§ Fundamentals of lattice theory, II.

Rmk. Nondegenerate lattices $L = (\mathbb{Z}^n, b_L)$ are quite convenient. First, $rk(L) = n$. Moreover, L naturally injects into the dual lattice $L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ as follows:

$$\begin{array}{ccc} L & \hookrightarrow & L^* \\ v & \longmapsto & b_L(v, -). \end{array}$$

For these reasons, unless otherwise stated, we will consider nondegenerate lattices.

Def. The discriminant group of a lattice L is the abelian group $A_L := L^*/L$.

Rmk. When computing A_L, the following identification is useful:

$$L^* \cong \{ v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b_L(v, w) \in \mathbb{Z} \ \forall w \in L \}.$$

Another useful tool is provided by the following.

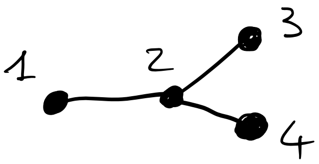
Thm. Let L be a lattice. Then

(i) $|A_L| = |\mathrm{discr}(L)|$;

(ii) If $L' \subseteq L$ is a sublattice s.t. $rk(L') = rk(L)$, then

$$|L/L'|^2 = \mathrm{discr}(L') / \mathrm{discr}(L).$$

Ex. Let $L = D_4$. As $\operatorname{discr}(D_4) = 4$, we have that $A_L \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$. $L = \langle e_1, e_2, e_3, e_4 \rangle$ with



. Note that $v := \frac{1}{2}(e_1 + e_3) \in L^*$ as $v \cdot e_i \in \mathbb{Z}$ $\forall i = 1, \dots, 4$. $w := \frac{1}{2}(e_1 + e_4) \in L^*$.

As $|\langle v+L, w+L \rangle| = 4$, $A_L = \langle v+L, w+L \rangle$ and $A_L \cong \mathbb{Z}_2^2$.

§ Back to K3 surfaces

Thm. Let X be a K3 surface. Then $(H^2(X,\mathbb{Z}),\smile)$ is a lattice isometric to $U^{\oplus 3}\oplus E_8^{\oplus 2}$.

Proof. By the universal coefficient theorem for cohomology (see Hatcher, Thm 3.2) we have that

$$H^2(X,\mathbb{Z})\cong H_2(X,\mathbb{Z})^*\oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{Z}).$$

Recall how to compute $\mathrm{Ext}^i(G,\mathbb{Z})$:

- Proj. resol. $\dots\to G_1\to G_0\to G\to 0$
- $0\to \mathrm{Hom}(G_0,\mathbb{Z})\to \mathrm{Hom}(G_1,\mathbb{Z})\to\dots$
- Cohomology at i-th position gives $\mathrm{Ext}^i(G,\mathbb{Z})$.

$H_1(X,\mathbb{Z})\cong \mathrm{Ab}(\pi_1(X))\cong 0$

$\Rightarrow \mathrm{Ext}^1(H_1(S,\mathbb{Z}),\mathbb{Z})\cong 0$

$\Rightarrow H^2(X,\mathbb{Z})\cong H_2(X,\mathbb{Z})^*$

As the right hand-side is torsion-free and $b_2(X)=22$, we have that $H^2(X,\mathbb{Z})\cong\mathbb{Z}^{22}$ as $\mathbb{Z}$-module.

The lattice $(H^2(X,\mathbb{Z}),\smile)$ is:

- Unimodular:

$$H^2(X,\mathbb{Z})\underset{\text{seen before}}{\cong} H_2(X,\mathbb{Z})^*\underset{\text{Poincaré duality}}{\cong} H^2(X,\mathbb{Z})^*$$
$$v\longmapsto \varphi_v,\ \text{where } \varphi_v(w)=v.w.$$

So $H^2(X,\mathbb{Z})$ is nondegenerate: let $v\in H^2(X,\mathbb{Z})$ s.t. $v.w=0\ \forall w\in H^2(X,\mathbb{Z})$. Then $\varphi_v=0$, which implies that $v=0$. So, $H^2(X,\mathbb{Z})\hookrightarrow H^2(X,\mathbb{Z})^*$. But since the discriminant group $H^2(X,\mathbb{Z})^*/H^2(X,\mathbb{Z})$ is trivial, we have that $H^2(X,\mathbb{Z})$ is unimodular.

