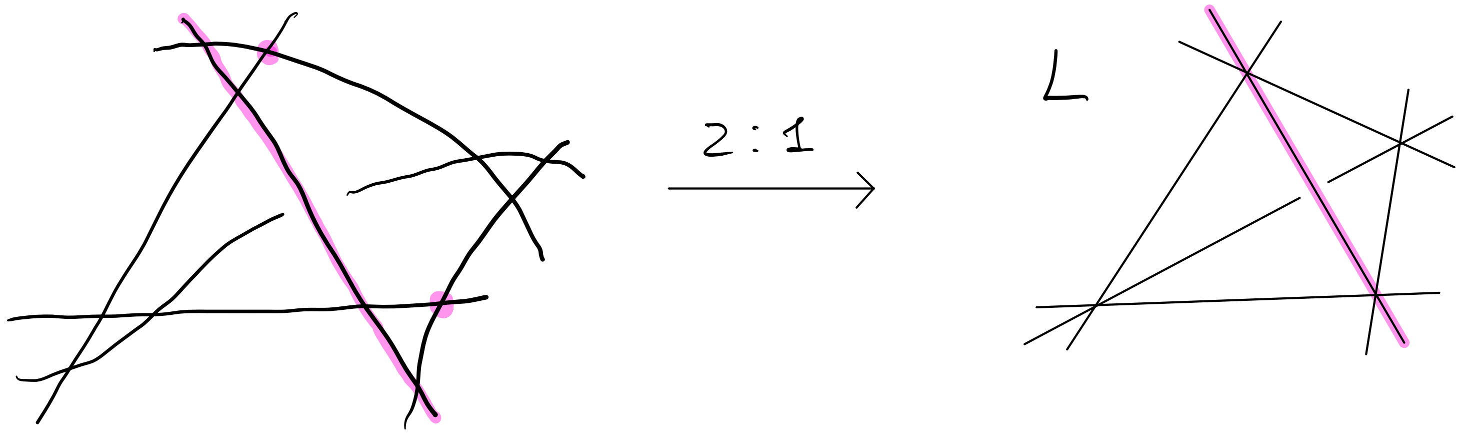
*Lemma.*

(a) $\forall t \in \overline{S}$ triple point, $|\nu^{-1}(t)| = 3$;

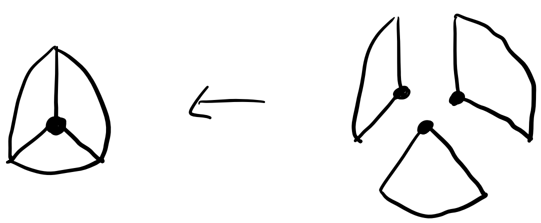
(b) $\forall p \in \overline{S}$ pinch point, $|\nu^{-1}(p)| = 1$;

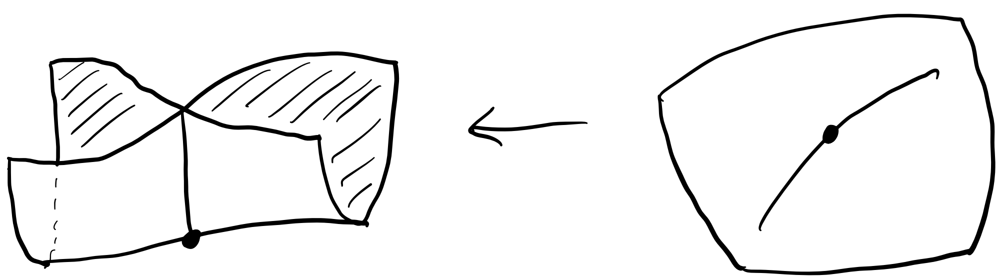
(c) $\forall d \in \overline{S}$ double crossing point, $|\nu^{-1}(d)| = 2$;

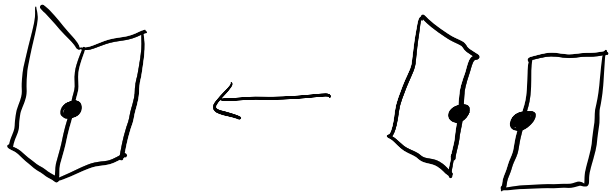
(d) Let $L := \sum_{i<j} l_{ij}$. Then $E := \nu^{-1}(L)$ is a configuration of 6 genus 1 curves $E = \sum_{i<j} e_{ij}$ with $\nu(e_{ij}) = l_{ij}$ as shown below:




Proof. (a), (b), (c) can be checked locally by computing explicitly the normalizations in each case (left as exercise). Qualitatively, the following gives an intuition of what is happening:

(a) 

(b) 

(c) 

(d) $\forall i<j$, $\nu^{-1}(l_{ij}) = e_{ij} \sqcup \{p\} \sqcup \{q\}$, where $e_{ij}$ is the double cover of $l_{ij} \cong \mathbb{P}^1$ branched at the 4 pinch points (hence, it is a genus 1 curve by Hurwitz's formula), and $\nu(p), \nu(q)$ are the two triple points on $l_{ij}$. $\square$

