Geometry of Enriques Surfaces.

§ Introduction.

An *algebraic surface* will be a smooth proj. connected 2-dim alg. var. $/\mathbb{C}$.

Castelnuovo's question (1895). S algebraic surf. with $q(S) := h^1(S, \mathcal{O}_S) = 0$ and $P_g(S) := h^0(S, \omega_S) = 0$. Then, is S rational? (i.e. birational to $\mathbb{P}^2$)

Enriques' answer (1896). No. We will soon discuss Enriques' example, which gave rise to what was later called an Enriques surface.

Rmks.

(a) Examples of Enriques surfaces also appeared in earlier work of Reye (1882).

(b) The correct rationality criterion is the following:

Thm (Castelnuovo, 1896). If $q(S) = 0$ and $P_2(S) := h^0(S, \omega_S^{\otimes 2}) = 0$, then S is rational.

(c) In fact, as we will see, an Enriques surface satisfies $P_2 = 1 \neq 0$.

Goal of this course.

1. Introduce Enriques surfaces, discuss examples and basic techniques to study them.
2. Projective realizations of Enriques surfaces and the non-degeneracy invariant.

(3) Moduli of Enriques surfaces and compactifica-
tions from the points of view of Hodge theory
and the minimal model program.

§ Enriques' example.

We discuss Enriques' counterexample to Castelnuovo's question. Along the way, we discuss some of the fundamental tools for the study of algebraic surfaces.

Consider $\mathbb{P}^3$ with coordinates $[X_0 : X_1 : X_2 : X_3]$.

__Def.__ Let $\overline{S} \subseteq \mathbb{P}^3$ be the surface given by the vanishing of

$$f(X_0, X_1, X_2, X_3) = \alpha X_0^2 X_1^2 X_2^2 + \beta X_0^2 X_1^2 X_3^2 + \gamma X_0^2 X_2^2 X_3^2 + \delta X_1^2 X_2^2 X_3^2 + X_0 X_1 X_2 X_3 q(X_0, X_1, X_2, X_3),$$

where q is a quadric and the coefficients are generic. This is called an __Enriques sextic.__

__Rmk.__ We can rescale $\alpha, \beta, \gamma, \delta$ to be 1 after the following change of coordinates:

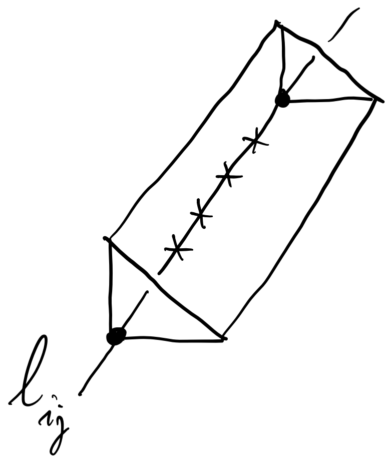
$$\begin{cases} X_0 = \sqrt{\delta}\, X_0' \\ X_1 = \sqrt{\gamma}\, X_1' \\ X_2 = \sqrt{\beta}\, X_2' \\ X_3 = \sqrt{\alpha}\, X_3' \end{cases}$$

and rescaling the equation by $\frac{1}{\alpha\beta\gamma\delta}$.

So we will work with Enriques sextics in the form:

$$f(X_0, X_1, X_2, X_3) = X_0^2 X_1^2 X_2^2 + X_0^2 X_1^2 X_3^2 + X_0^2 X_2^2 X_3^2 + X_1^2 X_2^2 X_3^2 + X_0 X_1 X_2 X_3 q(X_0, X_1, X_2, X_3).$$

Lemma. Let l_{ij} be the line $x_i = x_j = 0$. Then, $\overline{S}$ is singular precisely at $\bigcup_{i<j} l_{ij}$. Moreover, $\forall i \neq j$, along l_{ij} the surface $\overline{S}$ has 2 triple pts $(uvw=0)$, 4 pinch pts $(u^2 - v^2 w = 0)$, and away from these double crossing points $(uv=0)$.



