§ Generalities about cones.

Def. A cone $\sigma \subseteq \mathbb{R}^n$ is a non-empty convex subset satisfying

1) $\forall u, v \in \sigma$, $u + v \in \sigma$,
2) $\forall u \in \sigma$ and $\forall r \in \mathbb{R}_{\geq 0}$, $ru \in \sigma$. (Note that $\bar{0} \in \sigma$ automatically)

Attributes of cones. A cone $\sigma \subseteq \mathbb{R}^n$ is

not pointed cone.

- Closed if it is closed in the Euclidean topology.
- pointed provided it contains no line through $\bar{0}$.
- rational polyhedral if

$$\sigma = \left\{ \sum_{i=1}^{m} r_i v_i \;\middle|\; r_i \in \mathbb{R}_{\geq 0} \right\},$$

for some $v_i \in \mathbb{Q}^n$. (This implies closed.)

- full-dimensional if σ spans $\mathbb{R}^n$.

§ Cones of effective k-cycles.

X irr. smooth proj. var. of dim n. Let $0 \le k \le n$.
Recall the group homomorphisms

$$A_k(X) \to B_k(X) \to N_k(X)$$

Recall that $A_k(X)$ may not be finitely generated (for instance $A_0(X)$ where X is a curve of positive genus).

- However, $N_k(X)$ is a finitely generated abelian group (Fulton, "Intersection Theory", Example 19.1.4)
- Moreover, $N_k(X)$ is torsion free: assume $\alpha = [\sum n_Y Y]$ in $N_k(X)$ satisfies $m\alpha = 0$ for some integer $m \geq 2$. Then $(n-k)$-cycle $B \in Z_{n-k}(X)$, $(\sum n_Y Y) \cdot B = \frac{1}{m}(m \sum n_Y Y) \cdot B = 0$, so $\sum n_Y Y$ is numerically trivial, which implies $\alpha = 0$.
  So $N_k(X) \cong \mathbb{Z}^r$ for some $r \geq 0$. In particular, $N_k(X) \otimes \mathbb{R} =: N_k(X)_{\mathbb{R}}$ is a finite dim. real vector space.
- Def. Let
- $$\mathrm{Eff}_k(X) = \Big\{ \sum_{\substack{Y \subseteq X \\ \text{closed} \\ \text{irr. } k\text{-dim} \\ \text{subvar.}}} a_Y Y \;\Big|\; a_Y \in \mathbb{R}_{\geq 0} \text{ all zero but finitely many of them} \Big\} \Big/ \sim_{num}$$

It is clear from the definition that $\mathrm{Eff}_k(X)$ is a cone. It is called the <u>cone of effective k-cycles</u>. The closure $\overline{\mathrm{Eff}_k}(X)$ is also a cone called the <u>cone of pseudo-effective k-cycles</u>.

Rmk

1) $\mathrm{Eff}_k(X)$ is full-dimensional, as it spans $N_k(X)_{\mathbb{R}}$ (immediate from definition). Hence also $\overline{\mathrm{Eff}_k}(X)$ is full-dimensional.

2) $\mathrm{Eff}_k(X)$ is pointed. By contradiction, assume $\exists\, \alpha \in \mathrm{Eff}_k(X)\setminus\{0\}$ s.t. $-\alpha \in \mathrm{Eff}_k(X)\setminus\{0\}$.
Let $D \in A_{n-1}(X)$ be an ample divisor on X.
Then $\alpha \cdot D^k > 0$ & $(-\alpha)\cdot D^k > 0$ since both α and $-\alpha$ are effective.
Hence $0 < \alpha \cdot D^k + (-\alpha)\cdot D^k = (\alpha - \alpha)\cdot D^k = 0 \cdot D^k = 0$, which cannot be.
Also $\overline{\mathrm{Eff}_k}(X)$ is pointed: see Fulger-Lehmann, "Positive cones of dual cycle classes", 2017.

