$$\sum_{0\le i\le j\le 3} a_{ij} X_i X_j = 0 \quad \text{general quadric.}$$

Containing $l_{02} = V(X_0, X_2)$ is equivalent to $a_{11}, a_{13}, a_{33} = 0$.

So the quadrics are

$$a_{01} X_0 X_1 + a_{23} X_2 X_3 = 0,$$

which form a pencil. Let $Q_{[\lambda:\mu]} := V(\lambda X_0 X_1 + \mu X_2 X_3)$

The scheme theoretic intersection $\overline{S} \cap Q_{[\lambda:\mu]}$ can be described as follows:

$$\overline{S} \cap Q_{[\lambda:\mu]} = 2 l_{02} + 2 l_{12} + 2 l_{03} + 2 l_{13} + \overline{C}_{[\lambda:\mu]},$$

elliptic curve

$$\overline{C}_{[\lambda:\mu]} = \begin{cases} \text{smooth intersection of two quadrics} & \text{if } \lambda, \mu \neq 0 \\ 4\, l_{23} & \text{if } \lambda = 0 \\ 4\, l_{01} & \text{if } \mu = 0. \end{cases}$$

So, $\overline{C}_{[\lambda:\mu]}$ describes an elliptic pencil on $\overline{S}$.

$$C_{[\lambda:\mu]} := \nu^* \overline{C}_{[\lambda:\mu]} = \begin{cases} \cong \overline{C}_{[\lambda:\mu]} & \text{if } \lambda, \mu \neq 0 \\ 2\, e_{23} & \text{if } \lambda = 0 \\ 2\, e_{01} & \text{if } \mu = 0 \end{cases}$$

So, $C_{[\lambda:\mu]}$ is an elliptic pencil on the Enriques surface S (this explains why $e_{01}^2 = 0$, as $2e_{01}$ is the fiber of a fibration).

Also, we note two multiple fibers of multiplicity 2. Namely, $2e_{01}$ and $2e_{23}$.

§ Elliptic fibrations on Enriques surfaces: half-fibers.

Any elliptic fibration on an Enriques surface always has exactly two multiple fibers, and these multiplicities equal 2. We now prove this.

<u>Lemma</u>. Let $\Delta \subseteq \mathbb{C}$ be the unit disk centered at the origin. Let $\pi : X \to \Delta$ be a fibration. Assume that X_0 is a multiple fiber, i.e. $X_0 = nF$, $\exists\, n \geq 2$. Then, the line bundles $\mathcal{O}_X(F)$ and $\mathcal{O}_X(F)|_F$ are both torsion of order n.

<u>Proof</u>. We only observe that $\mathcal{O}_X(F)$ and $\mathcal{O}_X(F)|_F$ are torsion. For the rest, we refer to BHPVdV, Chapter II, Lemma 12.2.

$z : \Delta \longrightarrow \mathbb{C}$ be the inclusion. Then $z \circ \pi$ is a regular function on X s.t. $\operatorname{div}(z\circ\pi) = X_0 = nF$.

In particular $nF \sim 0$. Hence

$$\mathcal{O}_X(F)^{\otimes n} \cong \mathcal{O}_X(nF) \cong \mathcal{O}_X \Rightarrow \mathcal{O}_X(F) \text{ is torsion.}$$

Then,

$$\mathcal{O}_X(F)|_F^{\otimes n} \cong \mathcal{O}_X(F)^{\otimes n}\Big|_F \cong \mathcal{O}_X|_F \cong \mathcal{O}_F. \qquad \square$$

Restriction to F is pullback w.r.t. $F \hookrightarrow X$ and pullback is a homomorphism.

<u>Thm.</u> Let $f: Y \to \mathbb{P}^1$ be an elliptic pencil on an Enriques surface. Then

(1) f has exactly 2 multiple fibers, mF, $m'F'$, with F, F' curves on Y and $m, m' \in \mathbb{Z}_{\geq 2}$.

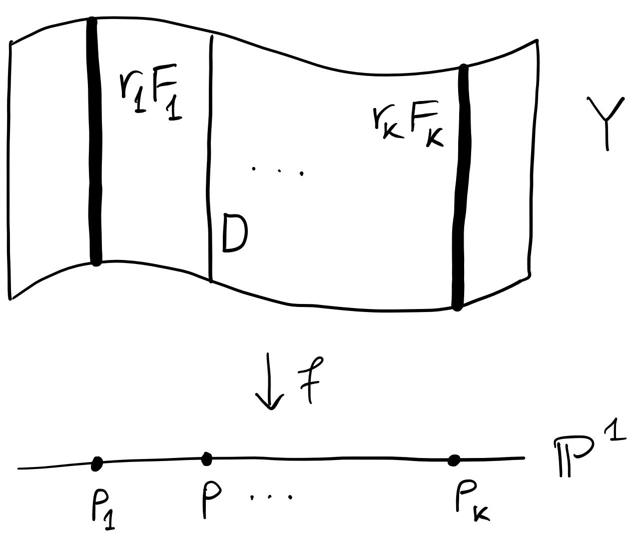
(2) $m = m' = 2$.