Proof. By studying the partial derivatives of f one can see that the singular locus is precisely $\bigcup_{i<j} l_{ij}$. We leave this as an exercise and instead we focus on identifying these singularities.

By the symmetries, it is enough to consider the line l_{01}, which has parametric form $x_0 = 0$, $x_1 = 0$, $x_2 = \lambda$, $x_3 = \mu$. To study the sing's of $\overline{S}$ along l_{01}, we fix $[\lambda : \mu] \in \mathbb{P}^1$ and look at the lowest degree part of $f(x_0, x_1, \lambda, \mu)$, where the variables are x_0, x_1:

$$x_0^2 x_1^2 \lambda^2 + x_0^2 x_1^2 \mu^2 + x_0^2 \lambda^2 \mu^2 + x_1^2 \lambda^2 \mu^2 + x_0 x_1 \lambda \mu \, q(x_0, x_1, \lambda, \mu).$$

The lowest degree part is: • a, b, c are appropriate coefficients of q.

$$\lambda^2\mu^2 x_0^2 + \lambda\mu\left(a\lambda^2 + b\lambda\mu + c\mu^2\right) x_0 x_1 + \lambda^2\mu^2 x_1^2$$

If the discriminant of this degree 2 homogeneous polynomial in x_0, x_1 is nonzero, then we have a double crossing singularity at $[0:0:\lambda:\mu]$.

Now consider the points where the discriminant vanishes:

$$\lambda^2\mu^2\left(a\lambda^2 + b\lambda\mu + c\mu^2\right)^2 - 4\lambda^4\mu^4 = 0$$

$$\Longleftrightarrow \quad \lambda^2\mu^2\left(\left(a\lambda^2 + b\lambda\mu + c\mu^2\right)^2 - 4\lambda^2\mu^2\right) = 0$$

$\lambda = 0$ yields the triple point $[0:0:0:1]$: the lowest degree part of $f(x_0, x_1, x_2, 1)$ is $x_0 x_1 x_2$.

Similarly, $\mu = 0$ yelds the triple point $[0:0:1:0]$.

The 4 solutions to $\left(a\lambda^2 + b\lambda\mu + c\mu^2\right)^2 - 4\lambda^2\mu^2 = 0$ yield the 4 pinch pts because the lowest degree part is a square and has degree 2:

$$\lambda^2\mu^2 x_0^2 + \lambda\mu\left(a\lambda^2 + b\lambda\mu + c\mu^2\right) x_0 x_1 + \lambda^2\mu^2 x_1^2$$

$$\Longrightarrow \quad \lambda^2\mu^2 x_0^2 \pm 2\lambda^2\mu^2 x_0 x_1 + \lambda^2\mu^2 x_1^2$$

$$\Longrightarrow \quad \left(\lambda\mu x_0 \pm \lambda\mu x_1\right)^2. \qquad \square$$

As all the singularities of $\bar{S}$ can be resolved via normalization, we have the following corollary.

<u>Corollary</u>. The normalization $\nu : S \to \bar{S}$ is a smooth surface.

S is Enriques' example: we show it satisfies $q = p_g = 0$ and that it is not rational. For this purpose we need a few preliminaries.

Lemma.

(a) $\forall\, t \in \overline{S}$ triple point, $|\nu^{-1}(t)| = 3$;

(b) $\forall\, p \in \overline{S}$ pinch point, $|\nu^{-1}(p)| = 1$;

(c) $\forall\, d \in \overline{S}$ double crossing point, $|\nu^{-1}(d)| = 2$;

(d) Let $L := \sum_{i<j} l_{ij}$. Then $E := \nu^{-1}(L)$ is a configuration of 6 genus 1 curves $E = \sum_{i<j} e_{ij}$ with $\nu(e_{ij}) = l_{ij}$ as shown below: