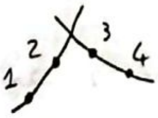
Thm (Hassett)

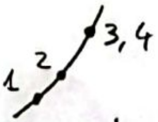
(1) $\overline{M}_{g,\underline{b}}$ is constant on each open chamber of the coarse decomposition. Moreover, this is the coarsest decomposition of $D_{g,n}$ with this property.

(2) $\overline{M}_{g,\underline{b}}$ together with the family $(\mathcal{X}, \mathcal{B}_1, \dots, \mathcal{B}_n) \to \overline{M}_{g,\underline{b}}$ is constant on each open chamber of the fine decomposition. Moreover, this is the coarsest decomposition of $D_{g,n}$ with this property.

Example. $\overline{M}_{0,(1,1,1,1)} \overset{\varphi}{\cong} \overline{M}_{0,(1,1,\frac{1}{2},\frac{1}{2})}$, so they belong in the same open chamber of the coarse decomposition.
However, the families are different: $p \in \overline{M}_{0,(1,1,1,1)}$ parametrizing



corresponds to $\varphi(p) \in \overline{M}_{0,(1,1,\frac{1}{2},\frac{1}{2})}$ parametrizing



. So the families are different.
So they belong to different open chambers of the fine decomposition.

Rmk. What happens when we cross a wall? See "Moduli of weighted hyperplane arrangements", Thm 1.1.6

Rmk. Hassett posed the problem of finding formulas for the number of open chambers of the coarse and fine decompositions. This was studied by Ascher-Dubé-Gershenson-Hou: "Enumerating Hassett's wall and chamber decomposition of the moduli space of weighted stable curves".

§ The singularities of the MMP.

<u>Intro</u>. We now move the first step towards the generaliz. in higher dimension of a Hassett stable weighted n-pointed curve. This is the notion of <u>stable pair</u>.
The current theory guarantees the existence of a proj. coarse mod. space paramet stable pairs with some fixed numerical invariants. Here is the definition of stable pair (not necessarily irred.)

<u>Def</u>. Let X be a variety, B $\mathbb{Q}$-divisor on X with coefficients in $[0,1]$. The pair (X,B) is called <u>stable</u> if

1) (X,B) is semi-log canonical;
2) K_X+B is ample.

Our next goal is to understand the meaning of semi-log canonical. First, we discuss the meaning of log canonical.

<u>Def</u>. Let X be a normal variety (in particular, X is irred.) and B a $\mathbb{Q}$-div. on X with coeff. in $[0,1]$ s.t. K_X+B is $\mathbb{Q}$-Cartier. Let $f: Y\to X$ be a birational morphism. Let $\{E_i\}_i$ be the set of irred. exceptional divisors of f. $\boxed{\dim E_i > \dim f(E_i)}$ Then $\exists$ rational numbers $a(E_i,X,B)$ such that

$$K_Y+f_*^{-1}B \sim_{\mathbb{Q}} f^*(K_X+B)+\sum_i a(E_i,X,B)E_i\,,$$

where:
- $f_*^{-1}B$ denotes the strict transform of B;
- If D_1 and D_2 are $\mathbb{Q}$-divisors, we say that $D_1\sim_{\mathbb{Q}} D_2$ provided $\exists m\in\mathbb{Z}_{>0}$ s.t. $mD_1\sim mD_2$, where "$\sim$" denotes linear equivalence.

The <u>discrepancy</u> of the pair (X,B) is defined to be

$$\operatorname{discrep}(X,B)=\inf\{a(E,X,B)\},$$

where we take inf over all birat. morph. $Y \to X$ and all irr exceptional divisors $E \subseteq Y$. We say that (X,B) is

- <u>terminal</u> if $\text{discrep}(X,B) > 0$;
- <u>canonical</u> if $\text{discrep}(X,B) \geq 0$;
- <u>Kawamata log terminal</u> if $\text{discrep}(X,B) > -1$ & $\lfloor B \rfloor = 0$;
- <u>purely log terminal</u> if $\text{discrep}(X,B) > -1$;
- <u>log canonical</u> if $\text{discrep}(X,B) \geq -1$.

<u>Remarks</u>.

