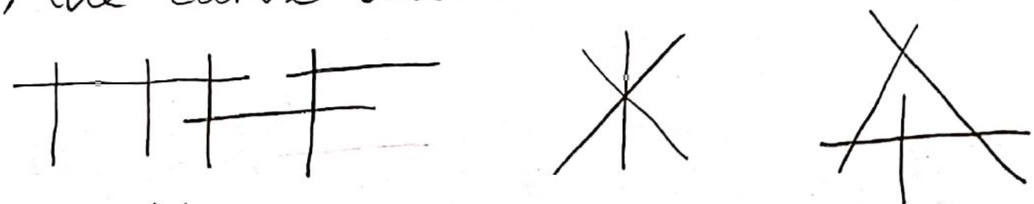
§ Stable n-pointed rational curves.

Def. A tree of proj. lines is a connected curve with the following properties:

(i) Each irr component, called twig, is iso. to $\mathbb{P}^1$.

(ii) Points of intersection of components, called nodes, are locally iso to $xy=0$ in $\mathbb{A}^2$.

(iii) There are no closed circuits: if a node is removed, the curve becomes disconnected.

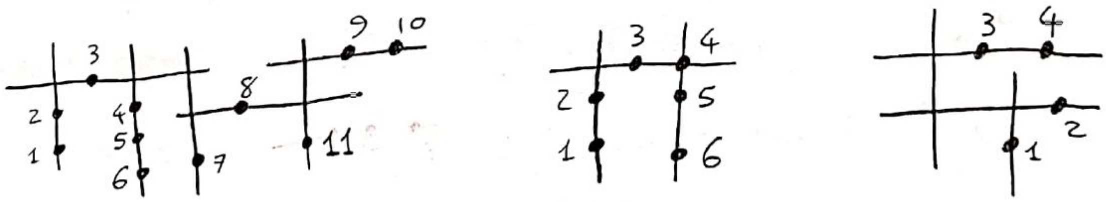


Yes no no

Def. Let $n \geq 3$. A stable n-pointed rational curve is a tree of proj. lines C with n distinct marked points such that

(i) the marked points are smooth in C,

(ii) every twig contains at least three special points (special means a marked point or a node).



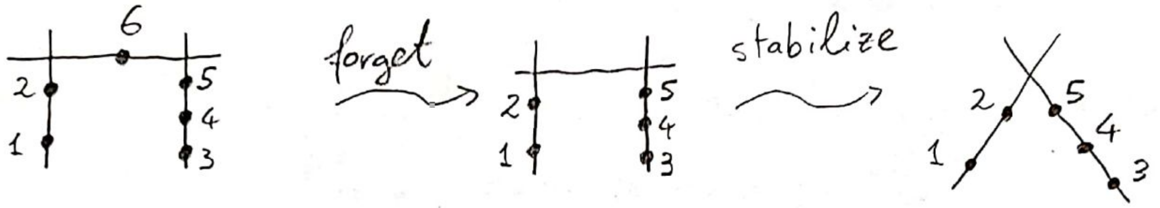
Yes no no

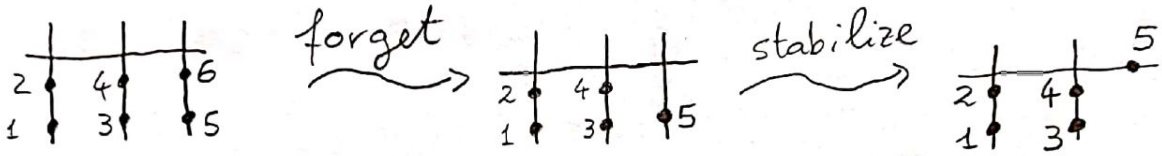
In a way analogous to n-pointed smooth rational curves, define the following notions for stable n-pointed rat. curves:

- $(C, p_1, \dots, p_n) \cong (C', p_1', \dots, p_n')$,
- Family $\pi: \mathcal{X} \to B$, σ_i

If $(C, p_1, \dots, p_n)$ is stable, then it defines a point $f_{n+1}(x) \in \overline{M}_{0,n}$. If $(C, p_1, \dots, p_n)$ is not stable, then we can modify it in a unique way to obtain a stable curve.

