§ Some remarks about pushforward and projection formula.

Pushforward.

Recall:

Def. Let $f: X \to Y$ be a proper map and let $A \subseteq X$ be a subvariety.

(a) If $\dim(f(A)) < \dim(A)$, then let $f_* A = 0$

(b) If $\dim(f(A)) = \dim(A)$, then $k(f(A)) \subseteq k(A)$ is a finite field extension and we let

$$f_* A = [k(f(A)) : k(A)] \cdot f(A)$$

(c) Extending by linearity, we obtain $f_*: Z(X) \to Z(Y)$

Thm. f_* above induces a map of Chow groups $f_*: A(X) \to A(Y)$.

Key idea. Assume f is onto. Let $g \in k(X) \setminus \{0\}$.

Then

$$f_* \operatorname{div}(g) = \begin{cases} 0 & \text{if } \dim(Y) < \dim(X) \\ \operatorname{div}(N(g)) & \text{if } \dim(Y) = \dim(X), \end{cases}$$

where $N(g)$ is the norm of g w.r.t. the finite field extension $\underset{N(g)}{\overset{}{k(Y)}} \subseteq \underset{g}{k(X)}$. More explicitly:

$N(g) :=$ determinant of the $k(Y)$-linear map

$$k(X) \longrightarrow k(X),$$
$$h \longmapsto gh.$$

Rmk. If $k(Y) = k(X)$, then $N(g) = g$.

Application. Linear and rational equivalence of divisors coincide.

Let $f \in k(X)\setminus\{0\}$, $\mathrm{div}(f) = \underset{\text{effective}}{D_1} - \underset{\text{effective}}{D_2}$. Let us show $D_1 \sim_{rat} D_2$

Let $\Gamma_f = \{(x, f(x)) \mid x \in X \text{ and } f \text{ is defined at } x\}$.

We have the following commutative diagram:

$$\begin{tikzcd} & X\times\mathbb{P}^1 \arrow[ddl] \arrow[ddr] & \\ & \overline{\Gamma}_f \arrow[u, phantom, "\cup" description] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, dashed, "f"'] & & \mathbb{P}^1 \end{tikzcd}$$

$p_2 \in k(\overline{\Gamma}_f)\setminus\{0\}$.

we would want $D_1\times\{0\} - D_2\times\{0\}$

$\mathrm{div}(p_2) = \langle \overline{\Gamma}_f \cap (X\times\{0\})\rangle - \langle \overline{\Gamma}_f \cap (X\times\{\infty\})\rangle =: RHS$

p_1 is birational, hence $k(X) \cong k(\overline{\Gamma}_f)$, and under this isomorphism p_2 corresponds to f. So $N(p_2) = f$. Therefore:

$p_{1*}RHS = p_{1*}\mathrm{div}(p_2) = \mathrm{div}(N(p_2)) = \mathrm{div}(f) = D_1 - D_2$.

Hence $RHS = D_1\times\{0\} - D_2\times\{\infty\}$, which implies $D_1 \sim_{rat} D_2$.

Conversely, let $\underset{\text{effective}}{D_1} \sim_{rat} \underset{\text{effective}}{D_2}$. So $\exists\, \Phi \subseteq X\times\mathbb{P}^1$ s.t.

$D_1\times\{0\} - D_2\times\{\infty\} = \langle \Phi \cap (X\times\{0\})\rangle - \langle \Phi \cap (X\times\{\infty\})\rangle$.

Consider the commutative diagram

$$\begin{tikzcd} & X\times\mathbb{P}^1 \arrow[ddl] \arrow[ddr] & \\ & \Phi \arrow[u, phantom, "\cup" description] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathbb{P}^1 \end{tikzcd}$$

$p_2 \in k(\Phi)\setminus\{0\}$ and $\mathrm{div}(p_2) = D_1\times\{0\} - D_2\times\{\infty\}$.

To conclude, note that

$$D_1 - D_2 = p_{1*}\left(D_1 \times \{0\} - D_2 \times \{\infty\}\right) = \underbrace{p_{1*}\, \mathrm{div}(p_2)}_{\text{principal}}.$$

Projection formula.

$f : X \to Y$ map of smooth quasi-proj var's.
If $\alpha \in A^k(Y)$; $\beta \in A_\ell(X)$, then

$$f_*\left((f^*\alpha)\cdot\beta\right) = \alpha \cdot f_*\beta \in A_{\ell-k}(X).$$

Useful application. $X, Y_1, \dots, Y_k \subseteq Z$ closed irr subvarieties, X smooth. Say you have to compute $[X]\cdot[Y_1]\cdot\ldots\cdot[Y_k]$. Let $i : X \hookrightarrow Z$ be the inclusion. Then

$$\underbrace{[X]\cdot[Y_1]\cdot\ldots\cdot[Y_k]}_{\text{intersection in } Z} = \left(i_*[X]\right)\cdot[Y_1]\cdot\ldots\cdot[Y_k]$$

$$= i_*\left(\overset{\text{identity}}{[X]} \cdot i^*\left([Y_1]\cdot\ldots\cdot[Y_k]\right)\right)$$

$$= i_*\left(i^*\left([Y_1]\cdot\ldots\cdot[Y_k]\right)\right)$$

$$= i_*\left(\underbrace{i^*([Y_1])\cdot\ldots\cdot i^*([Y_k])}_{\text{intersection in } X}\right).$$

§ Numerical equivalence.

