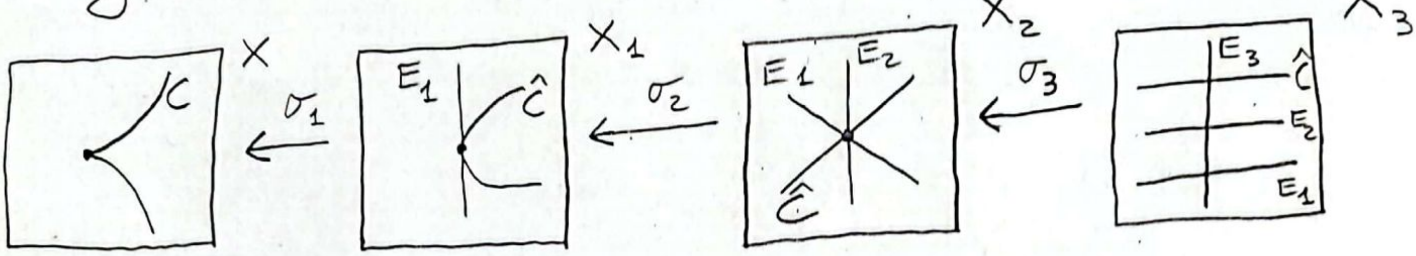
Ex. Let $X = \mathbb{A}^2$, $C = V(y^2 - x^3)$. Classify the sing's of the pair (X, C).

Solve. In a previous exercise we already computed a log resolution of (X, C):



In the previous exercise we already computed

$$K_{X_3} = E_1 + 2E_2 + 4E_3.$$

So let us write:

$$K_{X_3} + \sigma_*^{-1} C \sim_{\mathbb{Q}} \sigma^*(\overset{0}{K_X} + C) + \sum_{i=1}^{3} a_i E_i \Rightarrow$$

$$E_1 + 2E_2 + 4E_3 + \hat{C} \sim_{\mathbb{Q}} \underbrace{\sigma^* C}_{\text{need to compute!}} + \sum_{i=1}^{3} a_i E_i$$

$$\sigma^* C = (\sigma_1 \circ \sigma_2 \circ \sigma_3)^* C = \sigma_3^* \sigma_2^* \sigma_1^* C$$
$$= \sigma_3^* \sigma_2^* (\hat{C} + 2E_1) = \sigma_3^* (\hat{C} + E_2 + 2E_1 + 2E_2)$$
$$= \sigma_3^* (\hat{C} + 2E_1 + 3E_2)$$
$$= \hat{C} + E_3 + 2E_1 + 2E_3 + 3E_2 + 3E_3$$
$$= \hat{C} + 2E_1 + 3E_2 + 6E_3 .$$

So

$$E_1 + 2E_2 + 4E_3 + \cancel{\hat{C}} \sim_{\mathbb{Q}} \cancel{\hat{C}} + 2E_1 + 3E_2 + 6E_3 + \sum_{i=1}^{3} a_i E_i$$

$$\Rightarrow \begin{cases} a_1 = -1 \\ a_2 = -1 \\ a_3 = -2 \end{cases}$$

So (X, C) is not log canonical.

§ The log canonical threshold.

(Definition from "Rational and nearly rational varieties", Kollár, Smith, Corti.)

The following propositions will allow us to introduce the log canonical threshold and understand some of its properties. At the same time we will gain some intuition about what happens to the singularity of the pair (X, cD) as we vary the coefficient c.

Prop. Let X be a smooth variety and let D be an effective $\mathbb{Q}$-div. on X s.t. (X, D) is log canonical. Let $c \in [0,1] \cap \mathbb{Q}$. Then (X, cD) is log canonical.

Proof. Let $f: Y \to X$ be a log resolution of the pair (X, D). We have that

$$(1)\quad K_Y = f^* K_X + \sum_i a_i E_i \quad , \quad \overbrace{a_i > 0}^{X \text{ is terminal}} \ \forall i ,$$

$$(2)\quad f^* D \sim_{\mathbb{Q}} f_*^{-1} D + \sum_i b_i E_i \quad , \quad \underbrace{b_i > 0}_{\substack{\text{The centers of the blow} \\ \text{up } f \text{ are supported on} \\ \text{Sing}(D)}} \ \forall i .$$

Combining (1) and (2) we obtain

$$K_Y + f_*^{-1} D \sim_{\mathbb{Q}} f^*(K_X + D) + \sum_i (a_i - b_i) E_i$$

Since (X, D) is log canonical, we have that $a_i - b_i \geq -1 \ \forall i$.

