Introduction to Algebraic Geometry (MATH 8300) 08/25/2016

Lecture 4

TODAY:
- Basis for the Zariski topology on $\mathbb{A}^n$;
- Irreducible topological spaces, and application to Zariski topology.

Basis for the Zariski topology on $\mathbb{A}^n$:

Recall: Let X be a topological space. A basis $\mathcal{B}$ for the topology on X is a collection of open sets in X such that every open set in X can be written as the union of elements of $\mathcal{B}$. Equivalently, $\forall U \subseteq X$ open and $\forall p \in U$, $\exists B \in \mathcal{B}$ s.t. $p \in B \subseteq U$.

Def: Let $f \in k[x_1, \dots, x_n]$. Then define:

$$D(f) := \{ p \in \mathbb{A}^n \mid f(p) \neq 0 \} = \mathbb{A}^n \setminus Z(f).$$

Prop: The open sets $D(f)$ form a basis for the Zariski topology of $\mathbb{A}^n$.

proof 1: Let $U \subseteq \mathbb{A}^n$ be an open set and let $p \in U$. We want to find $f \in k[x_1, \dots, x_n]$ s.t. $p \in D(f) \subseteq U$. Observe that $U = \mathbb{A}^n \setminus Z(I)$, $\exists I \subseteq k[x_1, \dots, x_n]$ ideal. Since $p \notin Z(I) \Rightarrow \exists f \in I$ s.t. $f(p) \neq 0$. Therefore $p \in D(f)$ and $(f) \subseteq I \Rightarrow Z(f) \supseteq Z(I)$ $\Rightarrow \mathbb{A}^n \setminus Z(f) \subseteq \mathbb{A}^n \setminus Z(I) \Rightarrow D(f) \subseteq U$. $\square$

proof 2: Using Basissatz. $I = (f_1, \dots, f_n) \Rightarrow$

$$Z(I) = \bigcap_{i=1}^{n} Z(f_i) \Rightarrow$$

$$\mathbb{A}^n \setminus Z(I) = \mathbb{A}^n \setminus \bigcap_{i=1}^{n} Z(f_i) \Rightarrow$$

$$\mathbb{A}^n \setminus Z(I) = \bigcup_{i=1}^{n} \left(\mathbb{A}^n \setminus Z(f_i)\right) \Rightarrow$$

$$U = \bigcup_{i=1}^{n} D(f_i). \qquad \square$$

Proof 2 is also saying that you need only finitely many of these $D(f)$.

<u>Irreducible topological spaces:</u>

<u>Def</u>: A topological space X is irreducible if every time $X = C_1 \cup C_2$, C_1, C_2 closed, then $C_1 = X$ or $C_2 = X$. $S \subseteq X$ is irreducible if S is irreducible with its induced topology. $\emptyset$ is not considered irreducible by definition. X is reducible if it is not irreducible.

<u>Observation</u>: This definition is meaningless when one works in $\mathbb{R}^n$ with the Euclidean topology: $\forall S \subseteq \mathbb{R}^n$ which is not a point is reducible.

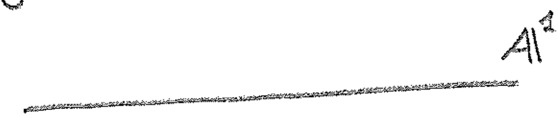
proof: Let $p, q \in S$, $p \neq q$, $m = \frac{p+q}{2}$, H hyperplane through m orthogonal to the segment $\overline{pq}$, H^+, H^- two closed half-spaces determined by H. Then $S = (S \cap H^+) \cup (S \cap H^-)$. $\square$

So this is probably why you never heard it before.

Examples:

(a) Let $k = \bar{k}$. Then the Zariski topology on $\mathbb{A}^1$ is the cofinite topology.