see Milnor-Stasheff's book: Characteristic classes

- Even by Wu's formula: let $w_2(T_X) \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ be the second Stiefel-Whitney class of the tangent bundle of X. Then, $\forall v \in H^2(X,\mathbb{Z})$, Wu's formula says that

$$v^2 \equiv w_2(T_X) \cdot v \pmod 2.$$

At the same time, $w_2(T_X) \equiv c_1(T_X) \pmod 2$.
Moreover, for a K3 surface X, $c_1(T_X) = 0$.
So, $v^2 \equiv 0 \pmod 2$.

So, the thesis follows by Milnor's theorem if we prove that the lattice has signature $(p,q)=(3,19)$.
The Hodge-Riemann bilinear relations imply that

$$p-q = \sum_{\substack{i+j \\ \text{even}}} (-1)^i h^{i,j} = h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2}$$
$$= 1 + 1 - 20 + 1 + 1 = -16,$$

$$p+q = 22 \Rightarrow (p,q) = (3,19).$$ $\square$

§ Enriques surfaces: general properties.

Recall the definition.

Def. An <u>Enriques surface</u> is a surface Y (smooth, conn, proj, 2-dim, alg. var.) such that $2K_Y \sim 0$ and $p_g(Y) = q(Y) = 0$.

Prop. Let Y be an Enriques surface. Then Y is minimal, $\kappa(Y) = 0$, $K_Y \not\sim 0$, $b_1(Y) = 0$, $b_2(Y) = 10$, and $h^{1,1}(Y) = 10$.

Proof. $2K_Y \sim 0 \Rightarrow K_Y \equiv 0 \Rightarrow S$ is minimal. $\kappa(S) = 0$ follows from the definition. $K_Y \not\sim 0$ becouse, otherwise, $p_g(Y) = 1$. $b_1(Y) = 2q(Y) = 0$. Noether's formula gives:

$$1 = \chi(\mathcal{O}_Y) = \frac{\chi_{top}(Y) + K_Y^2}{12} = \frac{2\overset{=1}{b_0} - \cancel{2b_1}^{\,0} + b_2}{12}$$

$$\Rightarrow b_2 = 12 - 2 = 10.$$

The Hodge decomposition gives that

$$H^2(Y, \mathbb{C}) \cong H^{2,0}(Y) \oplus H^{1,1}(Y) \oplus H^{0,2}(Y)$$

$$H^{2,0}(Y) \cong H^0(\Lambda^2 \Omega_Y) = H^0(\omega_Y) \cong 0$$

$$H^{0,2}(Y) = \overline{H^{2,0}(Y)} \cong 0$$

So, $H^{1,1}(Y)$ is 10 dimensional. $\square$

Covers induced by torsion line bundles.

Recall that a torsion line bundle $\mathcal{L}$ on an algebraic variety Y s.t. $\mathcal{L}^{\otimes n} \cong \mathcal{O}_Y$ with $n \in \mathbb{Z}_{>0}$ minimal, corresponds to an étale degree n cover $\pi : X \to Y$ such that $\pi^*\mathcal{L} \cong \mathcal{O}_X$. This goes as follows.

Let $\alpha : L^{\otimes n} \xrightarrow{\cong} Y \times \mathbb{C}$, L line bundle corresponding to $\mathcal{L}$, $p : L \to Y$ projection map. Define

Using that L is locally trivial: $U \times \mathbb{C}$

$$\begin{tikzcd} X := \{(y,v) \in L \mid \alpha(y, v^{\otimes n}) = (y,1)\} \arrow[r, phantom, "\subseteq"] \arrow[dr, bend right, "\pi = p|_X"'] & L \arrow[d, "p"] \\ & Y. \end{tikzcd}$$

Then $\pi : X \to Y$ is étale of degree n.

$\pi^* L = X \times_Y L$ is trivial because it has a nowhere vanishing section given by the diagonal $\Delta \subseteq X \times_Y X \subseteq X \times_Y L$.

Finally, the degree n étale cover $\pi : X \to Y$ is characterized by the fact that $\pi^*\mathcal{L} \cong \mathcal{O}_X$.