§ Preliminaries on birational geometry of algebraic surfaces.

Main reference: Beauville, Complex algebraic surfaces, chapter II.

In what follows, a surface S is a smooth projective connected 2-dim complex variety.

Thm (Blow up): S surface, $p \in S$. Then $\exists$ a surface $\hat{S}$ and a morphism $\sigma : \hat{S} \to S$, which are unique up to iso, such that

(1) $\sigma|_{\sigma^{-1}(S \setminus \{p\})}$ is an iso onto $S \setminus \{p\}$,

(2) $\sigma^{-1}(p)$, which we denote by E, is isomorphic to $\mathbb{P}^1$.

σ is called the blow up of S at p and E is the exceptional divisor.

Construction of σ. Take a neighborhood U of p on which there exist local coordinates x, y at p, so that p is the transverse intersection of $x=0$, $y=0$. Define:

$$\begin{tikzcd} \hat{U} := V(xB - yA) \subseteq U \times \mathbb{P}^1_{[A:B]} \arrow[dr, "\sigma"'] \arrow[d, phantom, "\circlearrowright"] & \\ & U \arrow[from=1-1, d, "\pi_1"] \end{tikzcd}$$

$\sigma^{-1}(p) = \{p\} \times \mathbb{P}^1$ e $\sigma|_{\sigma^{-1}(U \setminus \{p\})}$ is an iso.

$\hat{S}$ is obtained by gluing $\hat{U}$ and $S \setminus \{p\}$ along $U \setminus \{p\} \cong \hat{U} \setminus \sigma^{-1}(p)$.

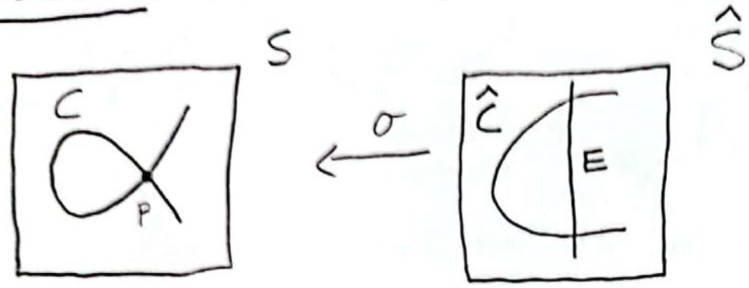
It is fundamental to understand how the blow up behaves with respect to curves passing through p.

Def. Let $C \subseteq S$ be a curve and $\sigma : \hat{S} \to S$ the blow up of S at p. The strict transform of C is the curve $\hat{C} \subseteq \hat{S}$ defined as follows:

$$\hat{C} := \overline{\sigma^{-1}(C \setminus \{p\})} \quad \text{(closure in } \hat{S}\text{)}.$$

Ex. Let $S = \mathbb{A}^2$, $C = V(x^2 - y^2 + x^3)$, $p = (0,0)$.
Compute $\hat{C}$ in the two affine charts of $\mathbb{A}^2 \times \mathbb{P}^1_{[A:B]}$ and understand how $\hat{C}$ and E are related.

Solve. Let us prove the following:



$C \cup E \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ is cut out by

$$\begin{cases} x^2 - y^2 + x^3 = 0 \\ xB - yA = 0 \end{cases}$$

In $A \neq 0$, let $\frac{B}{A} =: \beta$, so we obtain

$$\begin{cases} x^2 - y^2 + x^3 = 0 \Rightarrow x^2 - x^2\beta^2 + x^3\beta^3 = 0 \Rightarrow \underset{\text{exceptional locus}}{x^{②}} \overbrace{(1 - \beta^2 + x\beta^3)}^{\hat{C} \cap \{A \neq 0\}} = 0 \\ x\beta - y = 0 \Rightarrow y = x\beta \end{cases}$$

multiplicity of $p \in C$

So in $\{A \neq 0\}$, $\hat{C}$ and E are described by $1 - \beta^2 + x\beta^3 = 0$ & $x = 0$.
So $\hat{C} \cap E$ consists of the points $(x, \beta) = (0, \pm 1)$.

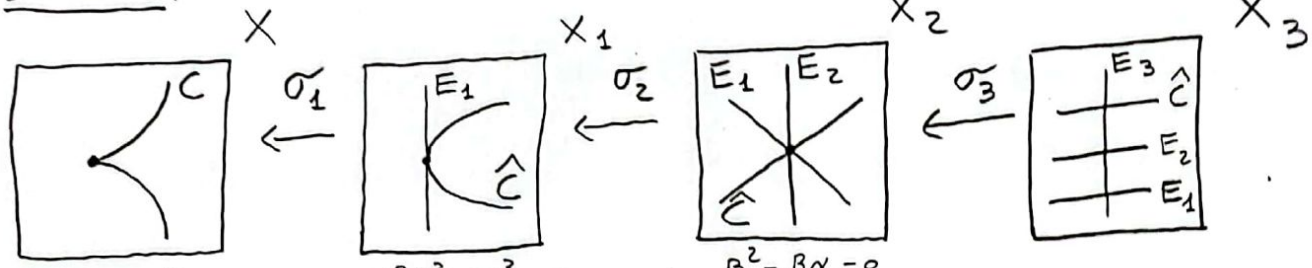
In $B \neq 0$, let $\frac{A}{B} =: \alpha$, so we obtain

$$\begin{cases} x^2 - y^2 + x^3 = 0 \Rightarrow y^2\alpha^2 - y^2 + y^3\alpha^3 = 0 \Rightarrow y^2(\alpha^2 - 1 + y\alpha^3) = 0 \\ x - y\alpha = 0 \Rightarrow x = y\alpha \end{cases}$$

So in $\{B \neq 0\}$, $\hat{C}$ and E are described by $\alpha^2 - 1 + y\alpha^3 = 0$ & $y = 0$.
So $\hat{C} \cap E$ consists of the same points we found before.

