§ The curve case. The following definition is due to Hassett, and generalizes the notion of Deligne-Mumford-Knudsen stable

Def. $n \in \mathbb{Z}_{\geq 0}$, $b_1, \dots, b_n \in (0,1] \cap \mathbb{Q}$. A weighted stable curve for the weight $\underline{b} = (b_1, \dots, b_n)$ is a pair $(X, B = \sum_{i=1}^{n} b_i B_i)$ where X is a reduced conn. proj. curve together with n pts $B_i \in X$ such that:

1) (Singularities) If $p \in X$ is a singular point, then p is a node (locally analytically at p, X is iso. to $\{xy=0\} \subseteq \mathbb{A}^2$). The points B_i are different from the nodes and may coincide provided

$$\sum_{B_i = x} b_i \leq 1, \ \forall x \in X.$$

2) (Numerical) The $\mathbb{Q}$-divisor $K_X + B$ is ample.

Rmk.

1) The case $B=0$ corresponds to taking $n=0$.
2) K_X is an invertible sheaf because X is nodal.
3) For a $\mathbb{Q}$-divisor D to be ample means that $\exists m \in \mathbb{Z}_{>0}$ s.t. mD is an ample divisor.
4) $K_X + B$ ample is equiv. to saying: $\forall X_j \subseteq X$ irr. comp., the degree of $(K_X+B)|_{X_j}$ is positive. This degree is given by

$$\deg(K_X+B)|_{X_j} = (2\underbrace{p_a(X_j)}_{\substack{\text{arithmetic genus} \\ \text{of the nodal} \\ \text{curve } X_j}} - 2) + \underbrace{|X_j \cap (\overline{X \setminus X_j})|}_{\substack{\text{number of nodes} \\ \text{of } X \text{ on } X_j \text{ in common} \\ \text{with other components.}}} + \sum_{B_i \in X_j} b_i .$$

Note that if $p_a(X_j) \geq 2$, then $\deg(K_X+B)|_{X_j} > 0$. So one only has to check positivity of $\deg(K_X+B)|_{X_j}$ for the irr. comp's X_j of arithmetic genus 0 or 1.

5) $p_a(X_j)$ can be computed combinatorially as follows. Let $X_j^\nu \to X_j$ be the normalization. Then

$$p_a(X_j) = \underbrace{g(X_j^\nu)}_{\text{geometric genus of } X_j^\nu} + \#\text{ nodes of } X_j.$$

More in general:

<u>Prop</u>. X nodal curve, $X = \bigcup_{j=1}^N X_j$ irr. components, $X_j^\nu \to X_j$ normaliz.

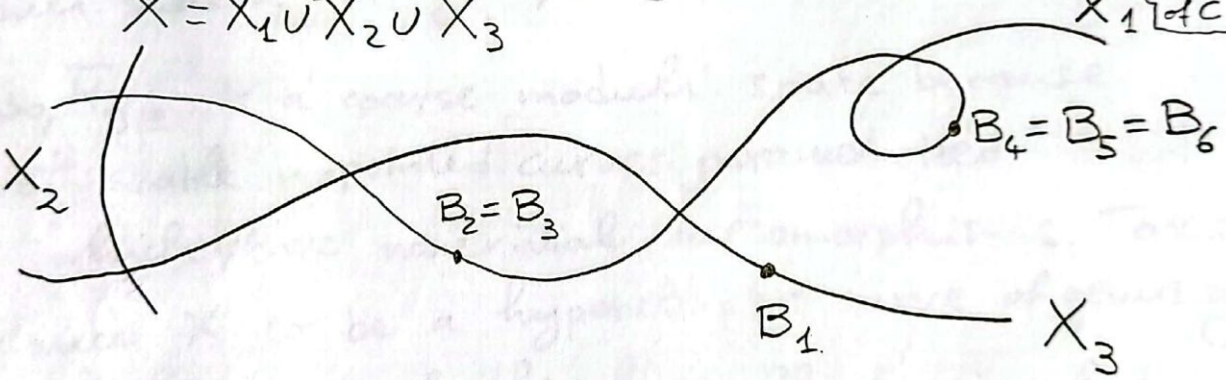
<u>Then</u> $p_a(X) = \sum_{j=1}^N g(X_j^\nu) + \#\left\{\begin{matrix}\text{nodes}\\ \text{of } X\end{matrix}\right\} - N + 1.$

$\boxed{\text{Proof idea. } p_a(X) = 1 - \chi(\mathcal{O}_X) \text{ together with the short exact sequence given by } 0 \to \mathcal{O}_C \to \nu_*\nu^*\mathcal{O}_C \to Q \to 0}$

Q: skyscraper sheaf on the nodes of C.

<u>Ex</u>. Consider the pair (X, B) given as follows:

$X = X_1 \cup X_2 \cup X_3$



$g(X_1^\nu) = 1$, $g(X_2) = 1$, $g(X_3) = 0$, $b_1 = \ldots = b_6 = \frac{1}{5}$.

The condition on singularities is satisfied (check).

The numerical condition is also satisfied:

- For X_1, $p_a(X_1) = 2$, so nothing to check;
- For X_2, $p_a(X_2) = 1$: $(2 \cdot 1 - 2) + 2 + 0 = 2 > 0$;
- For X_3, $p_a(X_3) = 0$: $(2 \cdot 0 - 2) + 3 + \frac{1}{5} = \frac{6}{5} > 0$.

