$$\dim \operatorname{im} \varphi_{|nK_S|} = \dim \operatorname{im} \varphi_{|K_S|}$$

$$= \dim \emptyset \overset{\text{set by convention}}{=} -\infty$$

$\mathbb{P}(H^0(K_S)) = \mathbb{P}(\emptyset) = \emptyset$

If $n \geq 1$ is even, we have that

$$\dim \operatorname{im} \varphi_{|nK_S|} = \dim \operatorname{im} \varphi_{|0|}$$

$$= \dim pt = 0$$

$\mathbb{P}(H^0(0)) = \mathbb{P}(H^0(\mathcal{O}_S)) = \mathbb{P}(\mathbb{C}) = pt$

So, $\kappa(S) = 0$. Hence S is not rational (rational surfaces have Kodaira dimension $-\infty$).

(c) From Noether's formula, we have that

$$\chi(\mathcal{O}_S) = \frac{\chi_{top}(S) + \underbrace{K_S^2}_{=0}}{12}$$

$=0$ because $2K_S = 0$

$$= \frac{b_0 - b_1 + b_2 - b_3 + b_4}{12}$$

Betti numbers of S.

$$\overset{\text{Poincaré duality}}{=} \frac{2b_0 - 2b_1 + b_2}{12}$$

$$= \frac{2 - 2b_1 + b_2}{12}$$

$$\overset{\text{Beauville Fact III.19}}{=} \frac{2 - 2(2q) + b_2}{12}$$

$$\overset{\text{Part (a)}}{\Rightarrow} 1 - q + \underbrace{p_g}_{=0} = \frac{2 - 4q + b_2}{12}$$

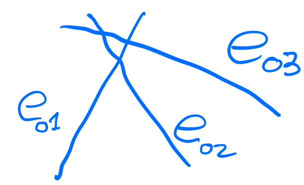
$$\Rightarrow 12 - 12q = 2 - 4q + b_2$$

$$\Rightarrow 8q = 10 - b_2 \Rightarrow q = \frac{10-b_2}{8}.$$

If we show that $b_2 \geq 3$, then $q < 1$, which implies that $q = 0$.

To show that $b_2 \geq 3$, consider the homology classes: e_{01}, e_{02}, e_{03}. These are independent in $H_2(S, \mathbb{Z})$. To prove this, suppose that $\exists\, a, b, c \in \mathbb{Z}$ s.t. $a e_{01} + b e_{02} + c e_{03} = 0$ in $H^2(S, \mathbb{Z})$. By intersecting with e_{01}, we obtain that

$$a e_{01}^2 + b e_{02} \cdot e_{01} + c e_{03} \cdot e_{01} = 0$$



$$\Rightarrow b + c = 0.$$

By intersecting also with e_{02} and e_{03}, we obtain that

$$\begin{cases} \quad b + c = 0 \\ a \quad + c = 0 \\ a + b \quad = 0, \end{cases}$$

whose only solution is $a = b = c = 0$. Hence, e_{01}, e_{02}, e_{03} are independent in $H^2(S, \mathbb{Z})$. $\square$

§ Enriques surfaces: general definition.

<u>Def.</u> An <u>Enriques surface</u> is a surface Y (smooth, conn, proj, 2-dim, alg. var.) such that $2K_Y \sim 0$ and $p_g(Y) = q(Y) = 0$.

<u>Ex.</u> Enriques' example S, normalization of the sextic $\bar{S} \subseteq \mathbb{P}^3$, is (... of course!) an example of Enriques surface, as we have shown that $2K_S \sim 0$ and $p_g(S) = q(S) = 0$.

As we will prove, any Enriques surface Y is not simply-connected. In particular, the universal cover $X \to Y$ is a different type of surface. Such universal covers are examples of K3 surfaces and many of the basic properties of Y can be understood from the geometry of X. So, we focus for a bit on K3 surfaces.

§ K3 surfaces.

Def. A K3 surface is a surface X (smooth, conn, proj, 2-dim, alg. var.) such that $K_X \sim 0$ (so that $p_g(X) = 1$) and $q(X) = 0$.

