§ Stable pairs: definition and examples.

We now have enough background to completely define stable pairs.

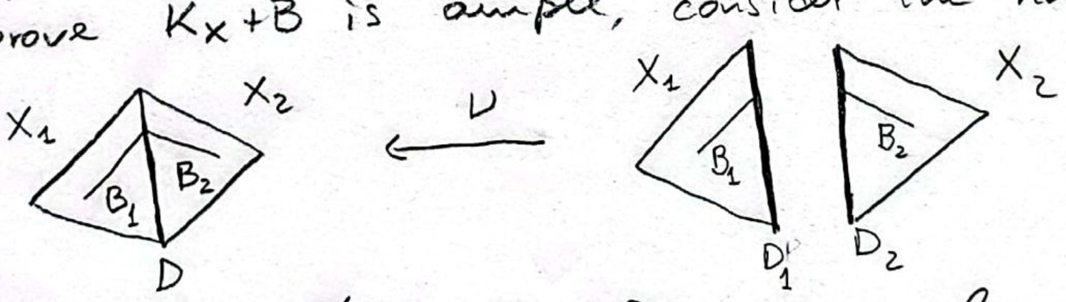
Recall these can be thought of as the higher dimensional analogue of Hassett's stable weighted pointed curves.

Def. Let X be a variety, B $\mathbb{Q}$-divisor on X with coeff's in $(0,1]$ s.t. K_X+B is $\mathbb{Q}$-Cartier. The pair (X,B) is called stable provided:

1) (X,B) is semi-log canonical;
2) K_X+B is ample.

Rmk. $\nu: X^\nu \to X$ normaliz. We have that K_X+B is ample $\Leftrightarrow$ $\nu^*(K_X+B) = K_{X^\nu} + \nu_*^{-1}B + D^\nu$ is ample. To prove the latter one can show that $K_{X_j^\nu} + (\nu_*^{-1}B)|_{X_j^\nu} + D_j^\nu$ is ample $\forall$ irr. comp. $X_j^\nu \subseteq X^\nu$.

Example 1. Consider $X = X_1 \cup X_2$ and $B = B_1 + B_2$ as in the previous example, which we know is slc. To prove K_X+B is ample, consider the normaliz.

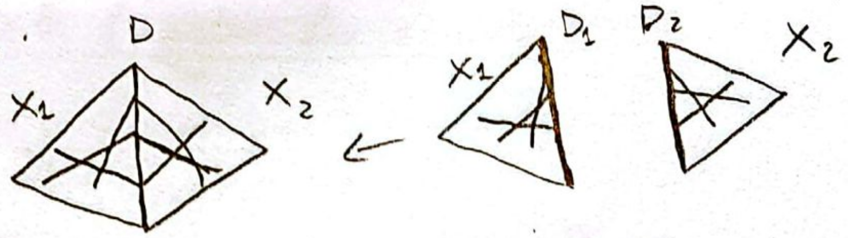


We prove that $K_{X_i} + B_i + D_i$ is ample, $i=1,2$. If $H \subseteq \mathbb{P}^2$ is a line, then

$$K_{X_i} + B_i + D_i \sim -3H + H + H = -H$$, which is not ample!

So (X,B) is not a stable pair. To "fix" this we can add more lines:

Example 2.



For $i = 1, 2$,

$$K_{X_i} + D_i + B|_{X_i} \sim -3H + H + 3H = H,$$ which is ample.

Example. Let $C \subseteq \mathbb{P}^2$ be a plane curve of degree $d \geq 3$ such that $(\mathbb{P}^2, \frac{3+\varepsilon}{d} C)$ is slc, $0 < \varepsilon < 1$, $\varepsilon \in \mathbb{Q}$. Then

$$K_{\mathbb{P}^2} + \frac{3+\varepsilon}{d} C \sim_{\mathbb{Q}} -3H + \frac{3+\varepsilon}{d} dH = \cancel{-3H} + \cancel{3H} + \frac{\varepsilon}{d} H = \frac{\varepsilon}{d} H,$$

which is ample. So, $(\mathbb{P}^2, \frac{3+\varepsilon}{d} C)$ is stable.

($\frac{3+\varepsilon}{d}$ is referred to as the Hacking weight.)

§ The moduli functor for stable pairs.

The theory of compact moduli of stable pairs was initiated by works of Kollár–Shepherd-Barron, Alexeev. By now there is a complete theory, which is the result of the work of numerous people.

We briefly discuss it first in the case of stable varieties, that is, the divisor of the pair is zero. We follow Kollár's book: "Families of varieties of general type".

- Stable varieties.

Def 3.40. S be a scheme over a field of characteristic 0.