__Lemma.__ The Weil divisor $4L$ on $\overline{S}$ is Cartier and $\nu^*(4L) = 2E$.

__Proof.__ Let $H_0 = \{x_0 = 0\} \subseteq \mathbb{P}^3$, which contains the lines $l_{01}, l_{02}, l_{03}$. Then

$$H_0|_{\overline{S}} = 2l_{01} + 2l_{02} + 2l_{03},$$

because $f(0, x_1, x_2, x_3) = x_1^2 x_2^2 x_3^2$. Similarly for $H_1, H_2, H_3$. Then we have that

$$\underbrace{(H_0 + H_1 + H_2 + H_3)|_{\overline{S}}}_{\text{Cartier}} = 4l_{01} + 4l_{02} + 4l_{03} + 4l_{12} + 4l_{13} + 4l_{23} = 4L$$

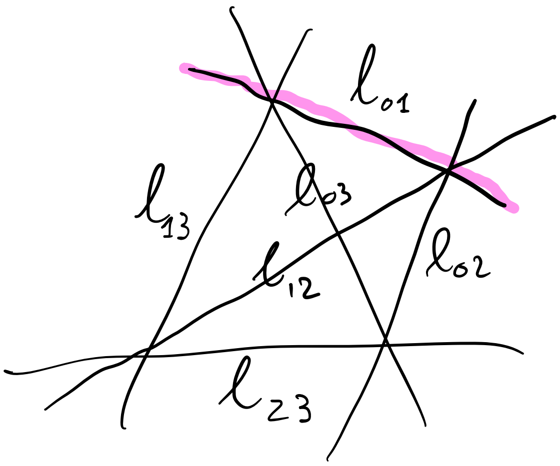
So, $4L$ is Cartier. To prove the last claim, we have that

$$\nu^*(4L) = mE, \quad \exists m \in \mathbb{Z}.$$

To determine $m$, we intersect both sides with the curve $e_{01}+e_{02}+e_{03}$ :

$$mE\cdot(e_{01}+e_{02}+e_{03}) = m3E\cdot e_{01}$$

It will be later proved that $2e_{01}$ is a fiber of an elliptic fibration. Hence, $(2e_{01})^2=0$ $\Rightarrow e_{01}^2=0$.




$$= m3\Big(\overset{0}{e_{01}^2} + e_{01}\overset{1}{\cdot} e_{02} + e_{01}\overset{1}{\cdot} e_{03} + e_{01}\overset{1}{\cdot} e_{12} + e_{01}\overset{1}{\cdot} e_{13}\Big)$$

$$= 12m$$

projection formula

$$\nu^*(4L)\cdot(e_{01}+e_{02}+e_{03}) = 4L\cdot\nu_*(e_{01}+e_{02}+e_{03})$$

definition of push-forward

$$= 4L\cdot(2l_{01}+2l_{02}+2l_{03})$$

$$= 4L\cdot H_0\big|_{\bar S}$$

$$= (H_0+\dots+H_3)\big|_{\bar S}\cdot H_0\big|_{\bar S}$$

$$= (H_0+\dots+H_3)\cdot H_0\cdot \bar S$$

$H\subseteq \mathbb{P}^3$ plane

$$= 4H\cdot H\cdot 6H = 24H^3 = 24$$

$\square$

Hence, $12m = 24 \Rightarrow m=2$.

<u>Thm.</u>

(a) $p_g(S)=0$;

(b) $S$ is not rational;

(c) $q(S)=0$.

Proof.

(a) The singularities of $\overline{S}$ allows to compute the canonical class of $S$ as:

$$K_S = \nu^*(K_{\overline{S}}) - E.$$

(See Kollár's "Singularities of the MMP", Section 5.1.)

$K_{\overline{S}}$ can be computed using the adjunction formula:

$$K_{\overline{S}} = \left(K_{\mathbb{P}^3} + \overline{S}\right)\Big|_{\overline{S}} = \left(-4H + 6H\right)\Big|_{\overline{S}} = 2H|_{\overline{S}}.$$

So, $|K_S| = |\nu^*(2H|_{\overline{S}}) - E|$ is the linear system of quadrics in $\mathbb{P}^3$ passing simply through $L$. As there are no such quadrics, $|K_S| = \emptyset$, which means that $p_g(S) = h^0(K_S) = 0$.

(b) $2K_S = \nu^*(4H|_{\overline{S}}) - 2E$

$\mathbb{P}(H^0(\mathcal{O})) = \mathbb{P}(\mathbb{C}) = pt$

$$\overset{\text{seen before}}{=} \nu^*(4L) - 2E$$

$$\overset{\text{Previous lemma}}{=} 2E - 2E = 0.$$

Recall that the Kodaira dimension of $S$, $\kappa(S)$, is defined as:

$\varphi_{|nK_S|}: S \dashrightarrow \mathbb{P}(H^0(nK_S))$

$$\kappa(S) = \max\left\{\dim \operatorname{im}\left(\varphi_{|nK_S|}\right) \,\middle|\, n \geq 1\right\}.$$

If $n \geq 1$ is odd, we have that

$$\dim \operatorname{im} \varphi_{|nK_S|} = \dim \operatorname{im} \varphi_{|K_S|}$$

$\mathbb{P}(H^0(K_S)) = \mathbb{P}(\emptyset) = \emptyset$

set by convention

$$= \dim \emptyset = -\infty$$

If $n \geq 1$ is even, we have that

$$\dim \operatorname{im} \varphi_{|nK_S|} = \dim \operatorname{im} \varphi_{|0|}$$

$\mathbb{P}(H^0(0)) = \mathbb{P}(H^0(\mathcal{O}_S)) = \mathbb{P}(\mathbb{C}) = pt$

$$= \dim pt = 0$$

So, $\kappa(S) = 0$. Hence $S$ is not rational (rational surfaces have Kodaira dimension $-\infty$).