

## § Some remarks about pushforward and projection formula.

### Pushforward.

Recall: Recall.

Def. Let  $f: X \rightarrow Y$  be a proper map and let  $A \subseteq X$  be a subvariety.

(a) If  $\dim(f(A)) < \dim(A)$ , then let  $f_* A = 0$

(b) If  $\dim(f(A)) = \dim(A)$ , then  $k(f(A)) \subseteq k(A)$  is a finite field extension and we let

$$f_* A = [k(f(A)) : k(A)] \cdot f(A)$$

(c) Extending by linearity, we obtain  $f_* : Z(X) \rightarrow Z(Y)$

Thm.  $f_*$  above induces a map of Chow groups  $f_* : A(X) \rightarrow A(Y)$ .

Key idea. Assume  $f$  is onto. Let  $g \in k(X) \setminus \{0\}$ .

Then

$$f_* \operatorname{div}(g) = \begin{cases} 0 & \text{if } \dim(Y) < \dim(X) \\ \operatorname{div}(N(g)) & \text{if } \dim(Y) = \dim(X), \end{cases}$$

where  $N(g)$  is the norm of  $g$  w.r.t. the finite field extension  $k(Y) \subseteq k(X)$ . More explicitly:

$N(g) :=$  determinant of the  $k(Y)$ -linear map

$$k(X) \longrightarrow k(X),$$
$$h \longmapsto gh.$$

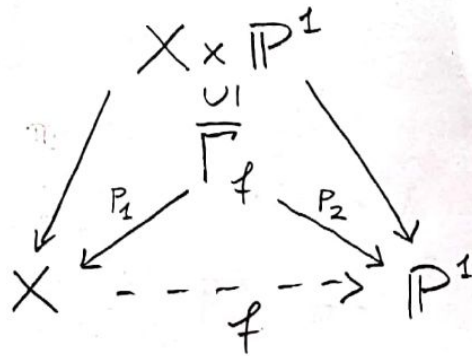
Remark. If  $k(Y) = k(X)$ , then  $N(g) = g$ .

Application. Linear and rational equivalence of divisors coincide.

Let  $f \in k(X) \setminus \{0\}$ ,  $\text{div}(f) = D_1 - D_2$ , let us show  $D_1 \sim_{\text{rat}} D_2$

Let  $\Gamma_f = \{(x, f(x)) \mid x \in X \text{ and } f \text{ is defined at } x\}$ .

We have the following commutative diagram:



$P_2 \in k(\overline{\Gamma}_f) \setminus \{0\}$ .

we would want  $D_1 \times \{0\} - D_2 \times \{\infty\}$

$\text{div}(P_2) = \langle \overline{\Gamma}_f \cap (X \times \{0\}) \rangle - \langle \overline{\Gamma}_f \cap (X \times \{\infty\}) \rangle =: \text{RHS}$

$P_1$  is birational, hence  $k(X) \cong k(\overline{\Gamma}_f)$ , and under this isomorphism  $P_2$  corresponds to  $f$ . So

$N(P_2) = f$ . Therefore:

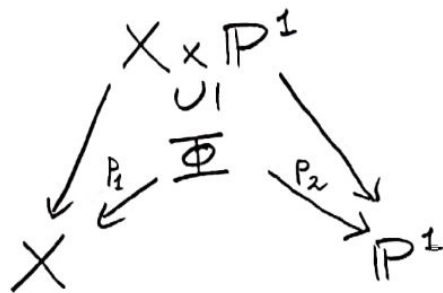
$P_{1*} \text{RHS} = P_{1*} \text{div}(P_2) = \text{div}(N(P_2)) = \text{div}(f) = D_1 - D_2$ .

Hence  $\text{RHS} = D_1 \times \{0\} - D_2 \times \{\infty\}$ , which implies  $D_1 \sim_{\text{rat}} D_2$ .

Conversely, let  $D_1 \sim_{\text{rat}} D_2$ , so  $\exists \Phi \in k(X \times \mathbb{P}^1)$  s.t.

$D_1 \times \{0\} - D_2 \times \{\infty\} = \langle \Phi \cap (X \times \{0\}) \rangle - \langle \Phi \cap (X \times \{\infty\}) \rangle$ .

Consider the commutative diagram



$P_2 \in k(\Phi) \setminus \{0\}$  and  $\text{div}(P_2) = D_1 \times \{0\} - D_2 \times \{\infty\}$ .

To conclude, note that

$$D_1 - D_2 = P_{1*} (D_1 \times \{0\} - D_2 \times \{0\}) = \underbrace{P_{1*} \operatorname{div}(P_2)}_{\text{principal}}$$

Projection formula.

$f: X \rightarrow Y$  map of smooth quasi-proj var's.

