$$H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

$$H^{2,0}(X) \cong H^0(\Lambda^2 \Omega_X) = H^0(\omega_X) \cong \mathbb{C}$$

$$H^{0,2}(X) = \overline{H^{2,0}(X)} \cong \mathbb{C}$$

So, $H^{1,1}(X)$ is 20 dimensional.

Finally, all K3 surfaces are diffeomorphic as differentiable 4-dimensional manifolds. So, $X$ is diffeomorphic to a smooth quartic $X_4 \subseteq \mathbb{P}^3$. Hence, $\pi_1(X) \cong \pi_1(X_4)$. To prove that $\pi_1(X_4) \cong \{1\}$, we can use the Lefschetz hyperplane theorem. To apply it, consider

$$X_4 \subseteq \mathbb{P}^3 \xrightarrow[\substack{v \\ \text{veronese deg } 4}]{|\mathcal{O}(4)|} \mathbb{P}^{\binom{4+3}{3}-1} \supseteq H$$

Roughly, the idea is to realize $S_4$ as a hyperplane section of something simply connected.

$\Rightarrow \quad v(X_4) = v(\mathbb{P}^3) \cap H$. Then,

$$\pi_1(X_4) \cong \pi_1(v(X_4)) = \pi_1(v(\mathbb{P}^3) \cap H)$$

$$\overset{\text{Lefschetz hyperplane theorem}}{\cong} \pi_1(v(\mathbb{P}^3))$$

$$\cong \pi_1(\mathbb{P}^3) \cong \{1\}. \qquad \square$$

**Rmk.** For a smooth proj alg. var $X$, recall

$\mathrm{Pic}(X)$ (line bundles $/\cong$) $\cong Cl(X) :=$ Divisors / linear equivalence

$NS(X) =$ Divisors / algebraic equivalence

$\mathrm{Num}(X) =$ Divisors / numerical equivalence

There are natural surjections:

$$\mathrm{Pic}(X) \to NS(X) \to \mathrm{Num}(X).$$

See Huybrechts' book on K3s, Ch 1, Prop 2.4

For a (proj) K3 surface, the above are isomorphisms. We will see, this will not be the case for Enriques surfaces.

The final goal for this section is to prove:

<u>Thm</u>. Let $X$ be a K3 surface. Then the lattice $(H^2(X,\mathbb{Z}), \cup)$ is isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$.

To prove this, we have to review a few fundamentals in lattice theory.

For the cup product "$\cup$" on $H^2(X,\mathbb{Z})$, see Hatcher's book, §3.2.

## § Fundamentals of lattice theory, I.

Def.

(1) A *lattice* is a pair $(L, b_L)$ where $L$ is a f.g. free $\mathbb{Z}$-module and $b_L : L \times L \to \mathbb{Z}$ is a symmetric bilinear form. We usually denote a lattice only by its underlying $\mathbb{Z}$-module $L$.

(2) A lattice $L' \subseteq L$ is a *sublattice of $L$* if

$$b_{L'} = b_L|_{L' \times L'}.$$

(3) Two lattices $L_1, L_2$ are *isometric* if $\exists$ a $\mathbb{Z}$-modules isomorphism $\varphi : L_1 \to L_2$ such that

$$b_{L_2}(\varphi(v), \varphi(w)) = b_{L_1}(v, w), \ \forall v, w \in L_1.$$

$\varphi$ is called an *isometry*.

Examples.

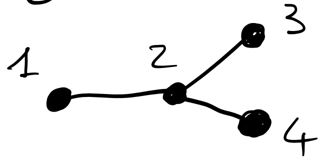
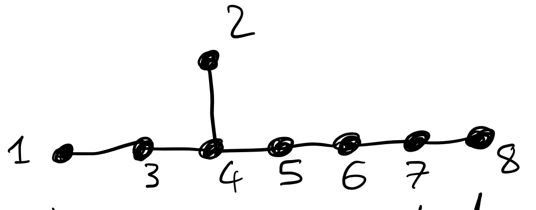
(1) Let $p, q \in \mathbb{Z}_{\geq 0}$. We denote by $I_{p,q}$ the lattice $\mathbb{Z}^{p+q}$ with symm. bil. form given by the matrix

$$\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right),$$

where $I_r$ denotes the identity matrix of size $r \times r$ and $0$ zero matrices of appropriate sizes.

(2) $U := \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$. It is called the <u>hyperbolic plane</u>.