Recall:

Theorem – Definition 1.5. If X is smooth quasi-proj var, then there exists a unique product structure on $A(X)$ satisfying the condition:

(*) If two subvarieties A, B of X are generically transverse, then

$$[A]\cdot[B] = [A \cap B].$$

This makes $A(X) = \bigoplus_{k=0}^{\dim(X)} A^k(X)$ into an associative, commutative, graded ring, called the Chow ring of X.

Def. Let X be a smooth projective n-dimensional variety. We say that $\alpha, \beta \in Z^k(X)$ are numerically equivalent provided $\deg(\alpha \cdot \gamma) = \deg(\beta \cdot \gamma)$ for all $\gamma \in Z^{n-k}(X)$. Let $\mathrm{Num}(X) \subseteq Z(X)$ be the subgroup generated by cycles which are numerically equivalent to zero. Define

$$N(X) = Z(X)/\mathrm{Num}(X).$$

Thm.

$$\underset{\text{already knew}}{\mathrm{Rat}(X) \subseteq} \mathrm{Alg}(X) \underset{\text{Fulton's "Intersection Theory", Chapter 19.}}{\subseteq} \mathrm{Num}(X).$$

In particular, we have homomorphisms

$$A(X) \longrightarrow B(X) \longrightarrow N(X).$$

Q. Is num. equiv really a new relation? In other words, can we find X, α, β such that $\alpha \sim_{num} \beta$, but $\alpha \not\sim_{alg} \beta$?

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Ex. (We will understand this example more later on!) Let X be an Enriques surface. Then $K_X \not\sim_{alg} 0$, but $K_X \sim_{num} 0$. $(\star)$

Ex. Let X be a genus one curve and let $a, b \in X$, $a \neq b$. We know that $a \not\sim_{rat} b$. But $a \sim_{num} b$ as $[a] \cdot [X] = 1 = [b] \cdot [X]$.

Rmk. $N_k(X)$ finitely generated (see Fulton's "Intersection Theory", Example 19.1.4).
In general, $A_k(X)$ may not be finitely generated. For instance, $A_0(X)$ were X is a genus one curve.

$(\star)$ In the next reading you will learn about K_X in general. I will now recall some preliminaries needed for the reading. $\longrightarrow$

§ Basics of vector bundles (needed for next reading assignment)

<u>Def</u>. Let X be a variety. A vector bundle of rank n over X is a variety V together with a map $f: V \to X$, an open covering $X = \bigcup_i U_i$, and isomorphisms $\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^n \times U_i$ s.t., for any i, j and for any open affine $W \subseteq U_i \cap U_j$, the automorphism $\psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n \times W$ on the coordinate ring $k[W][x_1, \dots, x_n]$ is given by a $k[W]$-linear automorphism.

Two vector bundles $(V, f, \{U_i\}, \{\psi_i\})$, $(V', f', \{U'_j\}, \{\psi'_j\})$ on X are isomorphic provided $\exists\, g: V \xrightarrow{\cong} V'$ s.t. $f' \circ g = f$ and s.t. $(V, f, \{U_i, U'_j\}, \{\psi_i, \psi'_j \circ g\})$ is also a vector bundle structure on X.

Operations with vector bundles.

Let V_1, V_2 be vector bundles on X of ranks r_1, r_2. By fiberwisely performing the named operations, one can globally define on X the following new vector bundles:

- $V_1 \oplus V_2 \longrightarrow X$ of rank $r_1 + r_2$;
- $V_1 \otimes V_2 \longrightarrow X$ of rank $r_1 \cdot r_2$;
- $V_1^\vee \longrightarrow X$ of rank r_1;
- $\Lambda^k V_1 \longrightarrow X$ of rank $\binom{r_1}{k}$.

<u>Def</u>. Rank 1 vector bundles are called line bundles.

Let $\mathrm{Pic}(X)$ denote the set of line bundles on X

modulo isomorphism. This comes with a binary operation:

$$\begin{aligned} \mathrm{Pic}(X) \times \mathrm{Pic}(X) &\longrightarrow \mathrm{Pic}(X) \\ ([L_1],[L_2]) &\longmapsto [L_1 \otimes L_2]. \end{aligned}$$

<u>Thm</u>. $(\mathrm{Pic}(X),\cdot)$ is a group. The identity element is $[\mathbb{A}^1 \times X]$. If $[L] \in \mathrm{Pic}(X)$, then $[L]^{-1} = [L^\vee]$.