Ex.

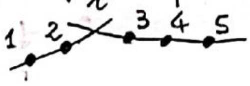




Thm (Knudsen) There exists a morphism $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ sending $(C, p_1, \dots, p_{n+1})$ to $(C', p_1', \dots, p_n')$, which is obtained from $(C, p_1, \dots, p_n)$ by contracting the unstable twigs. This is called forgetful map.

Rmk. $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ is the universal family of $\overline{M}_{0,n}$.

The idea is that given $(C, p_1, \dots, p_n) \in \overline{M}_{0,n}$, the fiber over it is described by $(C, p_1, \dots, p_n, p_{n+1})$ as p_{n+1} varies in C in a dense open subset, and when $p_{n+1} \in C$ is a node of p_i for $1 \leq \dots \leq n$, we need to stabilize.

Ex. Consider  $\in \overline{M}_{0,5}$. Then

- Iso. of two families $\mathcal{X} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} B$, $\mathcal{X}' \underset{\sigma_i'}{\overset{\pi'}{\rightleftarrows}} B$.

<u>Thm</u> (Knudsen) Let $n \geq 3$ and let

$$F : \mathrm{Sch}^{op} \longrightarrow \mathrm{Set}$$

$$B \longmapsto \left\{ \mathcal{X} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} B \;\middle|\; \begin{array}{l}\text{family of stable} \\ n\text{-pointed rat.} \\ \text{curves}\end{array} \right\} \Big/ \text{iso.}$$

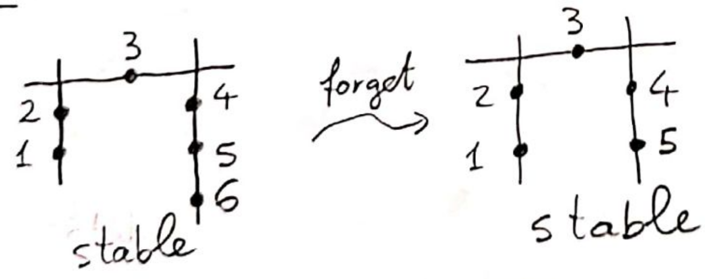
Then F is <u>finely</u> represented by a smooth proj. var. $\overline{M}_{0,n}$ containing $M_{0,n}$ as a dense open subset.

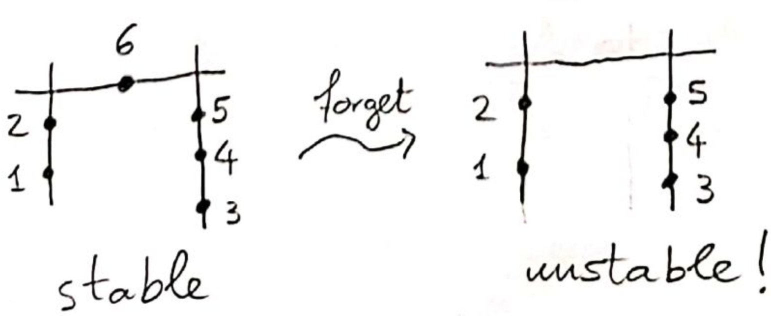
<u>Rmk</u>. As $M_{0,3} = \{pt\}$ and $M_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\}$, we have that $\overline{M}_{0,3} = \{pt\}$ and $\overline{M}_{0,4} = \mathbb{P}^1$.

§ <u>Forgetful morphisms</u>.

Let $n \geq 3$ and consider $\overline{M}_{0,n+1}$. Let $x \in \overline{M}_{0,n+1}$ and let $(C, p_1, \dots, p_{n+1})$ be the stable n-pointed rational curve parametrized by x. Consider $(C, p_1, \dots, p_n)$. This may or may not be stable!

<u>Ex</u>.





