§ Extension to the non-normal case: semi-log canonical singularities.

<u>Def</u>. Let (R,m) be a local k-algebra, char $k \neq 2$. We say that Spec$R$ has a <u>node</u> if $\hat{R} \cong k[[x,y]]/(x^2-ay^2)$ for some unit $a \in \hat{R}$.

<u>Example</u>. Let $R = \frac{\mathbb{C}[x,y]_{(x,y)}}{(x^2-y^2+y^3)}$. $X = \mathrm{Spec}R$ has a node because $\hat{R} \cong \frac{\mathbb{C}[[x,y]]}{(x^2-(1-y)y^2)}$, and $1-y \in \hat{R}$ is invertible.

We want to venture from normal varieties to possibly reducible varieties (these will be our degenerations). The weakening of normality that we will consider is called demi-normality.

<u>Def</u>. Let X be an equidimensional noetherian scheme. Then X is called <u>demi-normal</u> if X is S_2 and $\forall\, Y \subseteq X$ codimension one closed subscheme, $\mathcal{O}_{X,Y}$ is either regular or a node.

<u>Rmk</u>

- As nodal singularities are Gorenstein, then a demi-normal scheme X is Gorenstein in codimension one. This means that $\exists\, Z \subseteq X$ closed, $\mathrm{codim}_X(Z) \geq 2$ s.t. $X \setminus Z$ is Gorenstein. Therefore one can consider the canonical class K_X as <u>the closure</u> of the canonical class in $X \setminus Z$.
- The condition S_2 is one of the so-called Serre's conditions S_n, $n \in \mathbb{Z}_{\geq 0}$.

<u>Def</u>.

X satisfies S_n if $\forall\, p \in X$, depth $\mathcal{O}_{X,p} \geq \min\{n, \dim \mathcal{O}_{X,p}\}$.

depth $\mathcal{O}_{X,p}$ is the supremum of length of sequences $f_1, \dots, f_r \in m_p$ t.c. f_i is a nonzero divisor in $\mathcal{O}_{X,p}/(f_1, \dots, f_{i-1})$.

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In general, depth $\mathcal{O}_{X,p} \leq \dim \mathcal{O}_{X,p}$. Noeth. loc. rings for which equality holds are called Cohen-Macaulay. Equiv, these satisfy S_n, $\forall n$.

- S_2 means that X is connected in codimension 1. This means that one cannot disconnect X locally analytically by removing a subset of codimension ≥ 2. (This makes sense if X is reduced!)

Example. Let $X = V(x,y) \cup V(z,w) \subseteq \mathbb{A}^4$, two planes with the point $p=(0,0,0,0)$ in common. X is not S_2 because $X \setminus \{p\}$ is disconnected.

Q: How is demi-normal related to normal?

Def. A scheme X is R_1 if $\exists\, Z \subseteq X$ closed, $\operatorname{codim}_X(Z) \geq 2$ s.t. $X \setminus Z$ is smooth.

Thm (Serre's criterion for normality). A scheme X is normal $\iff$ is R_1 and S_2.

Rmk.

normal $\Rightarrow$ demi-normal.
$(R_1 + S_2)$

Def. Let X be a demi-normal scheme and $D \subseteq X$ the divisor obtained as the closures of the nodes of X. Let $\nu : X^\nu \to X$ be the normalization. Let $D^\nu := \nu^{-1} D$ with reduced scheme structure. Then D, D^ν are called the <u>conductors of ν</u>.
Note that $D^\nu \to D$ is generically of degree 2.

Prop. Let X be a demi-normal projective scheme, $\nu : X^\nu \to X$ normalization, D^ν, D conductor subschemes. Let B be an effective $\mathbb{Q}$-Cartier divisor s.t. $K_X + B$ is $\mathbb{Q}$-Cartier. Then

$$\nu^*(K_X + B) = K_{X^\nu} + \nu_*^{-1} B + D^\nu.$$

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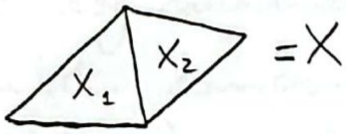
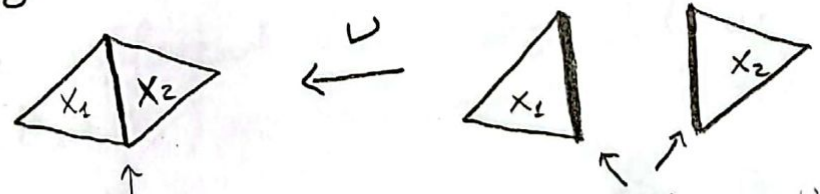
We now conclude with an example summarizing the notions of demi-normal, conductor, and the pullback formula with respect to normalization.

<u>Example</u>. Let $X_1 = \mathbb{P}^2 = X_2$ and $D_i \subseteq X_i$ a line, $i = 1, 2$. Let X be the gluing of X_1 and X_2 along $D_1 \cong D_2$. Moreover, we glue X_1 and X_2 transversely, so that X is nodal along $X_1 \cap X_2$. We can picture $\mathbb{P}^2$ as a triangle where we can think of the three edges as the three coordinate lines in $\mathbb{P}^2$. Then we can picture X as follows:



We have that X is demi-normal: it is nodal in codimension one and S_2 is guaranteed by the fact that X is not disconnected by removing closed subsets of codimension ≥ 2.