Ex. $X \subseteq \mathbb{P}^3$ smooth quartic hypersurface.

$$K_X \overset{\text{adjunction formula}}{=} \left(K_{\mathbb{P}^3} + X\right)\Big|_X \sim \left(-4H + 4H\right)\Big|_X = 0.$$

To show $q(X) = 0$, the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Then, in cohomology we have the long exact sequence

The cohomology $H^i(X, \mathcal{O}_{\mathbb{P}^r}(d))$ is well known. see Hartshorne Chapter III, Thm 5.1.

$$\cdots \to \underbrace{H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})}_{=0} \to H^1(X, \mathcal{O}_X) \to \underbrace{H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4))}_{=0} \to \cdots$$

Hence, $H^1(X, \mathcal{O}_X) \cong 0$. So, $q(X) = 0$.

Lemma. Let S be a surface with $K_S \equiv 0$ (numerically equivalent to 0). Then S is minimal: $\nexists\, C \subseteq S$, $C \cong \mathbb{P}^1$, s.t. $C^2 = -1$ (these are called (-1)-curves).

Proof. Homework. **Hint.** You have to use the so-called $\boxed{\text{genus formula}}$: If $C \subseteq S$ is a curve, then

$$\underbrace{g(C)}_{\text{geometric genus of } C} = 1 + \frac{C^2 + C \cdot K_S}{2}.$$

Prop. Let X be a K3 surface. Then, X is minimal, $\kappa(X) = 0$, $b_1(X) = 0$, $b_2(X) = 22$, $h^{1,1}(X) = 20$, and $\pi_1(X) = \{1\}$.

Proof. $K_X \sim 0 \Rightarrow K_X \equiv 0$, so minimality follows from the previous lemma. $\kappa(X) = 0$ follows from the definition of Kodaira dimension and the fact that $K_X \sim 0$. (Exercise.)

$$b_1(X) \underset{\substack{\uparrow \\ \text{Beauville's} \\ \text{Fact III.19}}}{=} 2q(X) = 0.$$

By Noether's formula, we have that

$$\underbrace{1 - q + p_g}_{2} = \chi(\mathcal{O}_X) = \frac{\chi_{top}(X) + K_X^2}{12} = \frac{\chi_{top}(X)}{12}$$

$$\Rightarrow \chi_{top}(X) = 24 \Rightarrow 2b_0 - \cancelto{0}{2b_1} + b_2 = 24$$

$$\Rightarrow 2 + b_2 = 24 \Rightarrow b_2 = 22.$$

The Hodge decomposition gives that

$$H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

$$H^{2,0}(X) \cong H^0(\Lambda^2 \Omega_X) = H^0(\omega_X) \cong \mathbb{C}$$

$$H^{0,2}(X) = \overline{H^{2,0}(X)} \cong \mathbb{C}$$

So, $H^{1,1}(X)$ is 20 dimensional.

Finally, all K3 surfaces are diffeomorphic as differentiable 4-dimensional manifolds. So, X is diffeomorphic to a smooth quartic $X_4 \subseteq \mathbb{P}^3$. Hence, $\pi_1(X) \cong \pi_1(X_4)$. To prove that $\pi_1(X_4) \cong \{1\}$, we can use the Lefschetz hyperplane theorem. To apply it, consider

$$X_4 \subseteq \mathbb{P}^3 \xrightarrow[v]{|\mathcal{O}(4)|} \mathbb{P}^{\binom{4+3}{3}-1} \supseteq H$$

veronese deg 4

Roughly, the idea is to realize S_4 as a hyperplane section of something simply connected.

$$\Rightarrow v(X_4) = v(\mathbb{P}^3) \cap H. \text{ Then,}$$

$$\pi_1(X_4) \cong \pi_1(v(X_4)) = \pi_1(v(\mathbb{P}^3) \cap H)$$

Lefschetz hyperplane theorem

$$\cong \pi_1(v(\mathbb{P}^3))$$

$$\cong \pi_1(\mathbb{P}^3) \cong \{1\}. \quad \square$$