$f : \mathcal{X} \to S$ is a stable family of varieties if:

1) f is flat and proper;
2) The fibers are slc;
3) $\forall m \in \mathbb{Z}$, $\omega_{\mathcal{X}/S}^{[m]} := \left(\omega_{\mathcal{X}/S}^{\otimes m}\right)^{**}$ is a flat family of divisorial sheaves,
4) $\omega_{\mathcal{X}/S}$ is f-ample.

$\forall s \in S$, $\omega_{\mathcal{X}/S}|_{\mathcal{X}_s}$ is ample.

flat over S

fibers are divisorial sheaves

A coherent sheaf L on a scheme X is called divisorial if L is S_2 and $\exists Z \subseteq X$ $\mathrm{codim}_X Z \geq 2$ s.t. $L|_{X \setminus Z}$ is invertible.

Thm 3.4. Let B be a base scheme of char 0. Let $d \in \mathbb{Z}_{>0}$, $\nu \in \mathbb{Q}_{>0}$. Let $\mathcal{SV}(d,\nu)$ be the functor of stable families $\mathcal{X} \to S$ over B of relative dimension d s.t. $\forall s \in S$, $K_{\mathcal{X}_s}^d = \nu$.

Then $\mathcal{SV}(d,\nu)$ has a coarse moduli space $SV(d,\nu) \to B$ which is projective over B.

- Stable pairs.

The definition of the moduli functor for stable pairs with a nonzero divisor is more involved. Here we point out some of its features. We fix d and ν as before, but also a vector $\underline{a}=(a_1,\dots,a_n)$ of coefficients for the divisor, $a_i \in [0,1]\cap\mathbb{Q}$ $\forall i$. We consider families

$$f:\left(\mathcal{X}, \mathcal{D}=\sum_{i=1}^{n} a_i \mathcal{D}_i\right) \longrightarrow S,$$

where $f:\mathcal{X}\to S$ is proper, flat, of relative dimension d, the $\mathcal{D}_i$ are relative $\mathbb{Z}$-divisors, and the fibers $(\mathcal{X}_s, \mathcal{D}_s)$ are stable pairs s.t. $(K_{\mathcal{X}_s}+\mathcal{D}_s)^d = \nu$. For all the details, see §8.7. By Thm 8.1, there exists projective coarse moduli space $SP(\underline{a},d,\nu)$ parametrizing stable pairs $(X, \sum_{i=1}^{n} a_i D_i)$, $\dim X = d$, $(K_X+\sum_{i=1}^{n}a_i D_i)^d = \nu$.

§ Compact moduli of weighted stable hyperplane arrangements.

Example. Let $r \geq 2$ and consider hyperplanes $H_1, \dots, H_n \subseteq \mathbb{P}^{r-1}$ which we take in g. l. p. Fix $\underline{b} = (b_1, \dots, b_n)$, $b_i \in (0,1] \cap \mathbb{Q}$. We know the pair $(\mathbb{P}^{r-1}, \sum_{i=1}^n b_i H_i)$ is log canonical. To be stable, we need the ampleness of

$$K_{\mathbb{P}^{r-1}} + \sum_{i=1}^n b_i H_i \sim_{\mathbb{Q}} (-(r-1)-1)H + \sum_{i=1}^n b_i H = \left(\sum_{i=1}^n b_i - r\right)H \,,$$

where $H \subseteq \mathbb{P}^{r-1}$ is any hyperplane. So we need $\sum_{i=1}^n b_i > r$.

Thm (Hacking-Keel-Tevelev, Alexeev). Let $r, n, \underline{b}$ as above. Then there exists a projective coarse moduli space $\overline{M}_{\underline{b}}(r,n)$ and a flat proj. family

$$\left(\mathcal{X}, \sum_{i=1}^n b_i \mathcal{B}_i\right) \to \overline{M}_{\underline{b}}(r,n)$$

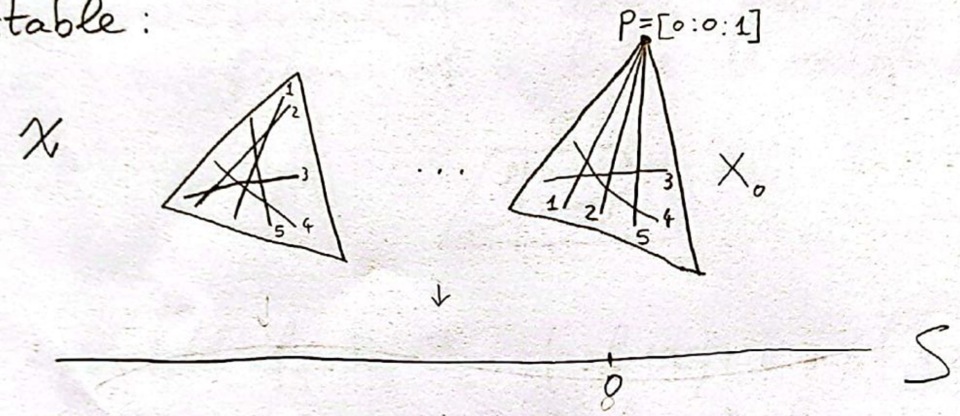
parametrizing stable pairs $(\mathbb{P}^{r-1}, \sum_{i=1}^n b_i H_i)$ and their stable degenerations $\left(\underset{\substack{\uparrow \\ \text{degeneration of } \mathbb{P}^{r-1}}}{X}, \sum_{i=1}^n b_i \underset{\substack{\uparrow \\ \text{"broken hyperplanes"}}}{B_i}\right)$, called weighted stable hyperplane arrangements.

