FIRST Seminar,
Looijenga semitoric compactifications

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1 Introduction

In this talk I will discuss the content of [L].

2 General idea of the paper

Let us briefly and informally give the general idea behind this paper. Let $\mathbb{D}$ be a bounded Hermitian[1] symmetric[2] domain of type IV. A concrete example of $\mathbb{D}$ which we will keep in the back of our mind throughout the whole talk is the period domain parametrizing polarized K3 surfaces of a given degree.

Example 2.1. Let $L_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$. Fix a primitive vector $v \in L_{\mathrm{K3}}$ and consider

$$\mathbb{D} \amalg \mathbb{D}' = \{[x] \in \mathbb{P}(v^{\perp} \otimes_{\mathbb{Z}} \mathbb{C}) \mid x \cdot x = 0, x \cdot \overline{x} > 0\}.$$

We select a preferred connected component: $\mathbb{D}$.

Let Γ be an arithmetic group[3] acting on $\mathbb{D}$. Let us also assume we are in the nice case where Γ is neat[4] and acts properly discontinuously[5] on $\mathbb{D}$. In particular, $\mathbb{D}/\Gamma$ is a complex analytic manifold (for this, neatness can be relaxed to torsion-freeness). Then we have the following theorems.

[1]A Hermitian manifold is a complex manifold X with a Hermitian inner product on each holomorphic tangent space T_xX which varies smoothly with $x \in X$. Important examples of these are Kähler manifolds.

[2]A bounded domain $\mathbb{D}$ is called symmetric if for any point $x \in \mathbb{D}$ there exists a holomorphic involution with x as an isolated fixed point.

[3]Let G be an algebraic group defined over $\mathbb{Q}$ together with a specified embedding $G \hookrightarrow GL(n, \mathbb{C})$. A subgroup $\Gamma \subset G(\mathbb{Q})$ is called arithmetic if it is commensurable with $G(\mathbb{Z}) := G(Q) \cap GL(n, \mathbb{Z})$, i.e. $\Gamma \cap G(\mathbb{Z})$ has finite index in Γ and $G(\mathbb{Z})$.

[4]Γ is neat if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of its elements is torsion-free.

[5]Let G be a group and X a topological space. An action $G \curvearrowright X$ is properly discontinuous if X is a locally compact space (i.e. every point in X has a compact neighborhood) and for every compact subset $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite

Theorem 2.2 (Baily-Borel)**.** *$\mathbb{D}/\Gamma$ is a quasi-projective variety. Indeed, $\mathbb{D}/\Gamma$ has a projective normal compactification $\overline{\mathbb{D}/\Gamma}^{BB}$ called the Baily-Borel compactification. The boundary of this compactification has dimension at most* 1 *(and usually it is highly singular).*

Theorem 2.3 (Ash-Mumford-Rapoport-Tai)**.** *For an appropriate choice τ of combinatorial data for each* 0*-dimensional stratum of $\overline{\mathbb{D}/\Gamma}^{BB}$, there exists a normal complete (possibly projective) compactification $\overline{\mathbb{D}/\Gamma}^{\tau}$ with divisorial boundary mapping birationally onto $\overline{\mathbb{D}/\Gamma}^{BB}$. These compactifications are called toroidal compactifications.*

An important problem in algebraic geometry is to provide toroidal compactifications with modular meaning in terms of degenerations of the object parametrized by $\mathbb{D}$.

In the paper in analysis a new family of compactifications of $\mathbb{D}/\Gamma$ is constructed. This family contains toroidal and Baily-Borel compactifications as special cases. These new compactifications are called *semitoric compactifications.* One has birational morphisms

$$\overline{\mathbb{D}/\Gamma}^{\tau} \to \overline{\mathbb{D}/\Gamma}^{\text{semitoric}} \to \overline{\mathbb{D}/\Gamma}^{BB},$$

which are isomorphisms on $\mathbb{D}/\Gamma$. Other specific birational modifications of these semitoric compactifications are constructed in relation to a given *hyperplane arrangement* on $\mathbb{D}$. We will mainly focus on these semitoric compactifications.

3 Setup: linear algebra

- Let V be a $\mathbb{C}$-vector space and let $\phi\colon V \times V \to \mathbb{C}$ be a symmetric bilinear form which is defined over $\mathbb{Q}$. For the applications we have in mind, one can start from a lattice L, take $V = L \otimes_{\mathbb{Z}} \mathbb{C}$ and define ϕ by extending the bilinear form on L.
- Assume that $\dim(V) = n + 2$ and that ϕ has signature $(2, n)$. Recall that the notion of signature of a symmetric bilinear form makes sense over $\mathbb{R}$ (which is our case), and 2 (resp. n) is the number of positive (resp. negative) eigenvalues.

