§ Blow up of $\mathbb{P}^2$ at the 9 base points of a pencil of cubics.

To understand this we need a preliminary lemma.

Lemma. Let X be a smooth proj. surface, $C \subseteq X$ irreducible curve.

- If $C^2 \leq 0$, then $[C]$ is in the boundary of $\overline{\mathrm{Eff}_1}(X)$;
- If $C^2 < 0$, then $[C]$ is extremal in $\overline{\mathrm{Eff}_1}(X)$.

Proof. For a curve C, $l_C : N_1(X)_{\mathbb{R}} \to \mathbb{R}$ s.t. $[D] \longmapsto C \cdot D$ defines a linear map, and $\{l_C \geq 0\} \subseteq N_1(X)_{\mathbb{R}}$ is a half-space.

- If $C^2 = 0$, then $\overline{\mathrm{Eff}_1}(X) \subseteq \{l_C \geq 0\}$ and $[C]$ is in the boundary of $\overline{\mathrm{Eff}_1}(X)$.
- If $C^2 < 0$, then $\overline{\mathrm{Eff}_1}(X)$ is generated by $\mathbb{R}_{\geq 0}[C]$ and $\overline{\mathrm{Eff}_1}(X) \cap \{l_C \geq 0\}$.

As $[C] \notin \{l_C \geq 0\}$, $[C]$ is extremal in $\overline{\mathrm{Eff}_1}(X)$. $\square$

Let us now get to the example, which is due to Nagata.

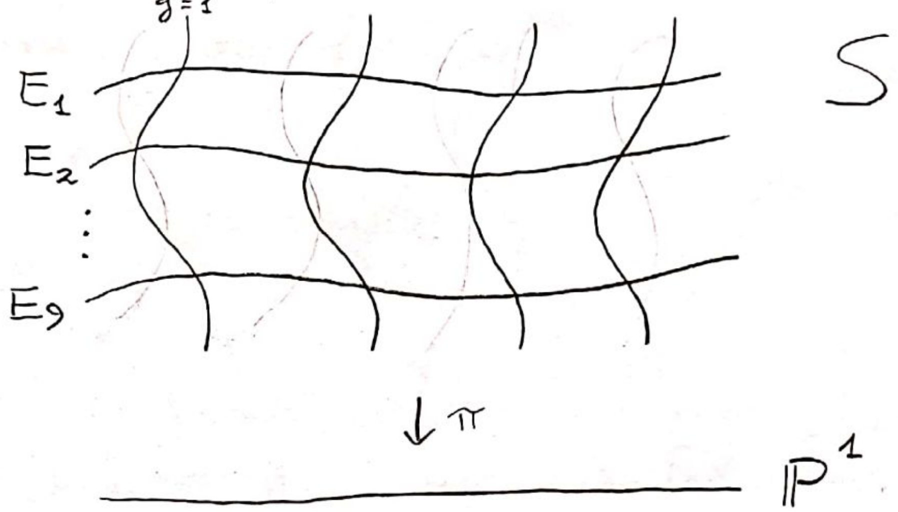
Let $[X:Y:Z] \in \mathbb{P}^2$ be coordinates.

Let $F(X,Y,Z)$, $G(X,Y,Z)$ be two homogeneous polynomials of deg 3 such that

- $V(F), V(G) \subseteq \mathbb{P}^2$ intersect exactly at 9 distinct points $P_1, \dots, P_9$.
- For all $[\lambda : \mu] \in \mathbb{P}^1$, $V(\lambda F + \mu G) \subseteq \mathbb{P}^2$ is an irreducible cubic curve.

This defines a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ which is undefined precisely at the points $P_1, \dots, P_9$.

Let $S = \mathrm{Bl}_{P_1, \dots, P_9} \mathbb{P}^2$. This resolves the indeterminacies of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, and we obtain genus 1 fibration $\pi: S \to \mathbb{P}^1$. If $E_1, \dots, E_9 \subseteq S$ are the exceptional divisors over $p_1, \dots, p_9$, these are sections of the fibration π.



The Mordell-Weil group of π is the group whose elements are the sections of π, and whose operation is fiberwisely the group law of the fibers. One has that

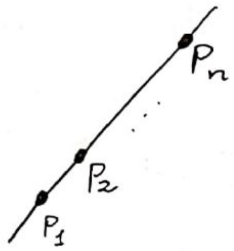
$$\operatorname{rank}(\operatorname{Pic}(S)) = \operatorname{rank}(MW(\pi)) + 2 + \sum_{\substack{F \subseteq S \\ \text{sing.} \\ \text{fiber}}} \left(\left(\# \begin{smallmatrix} \text{irr.} \\ \text{comp's} \\ \text{of } F \end{smallmatrix} \right) - 1 \right)$$

$\Rightarrow \operatorname{rank}(MW(\pi)) = 8$, so we have infinitely many sections of π, and these are all (-1)-curves. So $\overline{\mathrm{Eff}_1}(S)$ has infinitely many extremal rays. In particular, $\overline{\mathrm{Eff}_1}(S)$ is not rationally polyhedral.

