§ Definition of moduli functor

Main reference: Kock-Vainsencher, An Invitation to quantum Cohomology, chapter 0.

Roughly:

- A <u>moduli problem</u> wants to classify certain geom. objects up to notion of equivalence.
- A <u>moduli space</u> is a variety or scheme whose geometric points are in bijection with the set of equivalence classes of objects above, with some additional properties.

Fundamental for the formulation of a moduli problem are:

- The notion of <u>family of objects over a base scheme B</u>.
- This will typically be a morphism $\mathcal{X} \to B$, possibly with some extra structure, and the members of the family are the fibers.
- A notion of <u>equivalence of families</u>.

<u>Def</u>. A <u>moduli functor</u> is a contravariant functor (categories of schemes and sets)

$$F : \underline{Sch}^{op} \longrightarrow \underline{Set}$$
$$B \longmapsto F(B) = \{\text{families } \mathcal{X} \to B\}/_{\sim}$$

equivalence among families

defined on morphisms as follows. Say $f : B' \to B$, then we let $f^* : F(B) \to F(B')$ defined by
$$\begin{tikzcd} \mathcal{X}\times_B B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$
$f^*(\mathcal{X} \to B) = (\mathcal{X}\times_B B' \xrightarrow{\pi_2} B')$. That is,

<u>Remarks</u>.

(1) The adopted notion of equivalence among families should be compatible with f^* so that $f^* : F(B) \to F(B')$ is well defined. As a starter, one usually assumes that two families $\mathcal{X}_1 \to B_1$ and $\mathcal{X}_2 \to B_2$ are isomorphic if $\exists$ a commutative diagram

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{X}_2 \arrow[d] \\ B_1 \arrow[r, "\cong"] & B_2 \end{tikzcd}$$

In this case, then we have that $f^*(\mathcal{X}_1 \to B_1) \sim f^*(\mathcal{X}_2 \to B_2)$

(2) The definition satisfies the functor axioms.

- $id_B^* = id_{F(B)}$. If $(\mathcal{X} \to B) \in F(B)$, then $id_B^*(\mathcal{X} \to B) = (\mathcal{X}\times_B B \to B) \sim$

$\sim (\mathcal{X} \to B)$.

- Let $B'' \xrightarrow{g} B' \xrightarrow{f} B$. Then $(f \circ g)^* = g^* \circ f^*$. This is because

$$g^* \circ f^*(\mathcal{X} \to B) = g^*(\mathcal{X} \times_B B' \to B') = ((\mathcal{X} \times_B B') \times_{B'} B'' \to B'')$$
$$\sim (\mathcal{X} \times (B' \times_{B'} B'') \to B'') \sim (\mathcal{X} \times_B B'' \to B'') = (f \circ g)^* (\mathcal{X} \to B)$$

§ Fine moduli space.

Def. Let $F: \underline{Sch}^{op} \to \underline{Set}$ be a moduli functor. A fine moduli space for F is a scheme M together with a universal family $(\mathcal{U} \to M) \in F(M)$ s.t. $\forall B \in \underline{Sch}$ and $\forall (\mathcal{X} \to B) \in F(B)$, $\exists!\, f: B \to M$ s.t. $(\mathcal{X} \to B) \sim f^*(\mathcal{U} \to B)$.
In other words, there is a Cartesian square

$$\begin{array}{ccc} \mathcal{U} \times_M B \cong \mathcal{X} & \longrightarrow & \mathcal{U} \\ \downarrow & \square & \downarrow \\ B & \xrightarrow[\exists! f]{} & M. \end{array}$$

In other words, there is a bijection between $F(B)$ and $\mathrm{Hom}(B, M)$.

Rmk. • The points of M, $\mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), M)$, are in bijection with the iso. classes of objects we want to parametrize, which is $F(\mathrm{Spec}(\mathbb{C}))$.

- A universal family $\mathcal{U} \to M$ for a moduli problem M is unique up to equivalence, if it exists.
- Given a moduli functor, if the objects have nontrivial automorphisms, then in general a fine moduli space cannot exists as the next example shows.

Example. (Taken from Maarten Hoëve notes). $F: \underline{Sch}^{op} \to \underline{Set}$ s.t. $F(B) = \mathrm{Pic}(B)$. So the object we are parametrizing is $\mathbb{C}$.
Assume by contradiction that $\exists$ fine moduli space, M.
Consider the line bundle $\mathbb{C} \to \mathrm{Spec}(\mathbb{C})$. This induces a unique map $\mathrm{Spec}(\mathbb{C}) \to M$ which gives a $\mathbb{C}$-rational

point $x \in M$. Now let C be a curve with a non-trivial line bundle $V \to C$. This determines a unique morphism $f: C \to M$. Let $\{U_i\}_i$ be an open cover of C s.t. $V|_{U_i}$ is trivial $\forall i$. But then $f|_{U_i}$ is the constant map with image the point x.