So $\mathbb{A}^1$ is irreducible,

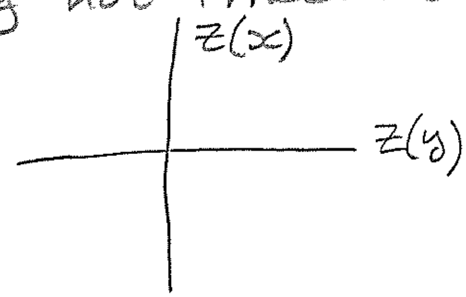


because if $\mathbb{A}^1 = X_1 \cup X_2$, X_1, X_2 closed, then the only possibility is that $X_1 = \mathbb{A}^1$ or $X_2 = \mathbb{A}^1$ (otherwise $|X_1 \cup X_2| < \infty$, but $|\mathbb{A}^1| = \infty$).

(b) We will show that if $k = \bar{k}$, then $\mathbb{A}^n$ is irreducible $\forall n$ (actually, $|k| = \infty$ suffices!).

(c) $Z(xy) \subset \mathbb{A}^2$ is definitely not irreducible.

$$Z(xy) = Z(x) \cup Z(y)$$



Observation: X topological space.

X irreducible $\Rightarrow$ X connected.

If X is disconnected, then $X = C_1 \cup C_2$, $C_1, C_2 \subsetneq X$ closed, $C_1 \cap C_2 = \emptyset$. Therefore X is reducible.

The converse is false, counterexample is (c)

In Zariski topology, irreducible closed sets are related to prime ideals.

<u>Thm</u>: Let $X \subseteq \mathbb{A}^n$ be a Zariski closed subset. Then X is irreducible $\iff I(X)$ is prime.

proof:

$(\Rightarrow)$ $f, g \in k[x_1, \dots, x_n] \setminus I(X)$, $fg \in I(X)$.

$$\begin{array}{l} Z(fg) = Z(f) \cup Z(g) \\ \quad \cup\!\mid \\ Z(I(X)) = X \end{array}$$

$$X = X_f \cup X_g, \qquad X_f := X \cap Z(f), \quad X_g := X \cap Z(g).$$

If we show that $X_f \subsetneq X$, we are done.

$X_f = X \Rightarrow X \cap Z(f) = X \Rightarrow Z(f) \supseteq X$

$\Rightarrow \underset{\substack{\cup \\ f}}{I(Z(f))} \subseteq I(X)$ $\quad \divideontimes$

$(\Leftarrow)$ $X = X_1 \cup X_2$, X_1, X_2 closed, $X_1 \subsetneq X$.

Want to show $X_2 \supseteq X$, or equivalently $I(X_2) \subseteq I(X)$. $g \in I(X_2)$. Also, let $f \in I(X_1) \setminus I(X)$. $fg \in I(X) \Rightarrow g \in I(X)$.

$\square$

Example: If $k=\bar{k}$ ($|k|=\infty$ suff.) then $I(\mathbb{A}^n)=(0)$, which is prime in $k[x_1,\dots,x_n] \Rightarrow \mathbb{A}^n$ is irreducible.

Corollary: If $k=\bar{k}$, then $Z(I)$ is irreducible $\Longleftrightarrow \sqrt{I}$ is prime.

proof:

($\Rightarrow$) $I(Z(I)) \overset{\text{Nullstellensatz}}{=} \sqrt{I}$ is prime by the previous theorem.

($\Leftarrow$) $\sqrt{I} \underset{\text{Nullstellensats}}{=} I(Z(I)) \Rightarrow Z(I)$ is irreducible by the previous theorem. $\square$

If there is time ↴

Observation: The previous corollary is false without $k=\bar{k}$. Here there are counterexamples.

"$\not\Leftarrow$" Let k be a finite field.

$\mathbb{A}^n_k$ has the discrete topology $\Rightarrow$ is reducible.

$\mathbb{A}^n_k = Z((0))$ with (0) prime.

"$\not\Rightarrow$" (Thanks to Ernest) $k=\mathbb{R}$, $n=1$.

$I=(x(x^2+1))$ is radical, not prime.

$Z(I)=\{0\}$ is irreducible.