(1) Lower is the discrepancy, worse are the singularities of (X,B). This is the intuition.

(2) If $\dim X = 1$, in which case X is a smooth curve, then the definition is vacuous. So the definition is meaningful for $\dim X \geq 2$.

(3) There exist pairs (X,B) which are not log canonical (These are "very bad" singularities). This happens if $\exists\ Y \to X$ and $E \subseteq Y$ exceptional divisor s.t. $a(E,X,B) < -1$.

(4) In general, it is not easy to compute $\text{discrep}(X,B)$ by the definition above because one has to consider all possible $E \subseteq Y \to X$ exceptional. In what follows, we first isolate a case where this is possible.

<u>Def</u>. A divisor $D = \sum_i D_i$ on a smooth variety X is <u>simple normal crossing</u> (snc) if D is reduced, any irr component D_i is smooth, and D is locally defined in a neighborhood of any point by an equation in local analytic coordinates of the type $z_1 \cdots z_k = 0$ with $k \leq \dim(X)$.

Ex. $D_i \subseteq \mathbb{P}^r$ hyperplanes. $D = \sum_i D_i$ is snc $\iff$ the hyperplanes are in general linear position.

Prop (Kollár-Mori book, Coroll 2.31). X normal variety, $D = \sum_i a_i D_i$ $\mathbb{Q}$-divisor, $0 < a_i \leq 1$. The following hold:

(1) $\mathrm{discrep}(X, D) \leq 1$;

(2) $-1 \leq \mathrm{discrep}(X, D)$ or $\mathrm{discrep}(X, D) = -\infty$. In other words, if $\exists\, E \subseteq Y \to X$ exceptional s.t. $a(E, X, D) < -1$, then $\mathrm{discrep}(X, D) = \inf\{a(E, X, D)\} = -\infty$.

(3) Assume that X is smooth and D is snc. Then

$$\mathrm{discrep}(X, D) = \min\left\{1, \min_i\{1 - a_i\}, \min_{\substack{i \neq j \\ D_i \cap D_j \neq \emptyset}}\{1 - a_i - a_j\}\right\}.$$

(4) If X is smooth, then $\mathrm{discrep}(X, 0) = 1$. In particular, $(X, 0)$ is terminal.

Ex. $D_1, \dots, D_n \subseteq \mathbb{P}^r$ hyperplanes in g.l.p., $n \geq 2$, $r \geq 2$. Then $\mathrm{discrep}\left(\mathbb{P}^r, \sum_{i=1}^n D_i\right) = -1$. So, the pair $\left(\mathbb{P}^r, \sum_{i=1}^n D_i\right)$ is log canonical.

As mentioned, in general it is not easy to compute discrep(X,D) to classify the singularities of (X,D). It is a highly nontrivial result that one can classify the singularities of (X,D) just by considering a specific birat. morphism $Y \to X$. The starting point is the following theorem of Hironaka.

<u>Theorem</u> (Hironaka, '64) Let X be an irr. complex var. and $D \subseteq X$ an effective Cartier divisor.

1) $\exists$ proj. birat. morphism $f: Y \to X$ where Y is smooth and $f_*^{-1} D \cup \mathrm{Exc}(f)$ is snc. Such morphism f is called a <u>log resolution</u> of the pair (X,D).

2) The smooth variety Y can be constructed as a sequence of blow ups along smooth centers supported in the singular loci of D and X. It follows that f is an isomorphism over $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(D))$.

<u>Theorem</u>. Let X be a normal var. and D a $\mathbb{Q}$-divisor on X with coeff's in $[0,1]$ s.t. $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f: Y \to X$ be a log resolution of (X,D) and write

$$K_Y + f_*^{-1} D = f^*(K_X + D) + \sum_i a_i E_i .$$

Then the pair is

- terminal if $a_i > 0 \ \forall i$;
- canonical if $a_i \geq 0 \ \forall i$;
- klt if $a_i > -1 \ \forall i$ and $\lfloor D \rfloor = 0$;
- plt if $a_i > -1 \ \forall i$;
- lc if $a_i \geq -1 \ \forall i$.

We now want to start working on some examples in the case of algebraic surfaces. For this we need some preliminaries.