Hence f is the constant map with image x.

As $V \cong \mathcal{U} \times_M C$ where $\mathcal{U} \to M$ is the universal family, then V would be the trivial vector bundle, which cannot be.

§ Categorical reformulation of fine moduli space.

Given a moduli functor F, we defined a fine moduli space M as a scheme together with a universal family $\mathcal{U}$ s.t. $\forall (\mathcal{X} \to B) \in F(B)$ can be recovered from $\mathcal{U} \to M$ after a unique pullback. We now translate this in more categorical language. This level of abstraction will allow to define coarse moduli spaces.

<u>Def.</u> Let $Y \in \underline{Sch}$. Let h_Y be the contravariant functor given by

$$h_Y : \underline{Sch}^{op} \longrightarrow \underline{Set}$$
$$B \longmapsto h_Y(B) = \mathrm{Hom}(B, Y)$$

and defined on morphisms as follows. Say $f : B' \to B$, then we let $f^* : h_Y(B) \to h_Y(B')$ defined as

$$f^*(B \xrightarrow{\varphi} Y) = (B' \xrightarrow{f} B \xrightarrow{\varphi} Y) = \varphi \circ f .$$

The functor axioms follow from the properties of composition of functions.

<u>Def.</u> Let F be a <u>moduli functor</u>. We say that F is <u>representable</u> if $\exists M \in \underline{Sch}$ such that F is naturally isomorphic to h_M. In this case M is called a <u>fine moduli space</u> for F.

<u>Recall</u>: Given functors $F, G : \mathcal{C}^{op} \to \mathcal{D}$, a natural transformation $\lambda : F \to G$ is a collections of maps $\lambda_C : F(C) \to G(C)$ for $C \in \mathcal{C}$ satisfying the following: given $f : C' \to C$ we have a commutative diagram

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\lambda_C"'] \arrow[dr, phantom, "\circlearrowright"] & F(C') \arrow[d, "\lambda_{C'}"] \\
G(C) \arrow[r, "G(f)"] & G(C')
\end{tikzcd}$$

We say that F and G are <u>naturally isomorphic</u> if $\exists$ natural transformation $\lambda : F \to G$ such that λ_C is an isomorphism $\forall C \in \mathcal{C}$.

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Yoneda's lemma. Let $F : \mathcal{C}^{op} \to \underline{Set}$ and $Y \in \mathcal{C}$. Then there is a natural bijection $Nat(h_Y, F) \leftrightarrow F(Y)$.

Proof. Exercise.

Prop. Let $F : \underline{Sch}^{op} \to \underline{Set}$ be a moduli functor and let $\lambda : h_Y \to F$ be a natural transformation. By Yoneda's lemma, λ corresponds to a unique $(\mathcal{X} \to Y) \in F(Y)$. Then λ is a natural iso. (in which case F is representable) $\iff$ Y is a fine moduli space for F with universal family $\mathcal{X} \to Y$. Proof. Exercise. □

§ Coarse moduli space.

Def. A coarse moduli space for a moduli functor F is a pair (M, ν), where $M \in \underline{Sch}$ and $\nu : F \to h_M$ a natural transformation such that

(i) If (M', ν') is another such pair,

$$\begin{tikzcd} & h_M \arrow[d, "\exists!"] \\ F \arrow[ur, "\nu"] \arrow[r, "\nu'"'] \arrow[ur, phantom, bend right, "\circlearrowleft"] & h_{M'} \end{tikzcd}$$

(ii) The set map $\nu_{Spec(\mathbb{C})} : F(Spec(\mathbb{C})) \to h_M(Spec(\mathbb{C}))$ is a bijection.

Rmk.

- A coarse moduli space (M, ν) is unique up to iso., if it exists.
- If $\mathcal{U} \to M$ is a fine mod space and $\lambda \in Nat(h_M, F)$ corresponds to $\mathcal{U} \to M$, then (M, λ^{-1}) is a coarse mod space.
- The geometric points of M parametrize the iso. classes of the objects we want to parametrize.

Useful observation. Let F be a moduli functor and assume it admits a coarse moduli space (M, ν). Let $(\mathcal{X} \to B) \in F(B)$. As $\nu_B : F(B) \to h_M(B)$, then let $f := \nu_B(\mathcal{X} \to B)$. Then $f : B \to M$, the geometric meaning of f is that

f sends a geometric point $b \in B$ to the geometric point of $\mathcal{M}$ parametrizing the class of the object $\mathcal{X}_b$, the fiber of $\mathcal{X} \to B$ over b.