$\underline{\text{Ex}}$. Let $X = \mathbb{A}^2$, $C = V(y^2 - x^3)$. Find a log resolution of (X, C).

$\underline{\text{Solve}}$.



$y^2 - x^3 = 0$

$x^2\beta^2 - x^3 = 0$

$\underbrace{x^2}_{E_1}\underbrace{(\beta^2 - x)} = 0$

$\beta^2 - \beta\alpha = 0$

$\underbrace{\beta}_{E_2}\underbrace{(\beta - \alpha)} = 0$

The log resolution is $\sigma := \sigma_1 \circ \sigma_2 \circ \sigma_3 : X_3 \longrightarrow X$

The divisor $E_1 + E_2 + E_3 + \hat{C}$ on X_3 is snc.

<u>Prop.</u> Let S be a surface, $p \in S$, $\sigma : \hat{S} \to S$ blow up of S at p. Then we have that $\sigma^* C = \hat{C} + mE$,

where m is the multiplicity of C at p.

(That is, if $p \notin C$, then $m = 0$. If $p \in C$, then in the complete local ring $\hat{\mathcal{O}}_{S,p}$ the curve C can be written as formal power series

$$f = f_m(x,y) + f_{m+1}(x,y) + \dots ,$$

where f_k are homogeneous polynomials of deg k in x, y and f_m is non zero.)

<u>Rmk.</u> If $p \in C$ is smooth, then $\sigma^* C = \hat{C} + E$.

<u>Prop.</u> S surface, $\sigma : \hat{S} \to S$ blow up at $p \in S$, $E \subseteq \hat{S}$ exceptional divisor.

(i) $\mathrm{Pic}(\hat{S}) \cong \mathrm{Pic}(S) \oplus \mathbb{Z}$;

(ii) Let D, D' be divisors on S. Then the following hold:

$(\sigma^* D) \cdot (\sigma^* D') = D \cdot D'$, $E \cdot \sigma^* D = 0$, $E^2 = -1$,

(iii) $K_{\hat{S}} = \sigma^* K_S + E$,

(iv) Let C be irr. smooth curve in S, $p \in C$. Then $\hat{C}^2 = C^2 - 1$.

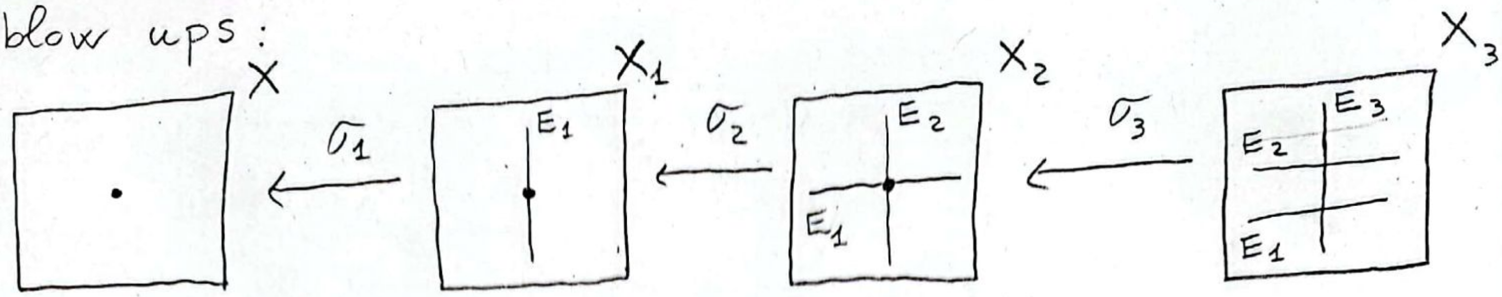
<u>Rmk.</u> (ii) $\Rightarrow$ (iv) by squaring $\sigma^* C = \hat{C} + E$.

<u>Rmk.</u> (iii) implies the following. Let D be a $\mathbb{Q}$-divisor on S with coeff's in $(0,1]$. Then $\mathrm{discrep}(S,D) \leq 1$.

To prove this, let $p \in S \setminus \mathrm{Supp}(D)$. Then

$$\mathrm{discrep}(S,D) = \min_{\substack{F \subseteq Y \to S \\ \text{exceptional}}} \{ a(F,S,D) \} \leq a(\underset{\substack{\uparrow \\ \text{except. div.} \\ \text{of blow up } S \text{ at } p}}{E}, S, D) = 1.$$

Ex. Let $X = \mathbb{A}^2$ and consider the following sequence of blow ups:



Compute K_{X_3} and E_1^2, E_2^2, E_3^2 on X_3.

Solve.

$E_1^2 = -3, \; E_2^2 = -2, \; E_3^2 = -1.$

$K_X = K_{\mathbb{A}^2} = 0$

$K_{X_1} = \sigma_1^* K_X + E_1 = E_1$

$K_{X_2} = \sigma_2^* E_1 + E_2 = (E_1 + E_2) + E_2 = E_1 + 2E_2$

$$
\begin{aligned}
K_{X_3} &= \sigma_3^*(E_1 + 2E_2) + E_3 = \sigma_3^* E_1 + 2\sigma_3^* E_2 + E_3 \\
&= (E_1 + E_3) + 2(E_2 + E_3) + E_3 \\
&= E_1 + 2E_2 + 4E_3 .
\end{aligned}
$$