If  $\alpha \in A^k(Y)$ ;  $\beta \in A_l(X)$ , then

$$f_*((f^*\alpha) \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X)$$

Useful application.  $X, Y_1, \dots, Y_k \subseteq Z$  closed irreducible subvarieties,  $X$  smooth. Say you have to compute  $[X] \cdot [Y_1] \cdot \dots \cdot [Y_k]$ . Let  $i: X \hookrightarrow Z$  be the inclusion. Then

$$\begin{aligned} [X] \cdot [Y_1] \cdot \dots \cdot [Y_k] &= (i_*[X]) \cdot [Y_1] \cdot \dots \cdot [Y_k] \\ &\underbrace{= i_*([X] \cdot i^*([Y_1] \cdot \dots \cdot [Y_k]))}_{\text{intersection in } Z} \\ &= i_*([X] \cdot \underbrace{i^*([Y_1] \cdot \dots \cdot [Y_k])}_{\text{identity}}) \\ &= i_*\left(i^*([Y_1] \cdot \dots \cdot [Y_k])\right) \\ &= i_*\left(\underbrace{i^*([Y_1]) \cdot \dots \cdot i^*([Y_k])}_{\text{intersection in } X}\right) \end{aligned}$$

## § Numerical equivalence.

Recall:

Theorem - Definition 1.5. If  $X$  is smooth quasi-proj var, then there exists a unique product structure on  $A(X)$  satisfying the condition:

(\*) If two subvarieties  $A, B$  of  $X$  are generically transverse, then

$$[A] \cdot [B] = [A \cap B].$$

This makes  $A(X) = \bigoplus_{k=0}^{\dim(X)} A^k(X)$  into an associative, commutative, graded ring, called the Chow ring of  $X$ .

Def. Let  $X$  be a smooth projective  $n$ -dimensional variety. We say that  $\alpha, \beta \in \mathbb{Z}^k(X)$  are numerically equivalent provided  $\deg(\alpha \cdot \gamma) = \deg(\beta \cdot \gamma)$  for all  $\gamma \in \mathbb{Z}^{n-k}(X)$ . Let  $\text{Num}(X) \subseteq \mathbb{Z}^k(X)$  be the subgroup generated by cycles which are numerically equivalent to zero. Define:

$$N(X) = \mathbb{Z}^k(X) / \text{Num}(X).$$

Thm already knew Fulton's "Intersection Theory", Chapter 19.

$$\text{Rat}(X) \subseteq \text{Alg}(X) \subseteq \text{Num}(X).$$

In particular, we have homomorphisms

$$A(X) \rightarrow B(X) \rightarrow N(X).$$

Q. Is num. equiv really a new relation? In other words, can we find  $X, \alpha, \beta$  such that  $\alpha \sim_{\text{num}} \beta$ , but  $\alpha \not\sim_{\text{alg}} \beta$ ?

Ex. (We will understand this example more later on!) Let  $X$  be an Enriques surface. Then  $K_X \neq 0$ , but  $K_X \sim_{\text{num}} 0$ . (☆)

Ex. Let  $X$  be a genus one curve and let  $a, b \in X, a \neq b$ . We know that  $a \neq b$ . But  $a \sim_{\text{num}} b$  as  $[a] \cdot [X] = 1 = [b] \cdot [X]$ .

Rmk.  $N_K(X)$  finitely generated (see Fulton's

- "Intersection Theory", Example 19.1.4).
- In general,  $A_K(X)$  may not be finitely generated.
- For instance,  $A_0(X)$  where  $X$  is a genus one curve.

(☆) In the next reading you will learn about  $K_X$  in general. I will now recall some preliminaries needed for the reading. →

## Basics of vector bundles (needed for next reading assignment)

Def. Let  $X$  be a variety. A vector bundle of rank  $n$  over  $X$  is a variety  $V$  together with a map  $f: V \rightarrow X$ , an open covering  $X = \bigcup_i U_i$ , and isomorphisms  $\psi_i: f^{-1}(U_i) \rightarrow \mathbb{A}^n \times U_i$  s.t., for any  $i, j$  and for any open affine  $W \subseteq U_i \cap U_j$ , the automorphism  $\psi_j \circ \psi_i^{-1}$  of  $\mathbb{A}^n \times W$  on the coordinate ring

$k[W][x_1, \dots, x_n]$  is given by a  $k[W]$ -linear automorphism.

Two vector bundles  $(V, f, \{U_i\}, \{\psi_i\}), (V', f', \{U'_j\}, \{\psi'_j\})$  on  $X$  are isomorphic provided  $\exists g: V \xrightarrow{\cong} V'$  s.t.  $f' \circ g = f$  and s.t.  $(V, f, \{U_i, U'_j\}, \{\psi_i, \psi'_j \circ g\})$  is also a vector bundle structure on  $X$ .

### Operations with vector bundles.

Let  $V_1, V_2$  be vector bundles on  $X$  of ranks  $r_1, r_2$ .

By fiberwisely performing the named operations, one can globally define on  $X$  the following new vector bundles:

•  $V_1 \oplus V_2 \rightarrow X$  of rank  $r_1 + r_2$ ;

•  $V_1 \otimes V_2 \rightarrow X$  of rank  $r_1 \cdot r_2$ ;

•  $V_1^\vee \rightarrow X$  of rank  $r_1$ ;

•  $\wedge^k V_1 \rightarrow X$  of rank  $\binom{r_1}{k}$ .

Def. Rank 1 vector bundles are called line bundles.

Let  $\text{Pic}(X)$  denote the set of line bundles on  $X$

modulo isomorphism. This comes with a binary operation:

$$\text{Pic}(X) \times \text{Pic}(X) \longrightarrow \text{Pic}(X)$$

$$([L_1], [L_2]) \longmapsto [L_1 \otimes L_2].$$

Thm.  $(\text{Pic}(X), \cdot)$  is a group. The identity element is  $[A^1_X]$ . If  $[L] \in \text{Pic}(X)$ , then

$$[L]^{-1} = [L^\vee].$$