§ The boundary of $\overline{M}_{0,n}$.

<u>Def.</u> The <u>boundary</u> of $\overline{M}_{0,n}$ is defined to be $\overline{M}_{0,n} \setminus M_{0,n}$. Its points parametrize reducible curves.

<u>Thm.</u> $\overline{M}_{0,n} \setminus M_{0,n}$ is a divisor with normal crossings.

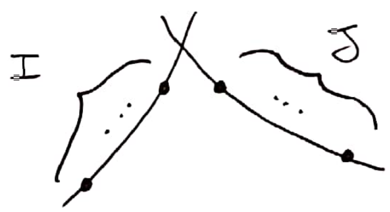
This means:

1) The irr. components are smooth;
2) If $D_1, \dots, D_k$ are irr. comp's, $k \leq n-3$, and $x \in D_1 \cap \dots \cap D_k$, then in local analytic coordinates at x, $D_1 \cup \dots \cup D_k$ is iso. to $z_1 \cdot \ldots \cdot z_k = 0$ in $\mathbb{A}^{n-3}$.

The irreducible components of $\overline{M}_{0,n} \setminus M_{0,n}$ are called <u>boundary divisors</u>.

<u>Combinatorial description of boundary divisors.</u>

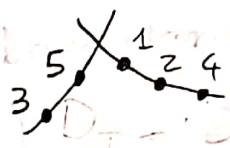
Let $D \subseteq \overline{M}_{0,n}$ be a boundary divisor. Then D parametrizes on a dense open subset stable n-pointed rational curves in the form



where $I \amalg J$ is a partition of $\{1, \dots, n\}$ with $|I|, |J| \geq 2$. So D is denoted by $D_{I,J} = D_{J,I}$.

In particular, $\overline{M}_{0,n}$ has $2^{n-1} - n - 1$ boundary divisors.

<u>Ex.</u> On $\overline{M}_{0,5}$, the boundary divisor $D_{35,124}$ param.

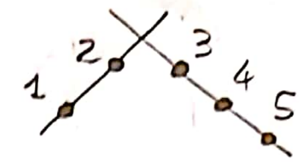


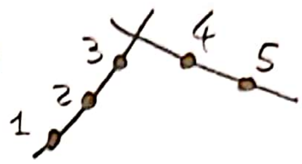
and its degenerations.

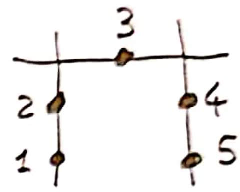
§ Relations in $A^*(\overline{M}_{0,n})$.

Guided discovery.

- Consider on $\overline{M}_{0,5}$ the boundary divisors $D_{12,345}$ and $D_{123,45}$. Do they intersect?

$D_{12,345}$ generically parametrizes 

$D_{123,45}$ generically parametrizes 

- We can see a common degeneration: 

- So $D_{12,345} \cap D_{123,45} \neq \emptyset$.

On the other hand $D_{12,345} \cap D_{13,245}$ seems to be empty as we cannot see this common degeneration.

- Consider $f_5 : \overline{M}_{0,5} \to \overline{M}_{0,4}$. We know $D_{12,34} \sim D_{13,24}$, so

$$\underbrace{f_5^* D_{12,34}}_{=\, D_{12,345} + D_{125,34}} \sim \underbrace{f^* D_{13,24}}_{=\, D_{135,24} + D_{13,245}}.$$

Def: Let $\delta_{I,J}$ be the class of $D_{I,J}$ in $A^*(\overline{M}_{0,n})$

Thm (Keel) $A^*(\overline{M}_{0,n})$ is generated by the boundary divisors modulo the following relations:

1) $\delta_{I,J} \cdot \delta_{K,L} = 0 \Leftrightarrow I \not\subseteq K$ and $I \not\subseteq L$ and $J \not\subseteq K$ and $J \not\subseteq L$.

2) $\forall$ distinct $i,j,k,l \in \{1,\dots,n\}$,

$$\sum_{\substack{i,j\in I\\ k,l\in J}} \delta_{I,J} = \sum_{\substack{i,k\in I\\ j,l\in J}} \delta_{I,J}.$$

Ex. On $\overline{M}_{0,5}$, $\delta_{12,345} \cdot \delta_{13,245} = 0$. Also, it holds that

$$\underbrace{\sum_{\substack{1,2\in I\\ 3,4\in J}} \delta_{I,J}}_{\delta_{12,345}+\delta_{123,45}} = \underbrace{\sum_{\substack{1,3\in I\\ 2,4\in J}} \delta_{I,J}}_{\delta_{13,245}+\delta_{135,24}}.$$

§ Cones of effective cycles on $\overline{M}_{0,n}$.

Def Let $0 < k < n-3$ (so $n \geq 5$). A boundary k-stratum in $\overline{M}_{0,n}$ is an irreducible k-dim subvariety obtained as intersection of boundary divisors. Let

$V_k(\overline{M}_{0,n}) :=$ cone generated by the classes of boundary k-strata $\subseteq \mathrm{Eff}_k(\overline{M}_{0,n})$.

Fulton's question: $V_k(\overline{M}_{0,n}) = \overline{\mathrm{Eff}}_k(\overline{M}_{0,n})$?

$n=5$ $V_1(\overline{M}_{0,5}) = \overline{\mathrm{Eff}}_1(\overline{M}_{0,5})$ (exercise).

$n=6$ $V_1(\overline{M}_{0,6}) = \overline{\mathrm{Eff}}_1(\overline{M}_{0,6})$ (Keel-McKernan)

$V_2(\overline{M}_{0,6}) \subsetneq \overline{\mathrm{Eff}}_2(\overline{M}_{0,6})$ (Keel, Vermeire)

Hassett-Tschinkel, Castravet: $\overline{\mathrm{Eff}}_2(\overline{M}_{0,6})$ is generated by

$V_2(\overline{M}_{0,6})$ and the Keel-Vermeire divisors.

- More in general, for $n \geq 7$, Keel and Vermeire showed that $V_{n-4}(\overline{M}_{0,n}) \subsetneq \overline{\mathrm{Eff}}_{n-4}(\overline{M}_{0,n})$, and it is hard to describe generators for $\overline{\mathrm{Eff}}_{n-4}(\overline{M}_{0,n})$.
  Castravet-Laface-Tevelev-Ugaglia, 2020: $\overline{\mathrm{Eff}}_{n-4}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$.
- For curves, $V_1(\overline{M}_{0,7}) \overset{KM}{=} \overline{\mathrm{Eff}}_1(\overline{M}_{0,7})$, but for $n \geq 8$ it is not known whether $V_1(\overline{M}_{0,n}) = \overline{\mathrm{Eff}}_1(\overline{M}_{0,n})$ (F-conjecture)
- For intermediate dimension $1 < k < n-4$, $V_k(\overline{M}_{0,n}) \subsetneq \overline{\mathrm{Eff}}_k(\overline{M}_{0,n})$ and no $\overline{\mathrm{Eff}}_k(M_{0,n})$ is fully described. See the paper: L S, "On the cone of effective 2-cycles on $\overline{M}_{0,7}$", Eur. J. Math. 1 (2015), no. 4, 669–694.