UMass, Reading Seminar in Algebraic Geometry, Homological mirror symmetry for elliptic curves

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The main reference for what follows is Polishchuck-Zaslow's paper [PZ98]. Recall the following general setup:

- $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$.
- $E_\tau := \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$.
- $E^\tau := \mathbb{R}^2/\mathbb{Z}^2$ with complexified Kähler class ω such that $\tau = \int_{E^\tau} \omega$. This is the mirror of E_τ.
- $q := e^{2\pi i \tau}$.
- $E_q := \mathbb{C}^*/x \sim qx$. Notice that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{e^{2\pi i -}} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ E_\tau & \xrightarrow{\cong} & E_q, \end{array}$$

  where the bottom isomorphism is given by $[z] \mapsto [e^{2\pi i z}]$.
- If $\varphi \colon \mathbb{C}^* \to \mathbb{C}^*$ is a holomorphic function, $\mathscr{L}_q(\varphi) = \mathbb{C}^* \times \mathbb{C}/(x, y) \sim (qx, \varphi(x)y)$ is a line bundle on E_q. Sometimes, we simply denote it by $\mathscr{L}(\varphi)$.
- Let $\varphi_0(z) = e^{-\pi i \tau - 2\pi i z}$. Recall $\mathscr{L}(\varphi_0)$ is our chosen degree 1 line bundle.
- Every line bundle on E_τ is isomorphic to $\mathscr{L}(t_x^* \varphi_0 \cdot \varphi_0^{n-1})$ for an appropriate choice of $x \in E_\tau$ and $n \in \mathbb{Z}$

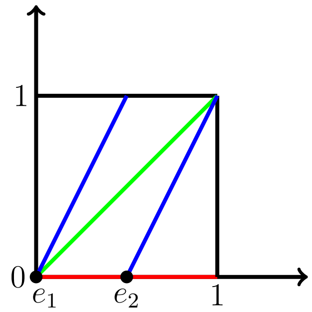

Figure 1: The special Lagrangians L_0 (in red), L_1 (in green), and L_2 (in blue).

- Finally, recall the following special case of the addition formula for theta functions:

$$\theta(\tau, z)\theta(\tau, z+x) = \theta(2\tau, x)\theta(2\tau, 2z+x) + \theta[1/2, 0](2\tau, x)\theta[1/2, 0](2\tau, 2z+x).$$

In the previous lecture, we defined $\Phi\colon \mathcal{D}^b(E_\tau) \to \mathcal{F}^0(E^\tau)$ on the subcategory of line bundles. Recall that Φ is bijective on Hom-sets and that

$$\Phi(\mathscr{L}(t^*_{\alpha\tau+\beta}\varphi_0 \cdot \varphi_0^{n-1})) = ((\alpha + t, (n-1)\alpha + nt), -2\pi i\beta dx).$$

Today, we check that Φ is compatible with composition of morphisms in a simple nontrivial example.

1 The simplest example

Let $\tau = iA$, $A \in \mathbb{R}_{>0}$. Let $\mathscr{L}_0 = \mathcal{O}, \mathscr{L}_1 = \mathscr{L}(\varphi_0), \mathscr{L}_2 = \mathscr{L}(\varphi_0)^2$. We have that

$$\begin{aligned}
\Phi(\mathscr{L}_0) &= \Phi(\mathscr{L}(t^*_{0\tau+0}\varphi_0 \cdot \varphi^{0-1})) = ((t, 0), 0),\\
\Phi(\mathscr{L}_1) &= \Phi(\mathscr{L}(t^*_{0\tau+0}\varphi_0 \cdot \varphi^{1-1})) = ((t, t), 0),\\
\Phi(\mathscr{L}_2) &= \Phi(\mathscr{L}(t^*_{0\tau+0}\varphi_0 \cdot \varphi^{2-1})) = ((t, 2t), 0).
\end{aligned}$$

These special Lagrangians with the trivial connection are pictured in Figure 1. What we want to check is that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(L_0, L_1) \times \mathrm{Hom}(L_1, L_2) & \xrightarrow{m_2} & \mathrm{Hom}(L_0, L_2)\\
\cong \downarrow & & \downarrow \cong\\
\mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1) \times \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) & \longrightarrow & \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_2).
\end{array}$$