One can show that $\overline{\mathrm{Eff}_1}(S)$ is generated by the (-1)-curves and $-K_S$. $\mathbb{R}_{\geq 0}(-K_S)$ is the only non-isolated extremal ray.

§ n-pointed smooth rational curves.

Def. An *n-pointed smooth rational curve* $(C, p_1, \dots, p_n)$ is a projective smooth rational curve C together with distinct points $p_1, \dots, p_n \in C$ called *marked points*.
We picture it as follows:



or simply by

Two n-pointed smooth rational curves $(C, p_1, \dots, p_n)$, $(C', p_1', \dots, p_n')$ are *isomorphic* provided there exists an isomorphism $\varphi: C \longrightarrow C'$ such that $\varphi(p_i) = p_i'$ $\forall i = 1, \dots, n$.

A *family* of n-pointed smooth rational curves is a flat proper morphism $\pi: \mathcal{X} \to B$ with n sections $\sigma_i: B \to \mathcal{X}$ such that for all $b \in B$, $(\pi^{-1}(b), \sigma_1(b), \dots, \sigma_n(b))$ is an n-pointed smooth rational curve.

Two families of n-pointed smooth rational curves $\pi: \mathcal{X} \underset{\sigma_i}{\leftrightarrows} B$, $\pi': \mathcal{X}' \underset{\sigma_i'}{\leftrightarrows} B$ are *isomorphic* provided there exists an isomorphism $\varphi: \mathcal{X} \to \mathcal{X}'$ giving a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathcal{X}' \arrow[dl, "\pi'"] \\
& B \arrow[ul, bend left, "\sigma_i"] \arrow[ur, bend right, "\sigma_i'"'] &
\end{tikzcd}$$

Thm. Let $n \geq 3$ and let $F : \underline{Sch}^{op} \to \underline{Set}$ be the functor for families of n-pointed smooth rational curves : for $B \in \underline{Sch}^{op}$, let

$$F(B) = \left\{ \mathcal{X} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} B \;\middle|\; \begin{matrix} \text{family of smooth } n\text{-pointed} \\ \text{rational curves} \end{matrix} \right\} / \text{iso}.$$

Then F is finely represented by

$$M_{0,n} = \left(\mathbb{P}^1 \setminus \{0,1,\infty\}\right)^{n-3} \setminus \text{diagonals}.$$

Rmk. The fact that $M_{0,n}$ is a fine moduli space is equivalent to the existence of a universal family $\mathcal{U}_{0,n} \underset{\mu_i}{\overset{\pi}{\rightleftarrows}} M_{0,n}$. This is given as follows. Let $\mathcal{U}_{0,n} = M_{0,n} \times \mathbb{P}^1$ and let $\pi : \mathcal{U}_{0,n} \to M_{0,n}$ be the projection. Define for $i = 1, \dots, n-3$,

$$\mu_i : M_{0,n} \longrightarrow \mathcal{U}_{0,n}$$
$$x = (x_1, \dots, x_{n-3}) \longmapsto (x, x_i),$$

$$\mu_{n-2} : M_{0,n} \longrightarrow \mathcal{U}_{0,n}$$
$$x \longmapsto (x, 0),$$

$$\mu_{n-1} : M_{0,n} \longrightarrow \mathcal{U}_{0,n}$$
$$x \longmapsto (x, 1),$$

$$\mu_n : M_{0,n} \longrightarrow \mathcal{U}_{0,n}$$
$$x \longmapsto (x, \infty).$$

Rmk. $M_{0,n}$ is a smooth $(n-3)$-dim. quasi-proj. variety, and it is not compact. One can compactify $M_{0,n}$ by allowing degenerations of n-pointed smooth rational curves where the curve is allowed to break.

§ Stable n-pointed rational curves.

Def. A tree of proj. lines is a connected curve with the following properties:

(i) Each irr. component, called twig, is iso. to $\mathbb{P}^1$.

(ii) Points of intersection of components, called nodes, are locally iso to $xy=0$ in $\mathbb{A}^2$.

(iii) There are no closed circuits: if a node is removed, the curve becomes disconnected.