Definition 3.1. A vector subspace $W \subset V$ is called *isotropic* if $\phi|_{W\times W}$ is identically zero (or equivalently, if the quadratic form induced by ϕ restricted to W is identically zero).

Observation 3.2. Let $W \subset V$ be a isotropic subspace. Then

$$W \subseteq W^{\perp} = \{v \in V \mid \phi(v, w) = 0 \text{ for all } w \in W\}.$$

This can be counterintuitive from the point of view of Euclidean geometry where we have a positive definite inner product (for instance, in our case it is false that $W \oplus W^{\perp} = V$). However, it still holds that

$$\dim(W) + \dim(W^{\perp}) = \dim(V).$$

Finally, observe that ϕ naturally induces a symmetric bilinear form on $W^{\perp}/W$ defined by

$$(x+W, y+W) \mapsto \phi(x,y).$$

It is easy to verify that this is well defined.

Proposition 3.3. *Let $W \subset V$ be an isotropic subspace defined over $\mathbb{R}$ (we callet it an $\mathbb{R}$-isotropic subspace for short). Then $0 \leq \dim(W) \leq 2$. An isotropic subspace of dimension 1 (resp. 2) is called isotropic line (resp. isotropic plane) and it is denoted by the letter I (resp. J). The bilinear form on $I^{\perp}/I$ (resp. $J^{\perp}/J$) has signature $(1, n-1)$ (resp. $(0, n-2)$).*

Isotropic subspaces of V are important for the following reason. The space

$$\{[v] \in \mathbb{P}(V) \mid \phi(v,v) = 0 \text{ and } \phi(v,\overline{v}) > 0\},$$

has two connected components exchanged by complex conjugation. Let us choose one of them and call it $\mathbb{D}$. If $O(\phi)$ is the group of isomorphisms $f\colon V \to V$ such that $\phi(f(v_1), f(v_2)) = \phi(v_1, v_2)$ for all $v_1, v_2 \in V$, let $\Gamma \subset O(\phi)$ be an arithmetic subgroup which is neat and which preserves $\mathbb{D}$. Then the 0-dimensional (resp. 1-dimensional) boundary strata of $\overline{\mathbb{D}/\Gamma}^{BB}$ correspond to Γ-orbits of $\mathbb{Q}$-isotropic lines (resp. $\mathbb{Q}$-isotropic planes).

4 Setup: the conical locus of $\mathbb{D}$

The combinatorial data necessary to compactify $\mathbb{D}/\Gamma$ is called *admissible decomposition of the conical locus of* $\mathbb{D}$. So, first, what is the conical locus of $\mathbb{D}$? As the name suggests, this is a disjoint union of cones living in a certain space.

Let I be a $\mathbb{Q}$-isotropic line. Then $I^{\perp}/I$ is a hyperbolic lattice lattice. The subset of $(I^{\perp}/I)(\mathbb{R})$ given by $x \cdot x > 0$ has two connected components which are open convex cones. Denote by C_I one of the two.

Let J be a $\mathbb{Q}$-isotropic plane. The subset of $\bigwedge^2 J(\mathbb{R}) \backslash \{0\}$ has two connected components which are open half lines. Denote by C_J one of the two.

The cones C_I, C_J can be chosen in a canonical way which depends on the choice of connected component $\mathbb{D}$ we made.

$\{0\}$ is also a $\mathbb{Q}$-isotropic space of V, and we define $C_{\{0\}}$ to be $\{0\}$.

Definition 4.1. The *conical locus of* $\mathbb{D}$ is the disjoint union

$$C(\mathbb{D}) = \coprod_{\substack{W \subset V \\ W \ \mathbb{Q}\text{-isotropic}}} C_W,$$

Observation 4.2. In which space does $C(\mathbb{D})$ live?

$$C(\mathbb{D}) \subset \mathfrak{so}(\phi) = \{f \in \mathrm{End}(V) \mid \phi(f(v_1), v_2) = -\phi(v_1, f(v_2)) \text{ for all } v_1, v_2 \in V\}.$$