Let us describe the Hom-sets involved. We have that:

- $\mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1) \cong H^0(\mathscr{L}(\varphi_0)) \cong \mathbb{C}\theta(\tau, z)$;

- $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) \cong H^0(\mathscr{L}(\varphi_0)) \cong \mathbb{C}\theta(\tau, z)$;
- $\mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_2) \cong H^0(\mathscr{L}(\varphi_0)^2) \cong \mathrm{Span}_{\mathbb{C}}\{\theta(2\tau, 2z), \theta[1/2, 0](2\tau, 2z)\}$;
- $\mathrm{Hom}(L_0, L_1) = \mathbb{C}e_1$;
- $\mathrm{Hom}(L_1, L_2) = \mathbb{C}e_1$;
- $\mathrm{Hom}(L_0, L_2) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$.

Let us describe the maps between Hom-sets:

$$\begin{aligned} \mathrm{Hom}(L_0, L_1) &\to \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1) \\ e_1 &\mapsto \theta(\tau, z), \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}(L_1, L_2) &\to \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) \\ e_1 &\mapsto \theta(\tau, z), \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}(L_0, L_2) &\to \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_2) \\ e_1 &\mapsto \theta(2\tau, 2z) \\ e_2 &\mapsto \theta[1/2, 0](2\tau, 2z). \end{aligned}$$

Next, we need to describe m_2, which is completely determined if we compute $m_2(e_1, e_1)$.

$$m_2(e_1, e_1) = C(e_1, e_1; e_1)e_1 + C(e_1, e_1; e_2)e_2,$$

where the coefficients are computed as follows:

$$C(e_1, e_1; e_1) = \sum_{[\phi]} e^{2\pi i \int \phi^* \omega},$$

where the sum run over all possible classes homolorphic maps from the closed unit disc to triangles bounded by L_0, L_1, L_2 with vertices given by the arguments of C. $C(e_1, e_1; e_2)$ is computed analogously. Note that $\int \phi^* \omega$ equals the area of the corresponding triangle multiplied by iA. We obtain that

$$C(e_1, e_1; e_1) = \sum_{n \in \mathbb{Z}} e^{-2\pi A n^2} = \theta(2\tau, 0), \ C(e_1, e_1; e_2) = \sum_{n \in \mathbb{Z}} e^{-2\pi A (n+1/2)^2} = \theta[1/2, 0](2\tau, 0).$$

Summarizing, the above diagram commutes provided the following equality of theta functions holds:

$$\theta(\tau, z)^2 = \theta(2\tau, 0)\theta(2\tau, 2z) + \theta[1/2, 0](2\tau, 0)\theta[1/2, 0](2\tau, 2z).$$

But this follows from the addition formula with $x = 0$.

2 Extending the definition of Φ

So far we defined the functor $\Phi\colon \mathcal{D}^b(E_\tau) \to \mathcal{F}^0(E^\tau)$ on line bundles. We now extend it to a special class of vector bundles. We start by observing first of all that all vector bundles on E_τ are obtained in the following way.

Definition 2.1. Let V be an r-dimensional complex vector space and let $A\colon \mathbb{C}^* \to \mathrm{GL}(V)$ be a holomorphic function. Define the following rank r vector bundle on E_q:

$$F_q(V, A) := \mathbb{C}^* \times V/(x, y) \sim (qx, A(x)y).$$

In the special case where $V = \mathbb{C}$ and $A = \varphi\colon \mathbb{C}^* \to \mathbb{C}^*$, the above vector bundle recovers the line bundle $\mathscr{L}(\varphi)$.

Definition 2.2. Let $\mathscr{L}(E_\tau)$ be the full subcategory of $\mathcal{D}^b(E_\tau)$ consisting of vector bundles in the form

$$\mathscr{L}(\varphi) \otimes F_q(V, e^N),$$

where V is a finite dimensional complex vector space, $N \in \mathrm{End}(V)$ a nilpotent endomorphism, and $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ a holomorphic map.

Proposition 2.3 ([PZ98, Proposition 2])**.** *Let V be a finite dimensional complex vector space, $N \in \mathrm{End}(V)$ a nilpotent endomorphism, and $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ a holomorphic map. Then*

$$H^0(\mathscr{L}(\varphi) \otimes F_q(V, e^N)) \cong H^0(\mathscr{L}(\varphi)) \otimes V.$$

The next definition extends Φ to $\mathscr{L}(E_\tau)$.