(3) $K_Y \sim F - F'$.

<u>Proof.</u> Let $r_1F_1, \dots, r_kF_k$ be the multiple fibers of f with $r_i \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{Z}_{\geq 0}$. The canonical bundle formula for an elliptic fibration is:

$(\star)$ $\omega_Y \cong \mathcal{O}_Y(-D) \otimes \mathcal{O}_Y\left(\sum_{i=1}^{k} (r_i - 1)F_i\right)$, where D is a general fiber.

(see BHPVdV, Coroll V.12.3)



Then,

$$\mathcal{O}_Y \cong \omega_Y^{\otimes 2} \cong \mathcal{O}_Y(-2D) \otimes \mathcal{O}_Y\left(\sum_{i=1}^{k} (2r_i-2) F_i\right). \quad (\star\star)$$

$\forall i$, restricting to F_i we obtain that

$$\mathcal{O}_{F_i} = \mathcal{O}_Y|_{F_i} \cong \mathcal{O}_Y((2r_i-2)F_i)\Big|_{F_i} \cong \mathcal{O}_Y(F_i)\Big|_{F_i}^{\otimes(2r_i-2)}.$$

From the previous lemma, we have that $\mathcal{O}_Y(F_i)|_{F_i}$ is torsion of order r_i, hence $r_i \mid 2r_i - 2$.

This implies that $r_i \mid 2$, which combined with $r_i \geq 2$ gives that $r_i = 2$. So, multiple fibers have multiplicity 2. Moreover, from $(\star\star)$ we obtain that

$$\begin{aligned} \mathcal{O}_Y &\cong \mathcal{O}_Y(-2D) \otimes \mathcal{O}_Y\left(\sum_{i=1}^{k} 2F_i\right) \\ &\underset{\forall i,\, 2F_i \sim D}{\cong} \mathcal{O}_Y(-2D) \otimes \mathcal{O}_Y(kD) \\ &\cong \mathcal{O}_Y((k-2)D). \end{aligned}$$

If $k-2 \neq 0$, then $(k-2)D \sim 0 \Rightarrow \exists h \in k(Y)$ s.t. has exactly one zero of mult. $(k-2)$ along D if $k-2>0$, or exactly one pole of mult. $2-k$ along D if $k-2<0$. However, $h: Y \to \mathbb{P}^1$ is surjective, so it must have both zeros and poles, which cannot be. So, $k-2=0 \Rightarrow k=2$. So, we have exactly 2 multiple fibers. This proves (1) and (2).

For (3), let $F := F_1$, $F' := F_2$. From $(\star)$ we have that

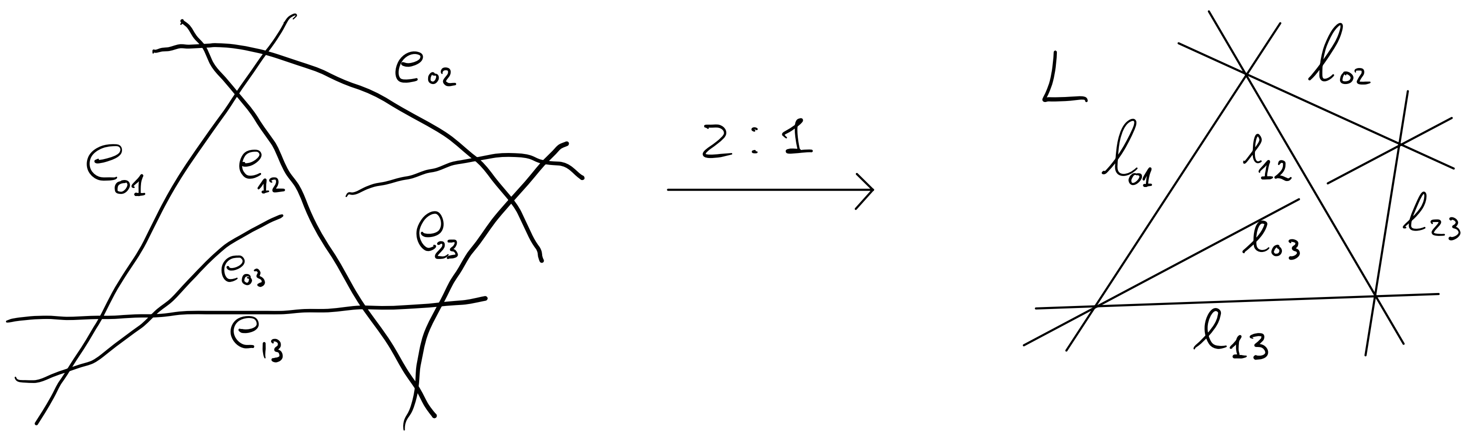
$$\omega_Y \cong \mathcal{O}_Y(-D)\otimes \mathcal{O}_Y(F+F')$$

$$\underset{D\sim 2F'}{\cong} \mathcal{O}_Y(F-F') \Rightarrow K_Y \sim F-F'. \qquad \square$$

<u>Def.</u> Let $f: Y \to \mathbb{P}^1$ be an elliptic fibration on an Enriques surface and let $2F, 2F'$ be its two multiple fibers. Then, F and F' are called <u>half-fibers</u> of the elliptic pencil.