Ex. Let $0 \leq k \leq n$, and let $L_k \subseteq \mathbb{P}^n$ be a k-dim linear subspace. Recall that $A_k(\mathbb{P}^n) = \mathbb{Z}[L_k]$. Note that $A_k(\mathbb{P}^n) = N_k(\mathbb{P}^n)$ as $\mathbb{P}^n$ is an HI space. So $N_k(\mathbb{P}^n)_{\mathbb{R}} = \mathbb{R}[L_k]$, and

$$\mathrm{Eff}_k(\mathbb{P}^n) = \mathbb{R}_{\geq 0}[L_k] = \overline{\mathrm{Eff}_k}(\mathbb{P}^n).$$

In what follows, we study more examples of $\mathrm{Eff}_k(X)$ with more intricate structure.

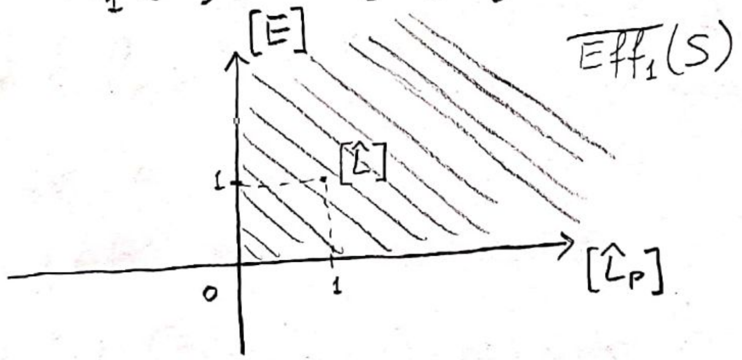
§ The cone of curves of $Bl_p \mathbb{P}^2$.

Let $p \in \mathbb{P}^2$. Let $\sigma : S \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at p.
Let $E \subseteq S$ be the exceptional divisor.
Let $L_p \subseteq \mathbb{P}^2$ be a line s.t. $p \in L_p$. $\hat{L}_p \subseteq S$ strict transform
We show that $\underbrace{\langle [\hat{L}_p], [E] \rangle}_{\text{cone generated by } E \text{ and } L_p} = \mathrm{Eff}_1(S)$.

Let $C \subseteq S$ be an irreducible curve which is not E.
Let $D := \sigma_* C$, which is a curve in $\mathbb{P}^2$ of a certain degree $d > 0$. So $D \sim d L_p$. Hence:

$$\sigma^* D = C + m_p(D) \cdot E$$
$$\wr$$
$$\sigma^* d L_p = d \sigma^* L_p = d(\hat{L}_p + E)$$

$\Rightarrow C \sim d \hat{L}_p + (d - m_p(D)) E$.
As $d \geq m_p(D)$, we have that $[C] \in \langle [\hat{L}_p], [E] \rangle$.
Hence, $\mathrm{Eff}_1(S) = \langle [\hat{L}_p], [E] \rangle = \overline{\mathrm{Eff}_1}(S)$.



Note:
$L \subseteq \mathbb{P}^2$ line, $p \notin L$. $L \sim L_p + E$.

Scansionato con CamScanner

§ Blow up of $\mathbb{P}^2$ at the 9 base points of a pencil of cubics.

To understand this we need a preliminary lemma.

Lemma. Let X be a smooth proj. surface, $C \subseteq X$ irreducible curve.

- If $C^2 \leq 0$, then $[C]$ is in the boundary of $\overline{\mathrm{Eff}}_1(X)$;
- If $C^2 < 0$, then $[C]$ is extremal in $\overline{\mathrm{Eff}}_1(X)$.

Proof. For a curve C, $l_C : N_1(X)_{\mathbb{R}} \to \mathbb{R}$ s.t.
$[D] \longmapsto C \cdot D$ defines a linear map, and
$\{l_C \geq 0\} \subseteq N_1(X)_{\mathbb{R}}$ is a half-space.

- If $C^2 = 0$, then $\overline{\mathrm{Eff}}_1(X) \subseteq \{l_C \geq 0\}$ and $[C]$ is in the boundary of $\overline{\mathrm{Eff}}_1(X)$.
- If $C^2 < 0$, then $\overline{\mathrm{Eff}}_1(X)$ is generated by $\mathbb{R}_{\geq 0}[C]$ and $\overline{\mathrm{Eff}}_1(X) \cap \{l_C \geq 0\}$.

As $[C] \notin \{l_C \geq 0\}$, $[C]$ is extremal in $\overline{\mathrm{Eff}}_1(X)$. □