Let us compute some of these stable degenerations.
For the geometry of $\overline{M}_{\underline{b}}(r,n)$, we refer to Alexeev's book: "Moduli of weighted hyperplane arrangements", chapter 5.

Example. $r=3$, $n=5$, $b_1=\ldots=b_5=1$. Let $S:=\mathbb{A}^1_t\setminus\{\pm 1\}$, $\mathcal{X}:=\mathbb{P}^2\times S\to S$. $[X_1,X_2,X_3]\in\mathbb{P}^2$ coordinate. Define the following divisors in $\mathcal{X}$:

$\mathcal{B}_1=V(X_1)$, $\mathcal{B}_2=V(X_2)$, $\mathcal{B}_3=V(X_3)$, $\mathcal{B}_4=V(X_1+X_2+X_3)$, and $\mathcal{B}_5=V(X_1-X_2+tX_3)$.

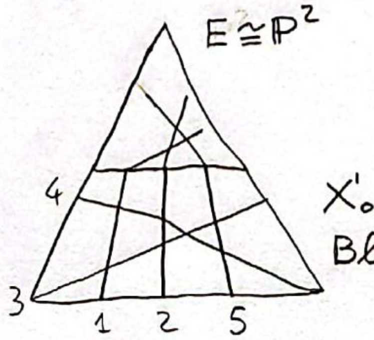
Consider $(\mathcal{X},\sum_{i=1}^{5}\mathcal{B}_i)\to S$. For $s\in S\setminus\{0\}$, the fiber over s is a stable pair because the 5 lines are in general linear position. The fiber over 0, $(X_0, B_1+\ldots+B_5)$ is not stable:



because the lines 1,2,5 are concurrent.

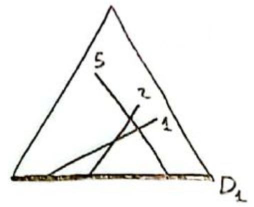
On the other hand, we know the stable limit exists within $\overline{M}_1(3,5)$ and one can find it after birat. modifications along the central fiber and base changes. In this case, it suffices to blow up $\mathcal{X}$ at $([0:0:1],0)$:

$\mathcal{X}'\to\mathcal{X}$ with exceptional divisor $E\cong\mathbb{P}^2$. The new central fiber of $\mathcal{X}'$ is given by:



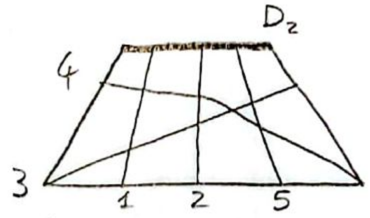
$X'_0 := Bl_P X_0 = Bl_{[0:0:1]}\mathbb{P}^2=\mathbb{F}_1$, the first Hirzebruch surface.

We need to prove this is stable.



$(E, D_1+L_1+L_2+L_3)$

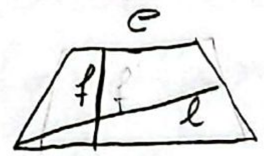
- log canonical
- $K_E+D_1+L_1+L_2+L_3$ is ample.



$(X_0', D_2+f_1+f_2+f_5+B_3+B_4)$

- log canonical
- Ampleness?

Let $e \subseteq X_0'$ be the exceptional divisor, $l \subseteq X_0'$ the strict transform of a line not through p, f the strict transform of a line through p



$$K_{X_0'}+D_2+f_1+f_2+f_5+B_3+B_4 \sim (-3l+e)+e+3f+2l = 2e+3f-l =$$
$$= 2(e+f)+f-l \sim 2l+f-l = l+f.$$

How to check $l+f$ is ample? As X_0' is a toric variety, it is enough to check that $(l+f)\cdot C>0$ for any torus fixed curve $C \subseteq X_0'$. (☆) Up to linear equivalence, these are precisely e, f, l.

$(l+f)\cdot e = 1 > 0$

$(l+f)\cdot f = 1 > 0$

$(l+f)\cdot l = 2 > 0$. ✓

(☆) These curves generate the closed cone of curves. So ampleness follows from Kleiman's criterion.