The normalization and the conductors are given by:



D $\qquad$ $D^\nu = D_1^\nu \sqcup D_2^\nu$,

$D_i^\nu \cong \ell_i$, $i = 1, 2$.

Finally, note that, if we set $B = 0$ in the pullback formula, we obtain that

$$\nu^* K_X = K_{X^\nu} + D^\nu = K_{X_1} + D_1^\nu + K_{X_2} + D_2^\nu .$$

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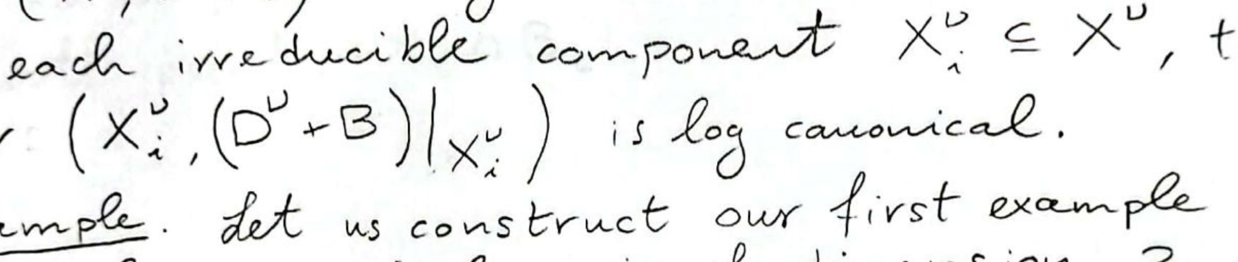
We are now ready to introduce the higher dimensional analogue of a stable weighted pointed curve from the point of view of the singularities.

<u>Def.</u> Let X be a demi-normal scheme with normalization $\nu : X^\nu \to X$ and conductors D, D^ν. Let B be an effective $\mathbb{Q}$-divisor on X whose support does not contain any irreducible component of D. Let B^ν be the divisorial part of $\nu^{-1}B$. The pair (X,B) is <u>semi-log canonical</u> (slc) provided

(1) $K_X + B$ is $\mathbb{Q}$-Cartier;

(2) $(X^\nu, D^\nu + B)$ is log canonical, which means that for each irreducible component $X_i^\nu \subseteq X^\nu$, the pair $(X_i^\nu, (D^\nu + B)|_{X_i^\nu})$ is log canonical.

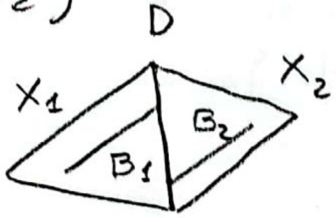
<u>Example</u>. Let us construct our first example of semi-log canonical pair of dimension 2. Let us start with $X = X_1 \cup X_2$, where $X_1 \cong X_2 \cong \mathbb{P}^2$ are glued along the lines $D_1 \cong D_2$. We know X is demi-normal. Let us endow X with a divisor B. Let $B_i \subseteq X_i$ be a line different from D_i. Let us glue $(X_1, D_1 + B_1)$ and $(X_2, D_2 + B_2)$ as follows:



We know X is demi-normal and B does not contain any irr. comp. of D, let us check the $\mathbb{Q}$-Cartierness of $K_X + B$. As K_X is Cartier, the fact that $K_X + B$ is $\mathbb{Q}$-Cartier is equivalent to B $\mathbb{Q}$-Cartier.

B is Cartier because it can be realized as the restriction of a hyperplane in $\mathbb{P}^3$ to $X = V(xy) \subseteq \mathbb{P}^3$.

<u>Rmk</u>. How about this other way of gluing $(X_1, D_1 + B_1)$ with $(X_2, D_2 + B_2)$:



In this case, (X, B) is not semi-log canonical because $K_X + B$ is not $\mathbb{Q}$-Cartier, as B is not $\mathbb{Q}$-Cartier. If it was, then $B|_{X_1}$ should be a divisor, but it is not because it has an isolate point: $X_1 \cap B_2$.

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§ Stable pairs: definition and examples.

We now have enough background to completely define stable pairs.

Recall these can be thought of as the higher dimensional analogue of Hassett's stable weighted pointed curves.

Def. Let X be a variety, B $\mathbb{Q}$-divisor on X with coeff's in $(0,1]$ s.t. K_X+B is $\mathbb{Q}$-Cartier. The pair (X,B) is called stable provided:

1) (X,B) is semi-log canonical;
2) K_X+B is ample.

Rmk. $\nu: X^\nu \to X$ normaliz. We have that K_X+B is ample $\iff$ $\nu^*(K_X+B) = K_{X^\nu} + \nu_*^{-1}B + D^\nu$ is ample. To prove the latter one can show that $K_{X_j^\nu} + (\nu_*^{-1}B)|_{X_j^\nu} + D_j^\nu$ is ample $\forall$ irr. comp. $X_j^\nu \subseteq X^\nu$.

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