<u>Ex.</u> Consider again the example given by the normalization of an Enriques' sextic $\nu: S \to \overline{S}$. Then we saw that the elliptic curves e_{ij}, $0\le i<j\le 3$, are half-fibers of elliptic fibrations.

More precisely:



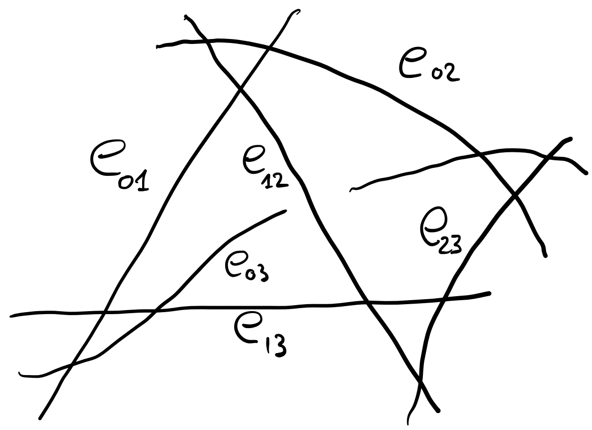
So, given $i<j$, $k<l$, $\{i,j,k,l\}=\{0,1,2,3\}$, then e_{ij}, e_{kl} are the two half-fibers of the same elliptic fibration.

Rmk. Let $f: Y \to \mathbb{P}^1$ be an elliptic fibration on an Enriques surface. Then f does not admit a section. To prove this, suppose $\exists\, \sigma: \mathbb{P}^1 \to Y$ s.t. $f \circ \sigma = id_{\mathbb{P}^1}$. Then, $S := \sigma(\mathbb{P}^1)$ is a curve on Y intersecting each fiber in one point transversely.
Let D be a generic fiber of f and let F, F' be the half-fibers of f. Then, $1 = D \cdot S = (2F) \cdot S$ $= 2(F \cdot S)$, which cannot be.

§ The non-degeneracy invariant of Enriques surfaces.

Def. Let Y be an Enriques surface. The non-degeneracy invariant of Y, denoted $nd(Y)$, is defined as the maximum m s.t. $\exists$ half-fibers $F_1, \dots, F_m$ with the property that $F_i \cdot F_j = 1 - \delta_{ij}$, where δ_{ij} is the Kroneker delta.

Ex. Consider again the example given by the normalization of an Enriques' sextic $\nu : S \to \overline{S}$. Consider the half-fibers e_{ij}, $1 \leq i < j \leq 3$:



Consider $F_1 := e_{01}$, $F_2 := e_{02}$, $F_3 := e_{12}$. Then $F_i \cdot F_j = 1 - \delta_{ij}$, so $nd(S) \geq 3$.

Prop. Let Y be an Enriques surface. Then, we have that $nd(Y) \leq 10$.

Proof. Homework.

Thm.

Let Y be an Enriques surface (over $\mathbb{C}$, as we always assumed so far). Then, we have that:

(1) $nd(Y) \geq 1$. This amounts to say that an Enriques surface always has an elliptic fibration.
(2) $nd(Y) \geq 3$ (Cossec, 1985).

(3) $nd(Y) \geq 4$ (Martin-Mezzedimi-Veniani, 2022).

The non-degeneracy invariant $nd(Y)$ gives information about the projective realizations of Y. An instance of this is illustrated by the next theorem.

<u>Thm</u>. Let Y be an Enriques surface and suppose that $F_1, \dots, F_{10}$ are half-fibers such that $F_i \cdot F_j = 1 - \delta_{ij}$ $\forall i,j$ (in particular, $nd(Y) = 10$). Then, the following hold:

1) $F_1 + \dots + F_{10}$ is divisible by 3 in $Pic(Y)$, i.e. $\exists\, \Delta \in Pic(Y)$ s.t. $F_1 + \dots + F_{10} = 3\Delta$;
2) $\Delta^2 = 10$;
3) Δ is a very ample divisor;
4) The linear system $|\Delta|$ induces an embedding $Y \xhookrightarrow{|\Delta|} \mathbb{P}^5$ realizing Y as the intersection of 10 cubic hypersurfaces.

<u>Rmk</u>.
① A polarization Δ like above is called an ample Fano polarization.

② If $nd(Y) < 10$, then in general one can construct a nef polarization L which may not be ample and induces a map to projective space s.t. the image has some singularities.

③ $nd(Y)$ also regulates the structure of the bounded derived category of the Enriques surface Y.

④ Given an Enriques surface Y, it is in general a hard question to compute $nd(Y)$.
(Depending on time, discuss known cases.)