Prop.

(a) Let Y be an Enriques surface and let $\pi : X \to Y$ be the double cover corresponding to the 2-torsion sheaf ω_Y. Then, X is a K3 surface, it is the universal cover, and $\pi_1(Y) \cong \mathbb{Z}_2$.

(b) Conversely, the quotient $X/\iota = Y$ of a K3 surface X by a fixed-point-free involution ι is an Enriques surface.

Proof.

(a) We need to show that $K_X \sim 0$ and $q(X) = 0$.

$$\omega_X \cong \pi^* \omega_Y \cong \mathcal{O}_X \Rightarrow K_X \sim 0.$$

canonical sheaf for an étale cover.

π is the étale cover associated to ω_Y.

$$\chi_{top}(X) = 2\chi_{top}(Y) = 24$$

Noether's formula

$$\Rightarrow \chi(\mathcal{O}_X) = \frac{\chi_{top}(X) + K_X^2}{12} = \frac{24+0}{12} = 2$$

$$\chi(\mathcal{O}_X) = 1 - q + \overset{1}{\cancel{p_g}}$$

$\Rightarrow q(X) = 0$. So, X is a K3 surface.

(b) We need to show that $2K_Y \sim 0$ and $p_g(Y) = q(Y) = 0$.

Let $\pi : X \to Y$ be the quotient map.

$\pi^* \omega_Y = \omega_X = \mathcal{O}_X \Rightarrow \pi$ is the étale cover associated to ω_Y, and since π has degree 2, we have that $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$, hence $2K_Y \sim 0$.

canonical sheaf for an étale cover

X is K3

$\chi_{top}(Y) = \frac{1}{2}\chi_{top}(X) = 12 \Rightarrow \chi(\mathcal{O}_Y) = 1 \Rightarrow p_g(Y) = q(Y)$.

To prove that $q(Y) = 0$, we use the theory of Albanese varieties (see Beauville, Chapter V)

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d] \\ \mathrm{Alb}(X) \arrow[r, "\mathrm{Alb}(\pi)"] & \mathrm{Alb}(Y) \end{tikzcd}$$

Functoriality. Rmk V.14 (2)

Rmk V.14 (1)

$\dim \mathrm{Alb}(X) = q(X) = 0,$

$\dim \mathrm{Alb}(Y) = q(Y),$

Rmk V.14 (3)

π surjective $\Rightarrow \mathrm{Alb}(\pi)$ surjective.

So, $\mathrm{Alb}(Y) = pt$, hence $q(Y) = 0$. $\square$

Prop. Let Y be an Enriques surface. Then:

(a) $\mathrm{Pic}(Y) \cong NS(Y) \cong H^2(Y, \mathbb{Z})$;

(b) $\mathrm{Num}(Y) \ncong \mathrm{Pic}(Y)$.

Proof. Consider the long exact sequence in cohomology associated to the exponential short exact sequence:

$$0 \to \mathbb{Z}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$$

$$\Rightarrow \ldots \to H^1(Y,\mathcal{O}_Y) \to H^1(Y,\mathcal{O}_Y^*) \to H^2(Y,\mathbb{Z}) \to H^2(Y,\mathcal{O}_Y) \to \ldots$$

$$\Rightarrow \quad 0 \to \mathrm{Pic}(Y) \xrightarrow{\cong} H^2(Y,\mathbb{Z}) \to 0,$$

$$\begin{tikzcd} \mathrm{Pic}(Y) \arrow[r, "\cong"] \arrow[dr, "c_1"'] & H^2(Y,\mathbb{Z}) \\ & NS(Y) \arrow[u] \end{tikzcd}$$

from which (a) follows.

(b) $\mathrm{Num}(Y)$ is torsion-free, while $\mathrm{Pic}(Y)$ has the 2-torsion element ω_S. □

Thm. Let Y be an Enriques surface. Then the torsion part T of $H^2(Y,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The quotient $H^2(Y,\mathbb{Z})_f := H^2(Y,\mathbb{Z})/T$ endowed with the cup product is a lattice isometric to $U \oplus E_8$, which is called the Enriques lattice.

Proof. By the universal coefficient theorem for cohomology, we have that

torsion free. torsion may only come from here.

$$H^2(Y,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(Y,\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(Y,\mathbb{Z}),\mathbb{Z})$$

$$H_1(Y,\mathbb{Z}) \cong Ab(\pi_1(Y)) \cong Ab(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_1(Y,\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Since $b_2(Y) = 10 \Rightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(Y,\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}^{10}$.

Hence $H^2(Y,\mathbb{Z}) \cong \mathbb{Z}^{10} \oplus (\mathbb{Z}/2\mathbb{Z})$.

