Geometry of curves in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $([X_0:X_1],[Y_0:Y_1])$ be coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$.
A curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is given by the vanishing locus of a bihomogeneous polynomial $F(X_0, X_1; Y_0, Y_1)$.
This means that F is homogeneous of a certain degree d_1 in X_0, X_1, and homogeneous of a certain degree d_2 in Y_0, Y_1. The pair (d_1, d_2) is called the bidegree of C, and F is bihomogeneous of bidegree (d_1, d_2).

Ex.

$$F = X_0 Y_0^2 + 2X_0 Y_0 Y_1 + 7X_0 Y_1^2 - 4X_1 Y_0^2 + X_1 Y_0 Y_1 - X_1 Y_1^2 = 0$$

is bihomogeneous of bidegree $(1,2)$
The vanishing $C = V(F) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defines an irreducible smooth curve of genus 0.

Hw. Let $p = [1:0]$, $q = [0:1]$, $r = [1:1] \in \mathbb{P}^1$.
Using the definition of rational equivalence (Def. 1.2) show that $p + q \sim_{rat} 2r$.
(Need to find $\Phi \subseteq \mathbb{P}^1_1 \times \mathbb{P}^1_2$ such that the fibers over two distinct $t_0, t_1 \in \mathbb{P}^1_1$ are $p+q$ and $2r$).

Different equivalences among cycles

Recall:

Def. Let X be a smooth projective variety. Let $Rat(X) \subseteq Z(X)$ be the subgroup generated by

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

where $t_0, t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not in any fiber $\{t\} \times X$. The group of rational equivalence classes is

$$A(X) = Z(X)/Rat(X).$$

§ Algebraic equivalence.

Def. Let X be a smooth projective variety. Let $Alg(X) \subseteq Z(X)$ be the subgroup generated by

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

where $t_0, t_1 \in C$ for some irr. proj. curve C and Φ is a subvariety of $C \times X$ not in any fiber $\{t\} \times X$. The group of algebraic equivalence classes is

$$B(X) = Z(X)/Alg(X).$$

Rmk. Let $\alpha, \beta \in Z(X)$. It is clear that if $\alpha \sim_{rat} \beta$, then $\alpha \sim_{alg} \beta$. In other words, $Rat(X) \subseteq Alg(X)$. Hence we have a homomorphism $A(X) \to B(X)$.

Q. Is alg. equiv. really a new relation? In other words, can we find examples of X, α, β such that $\alpha \sim_{alg} \beta$, but $\alpha \not\sim_{rat} \beta$?

<u>Prop</u>. Let X be a smooth projective variety. Then for all $a, b \in X$, $a \sim_{alg} b$.

<u>Proof</u>. Say X is a curve. Then define $\Phi := \{(x,x) \mid x \in X\}$. Then, by taking $C := X$,

$$< \Phi \cap (\{a\} \times X) > - < \Phi \cap (\{a\} \times X) > = a - b.$$

So $a - b \in Alg(X)$.

Now assume $\dim(X) \geq 2$. We make use of the following lemma:

<u>Lemma</u>: Let X be an irr. proj. var. $\dim(X) \geq 2$ and let $a, b \in X$. Then there exists an irreducible curve $C \subseteq X$ s.t. $a, b \in C$.

So let $C \subseteq X$ be an irr curve passing through a, b. Define

$$\Phi := \{(x,x) \mid x \in C\} \subseteq C \times X.$$

Then

$$< \Phi \cap (\{a\} \times X) > - < \Phi \cap (\{b\} \times X) > = a - b. \quad \square$$

<u>Proof of the lemma</u>. It is enough to show that there exists a prime divisor $Y \subseteq X$ s.t. $a, b \in Y$.
Let X' be the blow up of X at a and b.
Let $E_a, E_b \subseteq X'$ be the exceptional divisors over a and b.
X' is still projective, so fix an embedding $X' \subseteq \mathbb{P}^N$.
By Bertini's theorem, $\exists H \subseteq \mathbb{P}^N$ hyperplane such that $X' \cap H =: Y'$ is irreducible of codim 1.
Since Y' is ample, $E_a \cap Y' \neq \emptyset$ and $E_b \cap Y' \neq \emptyset$.

Then define Y as the image of Y' under the blow up morphism $X' \to X$. $\square$

<u>Ex.</u> Let X be a smooth proj. curve of genus $g \geq 1$.
Any two points on X are alg. equiv.
However, given any $a, b \in X$ with $a \neq b$, we have that $a \not\sim_{rat} b$.
Assume by cont. that $a \sim_{rat} b$. So $\exists f : X \dashrightarrow \mathbb{P}^1$ s.t. $\mathrm{div}(f) = a - b$. First, as X is a smooth curve, f extends to $X \to \mathbb{P}^1$.
Second, we prove that f is injective. Let $x, y \in X$, $x \neq y$, and assume $f(x) = f(y) \neq \infty$. Let $c := f(x) = f(y)$ and define $g := f - c$. Then
$\mathrm{div}(g) = m_x x + m_y y + D - b$ for some $m_x, m_y \in \mathbb{Z}_{>0}$ and effective divisor D
But then $\deg(\mathrm{div}(g)) \geq m_x + m_y - 1 > 0$.
This is impossible because the degree of a principal divisor on a smooth projective curve is zero (see Shafarevich, Basic AG, §3.2). So f is injective. Moreover, we observe that $f : X \to \mathbb{P}^1$ is also surjective as $f(X)$ is an irreducible closed subscheme of $\mathbb{P}^1$ which has at least two points.
So f is bijective. As $\mathbb{P}^1$ is normal, this implies that f is an isomorphism. ✓

<u>Rmk.</u> Say $g = 1$. Then X is an additive algebraic group acting on itself. Let $g, x \in X$ such that $g + x \neq x$. By above, $g + x \not\sim_{rat} x$. This does not contradict Thm 1.7 as X is projective.