(3) For any Dynkin diagram $A_n$ $(n \geq 1)$, $D_n$ $(n \geq 4)$, $E_n$ $(n = 6, 7, 8)$ we can associate a lattice by considering $\mathbb{Z}^n$ together with the bil. form given by the incidence matrix of the graph, but with $-2$'s along the diagonal. For example:

| Symbol | Dynkin diagram | Lattice |
| --- | --- | --- |
| $D_4$ |  | $\left(\mathbb{Z}^4, \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}\right)$ |
| $E_8$ |  (Bourbaki's notation) | $\left(\mathbb{Z}^8, \begin{pmatrix} -2 & 0 & 1 & 0 & \dots & 0 \\ & & & \vdots & & \\ & & & & & \end{pmatrix}\right)$ |

**Def.** $L$ lattice, $\{e_1, \dots, e_n\}$ $\mathbb{Z}$-basis for $L$. Then, the Gram matrix $L$ associated to $\{e_1, \dots, e_n\}$ is the $n \times n$ matrix $(b_L(e_i, e_j))_{1 \le i, j \le n}$.

**Def.** Let $L$ be a lattice and $M$ a Gram matrix for $L$. Then, we define the rank $rk(L)$, discriminant $discr(L)$, and signature $(p, q)$ to be resp. the rank, determinant, and signature of $M$. These do not depend on the choice of $M$: if $N$ is another Gram matrix, then $N = B^t M B$ for some integral invertible matrix $B$ (so, $\det(B) = \pm 1$).

positive negative

**Def:** A lattice $L$ is called

- Nondegenerate if $discr(L) \neq 0$;
- Unimodular if $discr(L) = \pm 1$;
- Indefinite if $p, q \geq 1$;
- Even if $\forall v \in L, b_L(v, v) \in 2\mathbb{Z}$;
- Odd if $\exists v \in L$ s.t. $b_L(v, v) \notin 2\mathbb{Z}$.

Examples.

- $I_{p,q}$ is odd and unimodular;
- $U$ is even, unimodular, of signature $(1,1)$;
- $E_8$ is even, unimodular, of signature $(0,8)$;

- The ADE lattices are even and negative definite. $E_8$ is the only unimodular one.

<u>Def.</u>

- Let $L = (\mathbb{Z}^n, b_L)$ be a lattice and $m \in \mathbb{Z}$. Then, $L(m) := (\mathbb{Z}^n, m b_L)$.
- Let $L_1 = (\mathbb{Z}^{n_1}, b_{L_1}), L_2 = (\mathbb{Z}^{n_2}, b_{L_2})$. Then $L_1 \oplus L_2 := (\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2}, b_{L_1} \oplus b_{L_2})$, where

$$b_{L_1} \oplus b_{L_2}\big((v_1, v_2), (w_1, w_2)\big)$$

$$:= b_{L_1}(v_1, w_1) + b_{L_2}(v_2, w_2).$$

<u>Thm</u> (Milnor). Let $L$ be a unimodular lattice of signature $(p,q)$. The following hold:

(i) If $L$ is odd, then $L \cong I_{p,q}$;

(ii) If $L$ is even and indefinite, then

$$L \cong \begin{cases} U^{\oplus p} \oplus E_8^{\oplus (q-p)/8} & \text{if } p < q \\ U^{\oplus p} & \text{if } p = q \\ U^{\oplus q} \oplus E_8(-1)^{\oplus (p-q)/8} & \text{if } p > q. \end{cases}$$

Examples.

(1) Let $L$ be a even unimodular lattice of signature $(1,9)$. Then $L \cong U \oplus E_8$.

(2) Let $L$ be a even unimodular lattice of signature $(3,19)$. Then $L \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$.

# § Fundamentals of lattice theory, II.

<u>Rmk.</u> Nondegenerate lattices $L = (\mathbb{Z}^n, b_L)$ are quite convenient. First, $\mathrm{rk}(L) = n$. Moreover, $L$ naturally injects into the dual lattice $L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ as follows:

$$\begin{aligned} L &\hookrightarrow L^* \\ v &\longmapsto b_L(v, -). \end{aligned}$$

For these reasons, unless otherwise stated, we will consider nondegenerate lattices.

<u>Def.</u> The <u>discriminant group</u> of a lattice $L$ is the abelian group $A_L := L^*/L$.

<u>Rmk.</u> When computing $A_L$, the following identification is useful:

$$L^* \cong \{ v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b_L(v, w) \in \mathbb{Z} \ \forall w \in L \}.$$

Another useful tool is provided by the following.