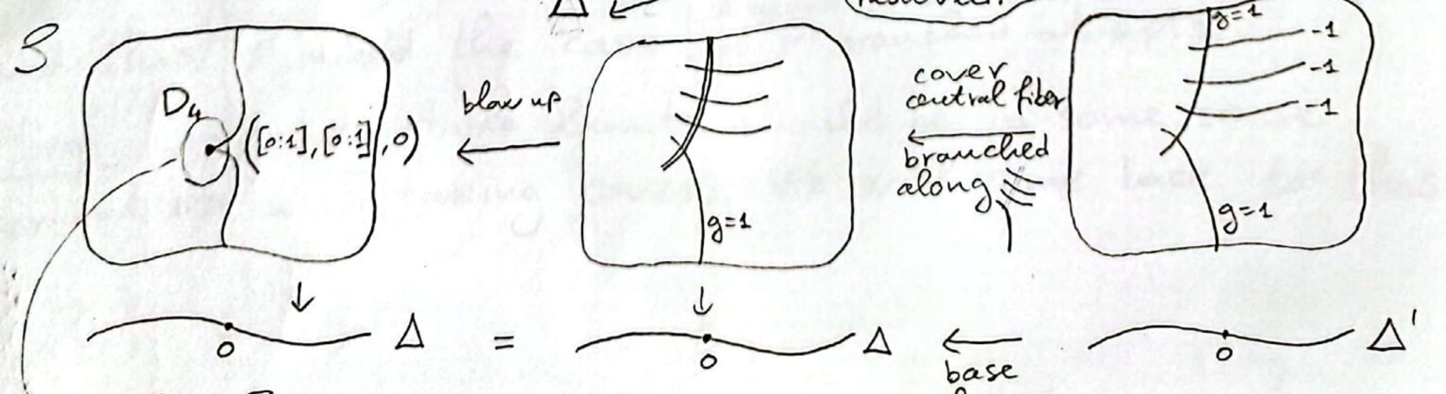
$$\mathcal{S} := V\left(Y_0^2 \prod_{i=1}^{3} (X_0 - \lambda_i X_1) + Y_1^2 \prod_{i=4}^{6} (X_0 - \lambda_i t X_1)\right) \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \Delta$$

- smooth fiber for $t \neq 0$
- not stable degen. for $t = 0$

$$\Big\downarrow$$

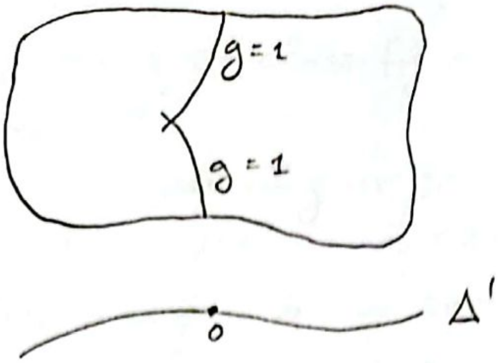
$$\Delta$$

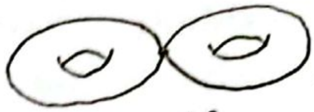
We will talk more later about how the D_4 surface singularity is resolved.



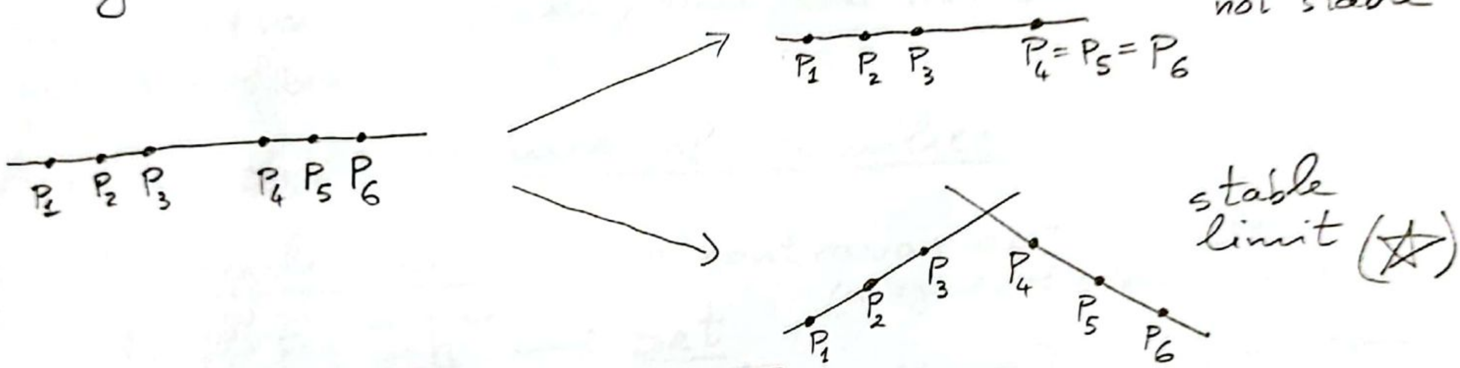
also $\mathcal{S}$ is singular. Locally analytically, $y^2 = x^3 + t^3$

The (-1)-curves can be contracted to smooth points, obtaining

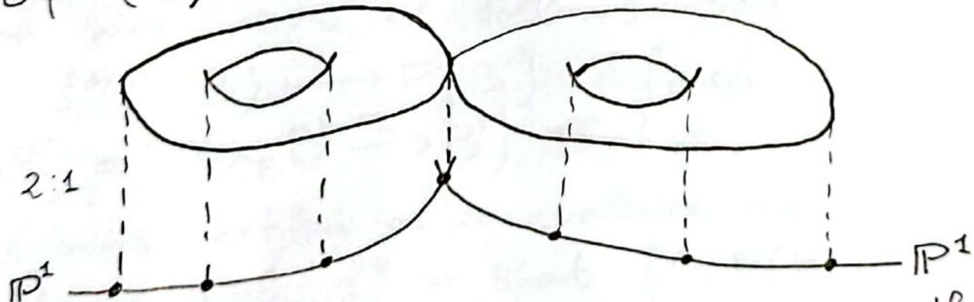


This is the stable limit! In other words, .

Rmk.
There is a quicker way to do it which involves the covering structure and curves with marked points



So the stable limit in $\overline{M}_2$ should be the double cover of (✫):



and this is indeed the case! A genus 1 curve is the double cover of $\mathbb{P}^1$ branched at 4 pts.

Moral. Computing stable limits should be in some sense compatible with taking covers. We will come back to this.