$H^2(Y,\mathbb{Z})_f$, endowed with the cup product, is a non-degenerate lattice of rank 10. Let us show it is even. Let $[C] \in H^2(Y,\mathbb{Z})_f = H^2(Y,\mathbb{Z})/T$.

Since $H^2(Y,\mathbb{Z}) \cong \mathrm{Pic}(Y)$, then up to changing the representative of $[C]$ we can assume that C is algebraic. So we can apply the Riemann-Roch formula and obtain that:

$$\chi(\mathcal{O}_Y(C)) = \underbrace{\chi(\mathcal{O}_Y)}_{=0} + \frac{C^2 - \cancelto{0}{K_Y \cdot C}}{2}$$

$\Rightarrow C^2 = 2\chi(\mathcal{O}_Y(C))$, which is even.

If (p,q) is the signature of $H^2(Y,\mathbb{Z})_f$, then the Hodge-Riemann bilinear relations give that:

$$p - q = \sum_{\substack{i+j \\ \text{even}}} (-1)^i h^{i,j} = h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2}$$
$$= 1 + 0 - 10 + 0 + 1 = -8.$$

Since $p+q = 10$, we argue that $(p,q) = (1,9)$.

Then, Milnor's theorem gives $H^2(Y,\mathbb{Z}) \cong U \oplus E_8$. □

<u>Remark</u>: $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ can be computed as follows.
Consider the left-exact contravariant functor $\mathrm{Hom}(-,\mathbb{Z})$.
Consider the proj. res. given by $0 \to \mathbb{Z} \xrightarrow{m_2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, where m_2 is multiplication by 2. Then, we remove $\mathbb{Z}/2\mathbb{Z}$ and apply the functor:

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{m_2^*} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0$$

$$\Rightarrow \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) / \mathrm{Im}(m_2^*) \cong \mathbb{Z}/2\mathbb{Z}.$$

§Elliptic fibrations on Enriques surfaces: definition and an example.

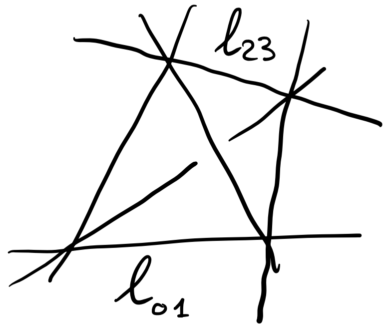
Def. On an alg. surf. Y, an *elliptic fibration* is a morphism $f : Y \to \mathbb{P}^1$ whose general fiber is a smooth elliptic curve. This is equivalent to the data of an *elliptic pencil* on Y, that is a base point free linear system $|D|$ of dim 1 (hence, a pencil) whose generic member is an elliptic curve.

Rmk. By saying elliptic fibration, we do not assume that $f : Y \to \mathbb{P}^1$ has a section, i.e., a morphism $\sigma : \mathbb{P}^1 \to Y$ s.t. $f \circ \sigma = id_{\mathbb{P}^1}$.

It turns out that an Enriques surface always admits an elliptic pencil (proving it is non-trivial). Moreover, elliptic pencils have a specific geometric structure we will describe. We start by discussing an example.

Ex. Consider the example of Enriques surface S given by the normaliz. $\nu : S \longrightarrow \overline{S}$ of an Enriques' sextic $\overline{S} \subseteq \mathbb{P}^3$.

Consider two skew lines $l_{01}, l_{01} \subseteq \mathbb{P}^3$ of the coordinate tetrahedron.



Consider the quadrics that pass through the remaining 4 lines: $l_{02}, l_{03}, l_{12}, l_{13}$.

$$\sum_{0 \le i \le j \le 3} a_{ij} X_i X_j = 0 \quad \text{general quadric.}$$

Containing $\ell_{02} = V(X_0, X_2)$ is equivalent to $a_{11}, a_{13}, a_{33} = 0$.

So the quadrics are

$$a_{01} X_0 X_1 + a_{23} X_2 X_3 = 0,$$

which form a pencil. Let $Q_{[\lambda:\mu]} := V(\lambda X_0 X_1 + \mu X_2 X_3)$

The scheme theoretic intersection $\overline{S} \cap Q_{[\lambda:\mu]}$ can be described as follows:

$$\overline{S} \cap Q_{[\lambda:\mu]} = 2\ell_{02} + 2\ell_{12} + 2\ell_{03} + 2\ell_{13} + \overline{C}_{[\lambda:\mu]},$$

elliptic curve

$$\overline{C}_{[\lambda:\mu]} = \begin{cases} \text{smooth intersection of two quadrics} & \text{if } \lambda, \mu \neq 0 \\ 4\,\ell_{23} & \text{if } \lambda = 0 \\ 4\,\ell_{01} & \text{if } \mu = 0. \end{cases}$$