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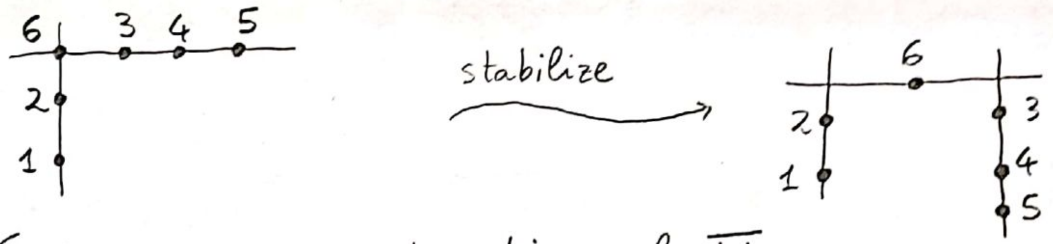


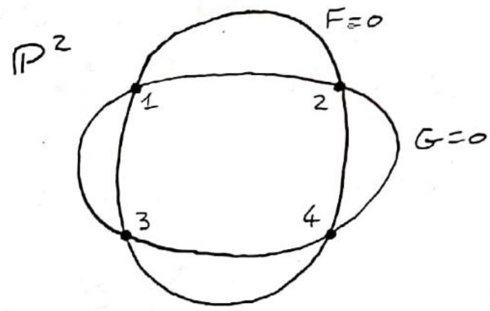
§ Kapranov's construction of $\overline{M}_{0,n}$.

Let $n \geq 5$. Kapranov showed that $\overline{M}_{0,n}$ can be realized as an appropriate blow up of $\mathbb{P}^{n-3}$.

Ex. Let us understand this for $\overline{M}_{0,5}$, which is going to be a blow up of $\mathbb{P}^2$.

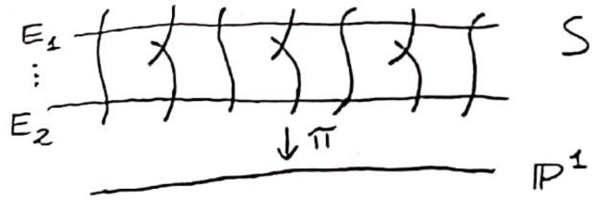
$P_1 = [1:0:0]$, $P_2 = [0:1:0]$, $P_3 = [0:0:1]$, $P_4 = [1:1:1]$.

$[X:Y:Z] \in \mathbb{P}^2$, $F(X,Y,Z)=0$ and $G(X,Y,Z)=0$ be two smooth independent conics through $P_1, \dots, P_4$.



$\lambda F + \mu G = 0$ all conics through $P_1, \dots, P_4$ for $[\lambda:\mu] \in \mathbb{P}^1$, which defines $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ undefined at $P_1, \dots, P_4$. If $S = Bl_{P_1,\dots,P_4}\mathbb{P}^2$, then

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$



$\pi : S \to \mathbb{P}^1$ is a fibration of curves of arithmetic genus zero. The exceptional divisors $E_1, \dots, E_4 \subseteq S$ are disjoint sections of π. One can check that $\pi : S \to \mathbb{P}^1$ can be identified with $f_5 : \overline{M}_{0,5} \to \overline{M}_{0,4}$, so $\overline{M}_{0,5} \cong Bl_4 \mathbb{P}^2$.

More in general, Kapranov's construction goes as follows. In $\mathbb{P}^{n-3}$ let

$$P_1=[1:0:\dots:0],\ P_2=[0:1:\dots:0],\ \dots,\ P_{n-2}=[0:0:\dots:1],\ P_{n-1}=[1:1:\dots:1].$$

- Let X_0 be the blow up of $\mathbb{P}^{n-3}$ at $P_1,\dots,P_{n-1}$;
- Let X_1 be the blow up of X_0 at the strict transforms of the lines $\overline{P_iP_j}$, $i\neq j$;
- Let X_2 be the blow up of X_1 at the strict transforms of the planes $\overline{P_iP_jP_k}$, i,j,k distinct;

$\vdots$

- Let X_{n-5} be the blow up of X_{n-6} at the strict transforms of the codim 2 linear subspaces $\overline{P_{i_1}\cdots P_{i_{n-4}}}$, $i_1,\dots,i_{n-4}$ distinct.

Then, $\overline{M}_{0,n}\cong X_{n-5}$.

<u>Corollary</u>. $\overline{M}_{0,n}$ is an HI space.