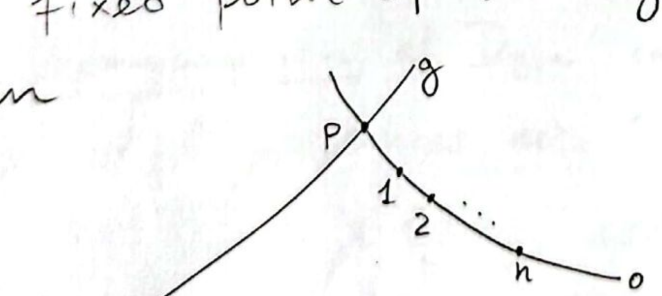
So (X, B) is a weighted stable curve. The genus g is equal to $(1+1+0) + 5 - 3 + 1 = 5$.

Rmk. The def. of weighted stable curve is due to Hassett, and generalizes Deligne-Mumford-Knudsen n-pointed stable curves. The latter are obtained if $B=0$ or all the weights $b_i = 1$.

As in the case of n-pointed stable curves, there exists a proj. coarse mod. space parametrizing weighted stable curves.

Thm (Hassett). Let $n, g \in \mathbb{Z}_{\geq 0}$ and $\underline{b} = (b_1, \dots, b_n)$, $b_1, \dots, b_n \in (0,1] \cap \mathbb{Q}$, s.t. $2g-2+\sum_{i=1}^{n} b_i > 0$. Then there exists an irr. proj. coarse mod space $\overline{M}_{g,\underline{b}}$ paramet. weighted stable curves for the weight $\underline{b}$ of arithmetic genus g. If $g=0$, then $\overline{M}_{0,\underline{b}}$ is a fine moduli space. For $g>0$, it is a coarse moduli space.

Rmk. For $g>0$, $\overline{M}_{g,\underline{b}}$ is a coarse moduli space because there exist stable n-pointed curves parametrized by $\overline{M}_{g,\underline{b}}$ which have nontrivial automorphisms. Take for instance X to be a hyperelliptic curve of genus g and $p \in X$ a fixed point of the hyperelliptic involution ι. Then



, $n \geq 2$, $b_1 + \dots + b_n > 1$,

has genus $(g+0)+1-2+1 = g$. A nontrivial automorphism is provided by the identity on $\mathbb{P}^1$ and ι on X.
(This example does not over all the possible $\overline{M}_{g,\underline{b}}$, but is a large class of examples.)

Scansionato con CamScanner

§ Birational geometry of $\overline{M}_{g,\underline{b}}$. Fix g and n.

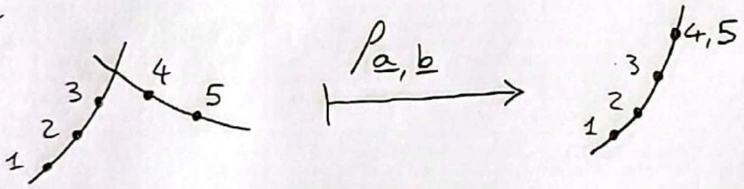
The spaces $\overline{M}_{g,\underline{b}}$ provide different compactifications of $M_{g,n}$, and hence are birational to each other. So it is natural to ask how $\overline{M}_{g,\underline{b}}$ changes as $\underline{b}$ varies and how $\overline{M}_{g,\underline{a}}$ and $\overline{M}_{g,\underline{b}}$ are related for different $\underline{a}, \underline{b}$.

<u>Thm</u> (Hassett) Suppose $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$ satisfy $b_j \le a_j \ \forall j = 1, \dots, n$. Then $\exists$ natural birational morphism $\rho^n_{\underline{a},\underline{b}} : \overline{M}_{g,\underline{a}} \to \overline{M}_{g,\underline{b}}$, called <u>reduction morphism</u>.

Given an element $(X, p_1, \dots, p_n) \in \overline{M}_{g,\underline{a}}$, $\rho_{\underline{a},\underline{b}}(X, p_1, \dots, p_n)$ is obtained by successively collapsing components of X along which $K_X + b_1 p_1 + \dots + b_n p_n$ fails to be ample.

<u>Ex</u>. $\overline{M}_{0,\underline{a}} \to \overline{M}_{0,\underline{b}}$ with $\underline{a} = (1,1,1,1,1)$, $\underline{b} = (1,1,1,\frac{1}{2},\frac{1}{2})$.

Then



<u>Def</u>. • The space of admissible weights $D_{g,n} \subseteq \mathbb{R}^n$ is

$$D_{g,n} := \left\{ (b_1, \dots, b_n) \in \mathbb{R}^n \;\middle|\; 0 < b_j \le 1 \ \forall j \text{ and } b_1 + \dots + b_n > 2 - 2g \right\}.$$

• A <u>chamber decomposition</u> of $D_{g,n}$ consists of a finite set $W = \{w_S\}_S$ of hyperplanes $w_S \subseteq D_{g,n}$. The connected components of $D_{g,n} \setminus \bigcup_S w_S$ are called <u>open chambers</u>.

• The <u>coarse</u> and <u>fine</u> chamber decompositions are respectively

$$W_c := \left\{ \sum_{j \in S} b_j = 1 \;\middle|\; S \subseteq \{1, \dots, n\} \ \& \ 2 < |S| < n-2 \right\},$$

$$W_f := \left\{ \sum_{j \in S} b_j = 1 \;\middle|\; S \subseteq \{1, \dots, n\} \ \& \ 2 \le |S| \le n-2 \right\}.$$