How do you see this? There is a natural identification $\mathfrak{so}(\phi) \equiv \bigwedge^2 V$ given by $v_1 \wedge v_2 \mapsto f_{v_1,v_2}$ such that $f_{v_1,v_2}(v) = \phi(v_1, v)v_2 - \phi(v_2, v)v_1$ (many thanks to Ernest Guico for explaining this to me). Obviously $\bigwedge^2 J \subset \bigwedge^2 V$. In addition, $I^\perp/I$ can be naturally embedded in $\bigwedge^2 V$ by considering $I \otimes_{\mathbb{C}} (I^\perp/I)$ instead. More explicitly, the map

$$x \otimes (y + I) \mapsto x \wedge y,$$

gives an embedding $I \otimes_{\mathbb{C}} (I^\perp/I) \hookrightarrow \bigwedge^2 V$. Therefore, we can see how $C(\mathbb{D})$ is contained in $\mathfrak{so}(\phi) \equiv \bigwedge^2 V$. Also, observe that the elements in $C(\mathbb{D}) \subset \mathfrak{so}(\phi)$ are nilpotents.

5 Admissible decomposition of $C(\mathbb{D})$

Notation 5.1. Let $W \subset V$ be a $\mathbb{Q}$-isotropic space. Then let $C_{W,+}$ be the convex hull of the $\mathbb{Q}$-vectors in $\overline{C}_W$.

Definition 5.2. An *admissible decomposition of* $C(\mathbb{D})$ is a Γ-invariant collection Σ of closed convex cones contained in $C_{I,+}$ such that, for any I $\mathbb{Q}$-isotropic lines such, we have that

- If $\sigma \in \Sigma|_{C_{I,+}}$ and τ is a face of σ, then $\tau \in C_{I,+}$;
- If $\sigma, \tau \in \Sigma|_{C_{I,+}}$, then σ and τ meet along a common face;
- $\cup_{\sigma \in \Sigma|_{C_{I,+}}} \sigma = C_{I,+}$;
- If $\sigma \subset C_{I,+}$ is a rational finite closed convex cone, then $\sigma \cap \Sigma|_{C_{I,+}}$ is a finite fan.

Σ has to satisfy the following compatibility condition: for any $\mathbb{Q}$-isotropic plane J, the *support space* (we omit the formal definition of it for simplicity, but we give an idea) of $C_{J,+}$ in $C_{I,+}$ has to be independent from the choice of $\mathbb{Q}$-isotropic line $I \subset J$.

Theorem 5.3 (Looijenga)**.** *Let Σ be an admissible decomposition of $C(\mathbb{D})$. Then there exists a normal complete (possibly projective) compactification $\overline{\mathbb{D}/\Gamma}^{\Sigma}$ associated to Σ. If Σ_1, Σ_2 are two such decompositions and Σ_2 refines Σ_1, then we have a birational morphism*

$$\overline{\mathbb{D}/\Gamma}^{\Sigma_2} \to \overline{\mathbb{D}/\Gamma}^{\Sigma_1}.$$

Example 5.4.
- $\overline{\mathbb{D}/\Gamma}^{BB} = \overline{\mathbb{D}/\Gamma}^{\Sigma}$ where Σ is the admissible decomposition of $C(\mathbb{D})$ induced by the cones $C_{I,+}$ (so there are no subdivisions).

- Toroidal compactifications can also be recovered as $\overline{\mathbb{D}/\Gamma}^{\Sigma}$ if $\Sigma|_{C_{I,+}}$ is a fan for all I $\mathbb{Q}$-isotropic lines.

Observation 5.5. Semitoric compactifications are birational modifications of $\overline{\mathbb{D}/\Gamma}^{BB}$.

Example 5.6. Another class of examples of semitoric compactifications comes from hyperplane arrangements. Let $\mathscr{H} = \{H_i\}_i$ be a collection of hyperplanes in V of signature $(2, n-1)$ (these determine nonempty hyperplane sections $\mathbb{D} \cap \mathbb{P}(H_i)$). Then, for any given $\mathbb{Q}$-isotropic line I, the hyperplanes $H_i \supset I$ determine a decomposition of $C_{I,+}$. Therefore, for an appropriate choice of $\mathscr{H}$, we can obtain an admissible decomposition $\Sigma(\mathscr{H})$ of $C(\mathbb{D})$, hence a semitoric compactification $\overline{\mathbb{D}/\Gamma}^{\Sigma(\mathscr{H})}$.

Remark 5.7. Conclude with how this connects to my research.

References

[L] Looijenga, E.: *Compactifications defined by arrangements, II: locally symmetric varieties of type IV*. Duke Math. J. 119 (2003), no. 3, 527–588.