Definition 2.4. On objects,

$$\begin{aligned} \Phi\colon \mathscr{L}(E_\tau) &\to \mathcal{F}^0(E^\tau) \\ \mathscr{L}(t^*_{\alpha\tau+\beta}\varphi_0 \cdot \varphi_0^{n-1}) \otimes F_q(V, e^N) &\mapsto (\Lambda, A), \end{aligned}$$

where:

- $\Lambda = (\alpha + t, (n-1)\alpha + nt)$ is a line parametrized by t, and
- $A = (-2\pi i\beta \,\mathrm{id}_V + N)dx$ is a flat connection of the line.

On morphisms, consider

$$\mathrm{Ob}_1 = \mathscr{L}(t^*_{\alpha_1\tau+\beta_1}\varphi_0 \cdot \varphi_0^{n_1-1}) \otimes F_q(V_1, e^{N_1}) \to \mathscr{L}(t^*_{\alpha_2\tau+\beta_2}\varphi_0 \cdot \varphi_0^{n_2-1}) \otimes F_q(V_2, e^{N_2}) = \mathrm{Ob}_2\,.$$

Let $\Phi(\mathrm{Ob}_i) = (\Lambda_i, A_i)$, $i = 1, 2$. We want to define a morphism $(\Lambda_1, A_1) \to (\Lambda_2, A_2)$ in a functorial way. We have that

$$\mathrm{Hom}(\mathrm{Ob}_1, \mathrm{Ob}_2) \cong H^0(\mathscr{L}(t^*_{\alpha_2\tau+\beta_2}\varphi_0 \cdot \varphi_0^{n_2-1}) \otimes \mathscr{L}(t^*_{\alpha_1\tau+\beta_1}\varphi_0 \cdot \varphi_0^{n_1-1})^\vee \otimes F_q(V_2, e^{N_2}) \otimes F_q(V_1, e^{N_1})^\vee)$$

$$
\begin{aligned}
&\cong H^0(\mathscr{L}(t^*_{\alpha\tau+\beta}\varphi_0^{n_2-n_1}) \otimes F_q(V_1^\vee \otimes V_2, e^{\mathbf{1}\otimes N_2 - N_1^\vee \otimes \mathbf{1}}))\\
&\cong H^0(\mathscr{L}(t^*_{\alpha\tau+\beta}\varphi_0^{n_2-n_1})) \otimes V_1^\vee \otimes V_2\\
&= \mathrm{Span}_{\mathbb{C}}\{f_k = \theta[k/(n_2-n_1), 0]((n_2-n_1)\tau, (n_2-n_1)(z+\alpha\tau)\beta) \mid k \in \mathbb{Z}/(n_2-n_1)\mathbb{Z}\}\\
&\quad \otimes V_1^\vee \otimes V_2,
\end{aligned}
$$

where $\alpha = \frac{\alpha_2-\alpha_1}{n_2-n_1}, \beta = \frac{\beta_2-\beta_1}{n_2-n_1}$. Moreover,

$$
\mathrm{Hom}((\Lambda_1, A_1), (\Lambda_2, A_2)) = \bigoplus_{e_k \in \Lambda_1 \cap \Lambda_2} V_1^\vee \otimes V_2 \cdot e_k,
$$

where

$$
e_k = \left(\frac{k+\alpha_2-\alpha_1}{n_2-n_1}, \frac{n_1 k + n_1\alpha_2 - n_2\alpha_1}{n_2-n_1} \right), \; k \in \mathbb{Z}/(n_2-n_1)\mathbb{Z}.
$$

So at the level of morphisms, we define

$$
\Phi(f_k \otimes T) = e^{-\pi i \tau \alpha^2 (n_2-n_1)} e^{\alpha(N_2 - N_1^\vee - 2\pi i (n_2-n_1)\beta)} T e_k.
$$

The next step is to extend Φ to all vector bundles.

References

[PZ98] Alexander Polishchuk and Eric Zaslow. *Categorical mirror symmetry: the elliptic curve.* Adv. Theor. Math. Phys. 2 (1998), no. 2, 443–470.