Introduction to Algebraic Geometry (MATH 8300) 08/23/2016

Lecture 3

TODAY: Nullstellensatz.

Recall: Let k be any field.

$$\left\{\begin{matrix}\text{Ideals in}\\ k[x_1,\dots,x_n]\end{matrix}\right\} \underset{I(-)}{\overset{Z(-)}{\rightleftarrows}} \left\{\begin{matrix}\text{Zariski closed}\\ \text{subsets of } \mathbb{A}^n\end{matrix}\right\}$$

- $Z(-)$ is surjective by definition of Zariski closed subset,
- $Z(-)$ is not injective: $Z(x) = Z(x^2)$;
- So we need to restrict our attention to radical ideals of $k[x_1,\dots,x_n]$.

  However, if J is radical, it is still not true in general that $J = I(Z(J))$

  Ex: $J = (x^2+1) \subset \mathbb{R}[x]$.
  J is radical,
  $I(Z(J)) = \mathbb{R}[x]$.

  Need: $k = \bar{k}$.

Nullstellensatz: Let $k=\bar{k}$. If $J \subseteq k[x_1,\dots,x_n]$ radical, then $J = I(Z(J))$. Equivalently, let I be any ideal, then $\sqrt{I} = I(Z(I))$.

Observation: For any k, if $I \subseteq k[x_1, \dots, x_n]$ is an ideal, then it is always true that

$$\sqrt{I} \subseteq I(Z(I)).$$

proof: $f \in \sqrt{I} \Rightarrow f^N \in I$, $\exists N \in \mathbb{Z}_{>0}$. Let $a \in Z(I)$. We want to show that $f(a) = 0$. But $f^N(a) = 0$ by definition, so that $f(a) = 0$. Since a is arbitrary, $f \in I(Z(I))$. $\square$

So the Nullstellensatz is really about the other containment. We will see how the Nullstellensatz (geometry) relies on a purely algebraic fact.

Algebraic interlude:

R ring, commutative with unit.

Def: An ideal $P \subsetneq R$ is prime if $\forall a, b \in R$ s.t. $ab \in P$, then $a \in P$ or $b \in P$.

Def: An ideal $M \subsetneq R$ is maximal if, given an ideal $I \subseteq R$ s.t. $M \subsetneq I$, then $I = R$.

Prop: Maximal $\Rightarrow$ Prime.

Prop: $I \subsetneq R$ ideal $\Rightarrow \exists$ maximal ideal $M \supseteq I$
(it uses Zorn's lemma & the unit)

Prop: $I \subsetneq R$ is prime (resp. maximal) $\Leftrightarrow R/I$ is a domain (resp. a field).

Corollary: Prime ideals are radical.

proof: From previous class, I radical $\Leftrightarrow$ R/I has no nilpotents. Then use previous proposition.

Corollary: $k[x_1, \dots, x_n]$, $a \in \mathbb{A}^n$. Then

(i) $(x_1 - a_1, \dots, x_n - a_n)$ is maximal;

(ii) $I(\{a\}) = (x_1 - a_1, \dots, x_n - a_n)$.

proof:

(i) $$\frac{k[x_1, \dots, x_n]}{(x_1 - a_1, \dots, x_n - a_n)} \cong k.$$

(ii) $(\supseteq)$ ✓

$(\subseteq)$ (Thanks to Makoto for making this proof shorter).

$1 \notin I(\{a\})$ and $(x_1 - a_1, \dots, x_n - a_n) \subseteq I(\{a\})$. Therefore $(x_1 - a_1, \dots, x_n - a_n) = I(\{a\})$ by part (i).

$\square$

Observation: Consider $k[x_1, \dots, x_n]$, $k = \overline{k}$, and assume the Nullstellensatz holds. Let $M \subset k[x_1, \dots, x_n]$ be a maximal ideal.

Then:

$$M = I(Z(M)) \Rightarrow$$
$$Z(M) \neq \phi \Rightarrow$$
$$Z(M) \supseteq \{(a_1, \dots, a_n)\} \Rightarrow$$
$$M = I(Z(M)) \subseteq I(a) = (x_1 - a_1, \dots, x_n - a_n) \Rightarrow$$
$$M = (x_1 - a_1, \dots, x_n - a_n).$$

Algebraic Nullstellensatz: Let $M \subset k[x_1, \dots, x_n]$ be a maximal ideal, $k = \overline{k}$. Then $\exists a_1, \dots, a_n \in k$ s.t. $M = (x_1 - a_1, \dots, x_n - a_n)$.

We proved that:

Nullstellensatz $\Rightarrow$ Algebraic Nullstellensatz.

Thm: The Algebraic Nullstellensatz implies the Nullstellensatz.

proof: Want to show: $I(Z(I)) \subseteq \sqrt{I}$.

Set up: $I = (f_1, \dots, f_k)$, $g \in I$.

Rabinowitz: $J := (f_1, \dots, f_k, yg - 1) \subseteq k[x_1, \dots, x_n, y]$

$$\mathbb{A}^{n+1} \supseteq Z(J) = \phi \Rightarrow J = k[x_1, \dots, x_n, y]$$

$\uparrow$

Algebraic Nullstellensatz.

$$1 \in J \implies 1 = h_1 f_1 + \dots + h_k f_k + h(yg - 1)$$

Set $y = \frac{1}{z} \Rightarrow 1 = h_1^{(1)} f_1 + \dots + h_k^{(1)} f_k + h^{(1)}\left(\frac{g}{z} - 1\right)$

Clear denominators $\Rightarrow z^N = h_1^{(2)} f_1 + \dots + h_k^{(2)} f_k + h^{(2)}(g - z)$

Set $z = g \Rightarrow g^N = h_1^{(3)} f_1 + \dots + h_k^{(3)} f_k$. $\square$

We will assume the algebraic form of the Nullstellensatz (to prove it, one can use Noether normalization lemma)

<u>Conclusion</u>:

$$\left\{\begin{array}{l}\text{radical ideals}\\ \text{in } k[x_1, \dots, x_n]\end{array}\right\} \underset{I(-)}{\overset{Z(-)}{\rightleftarrows}} \left\{\begin{array}{l}\text{Zariski closed}\\ \text{subsets of } \mathbb{A}^n\end{array}\right\}$$

Are bijective maps inverse of each other.