Consider

$$(3)\quad f^*(cD) \sim_{\mathbb{Q}} f_*^{-1}(cD) + \sum_i c b_i E_i .$$

Combining (1) and (3) we obtain

$$K_Y + f_*^{-1} cD \sim_{\mathbb{Q}} f^*(K_X + cD) + \sum_i (a_i - c b_i) E_i .$$

Since $a_i - c b_i \geq a_i - b_i \geq -1$, we can conclude that also (X, cD) is log canonical. $\square$

Rmk. An analogous argument shows that if (X,D) is canonical (resp. terminal, log terminal), then (X,cD) is canonical (resp. terminal, log terminal).

Prop. Let X be a smooth variety and let D be an effective $\mathbb{Q}$-div. on X. Then $\exists\, c \in (0,1] \cap \mathbb{Q}$ s.t. (X,cD) is log canonical.

Proof. Let $f: Y \to X$ be a log resolution of (X,D). We have that

(1) $K_Y = f^*K_X + \sum_i a_i E_i$, $a_i > 0\ \forall i$,

(2) $f^*D \sim_{\mathbb{Q}} f_*^{-1} D + \sum_i b_i E_i$, $b_i > 0\ \forall i$.

(3) $f^*(cD) \sim_{\mathbb{Q}} f_*^{-1}(cD) + \sum_i c b_i E_i$

$\Rightarrow K_Y + f_*^{-1} cD \sim_{\mathbb{Q}} f^*(K_X + cD) + \sum_i (a_i - cb_i) E_i$. For (X,cD) to be log canonical we would like $a_i - cb_i \geq -1\ \forall i$. Hence, $c \leq \frac{1+a_i}{b_i}\ \forall i$. So let us set $m := \min_i \left\{\frac{1+a_i}{b_i}\right\}$. We have two cases:

$m \leq 1$. Let $c := m$. Then, $\forall i$, $c \leq \frac{1+a_i}{b_i} \Rightarrow -1 \leq a_i - cb_i$, so (X,cD) is log canonical.

$m > 1$. Let $c := 1$. Then, $\forall i$, $1 \leq \frac{1+a_i}{b_i} \Rightarrow -1 \leq a_i - b_i$ $\Rightarrow (X,D)$ is log canonical. $\square$

We can finally give the following definition.

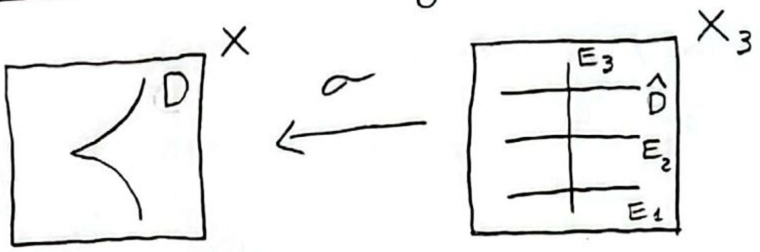
Def. The log canonical threshold of an effective divisor D on a smooth variety X is defined as

$$\tau := \sup \{ c \in (0,1] \mid (X,cD) \text{ is log canonical} \}.$$

By the previous propositions and proofs, we know that τ exists and is rational. Also, $\forall\, 0 \leq c \leq \tau$, (X,cD) is log canonical.

Ex. Let $X = \mathbb{A}^2$, $D = V(y^2 - x^3)$. Find the log canonical threshold of (X, D).

Solve. Recall a log resolution is provided by



We also computed that: (1) $K_{X_3} = E_1 + 2E_2 + 4E_3$.

(2) $\sigma^* D = \hat{D} + 2E_1 + 3E_2 + 6E_3$.

Therefore:

$$K_{X_3} + \sigma_*^{-1} cD \sim_{\mathbb{Q}} \sigma^*(\overset{0}{K_X} + cD) + \sum_{i=1}^{3} a_i E_i \Rightarrow$$

$$E_1 + 2E_2 + 4E_3 + \cancel{c\hat{D}} \sim_{\mathbb{Q}} \cancel{c\hat{D}} + 2cE_1 + 3cE_2 + 6cE_3 + \sum_{i=1}^{3} a_i E_i$$

$$\Rightarrow \begin{cases} a_1 = 1 - 2c \geq -1 & \Rightarrow c \leq 1 \\ a_2 = 2 - 3c \geq -1 & \Rightarrow c \leq 1 \\ a_3 = 4 - 6c \geq -1 & \Rightarrow \boxed{c \leq 5/6} \end{cases}$$

So the log canonical threshold of (X, D) is $5/6$.

The following general result confirms the above calculation.

Prop. (Prop 6.39 in "Rational and nearly rational varieties")
Let $C \subseteq \mathbb{A}^2$ be the curve $C = V(x^b + y^a)$ with $a, b \geq 2$.
Then the log canonical threshold of $(\mathbb{A}^2, C)$ is $\frac{1}{a} + \frac{1}{b}$.