§ Relation with singular homology.

Recall. Let X be a top. space. Let Δ_K be the K-dim simplex

$$\Delta_K := \{ x \in \mathbb{R}^K \mid x_i \geq 0 \ \& \ \sum_{i=1}^{K} x_i \leq 1 \}.$$

A singular K-simplex of X is a continuous map $\sigma : \Delta_K \to X$. $C_K := \{ \sum a_\sigma \sigma \mid a_\sigma \in \mathbb{Z}$ all zero but finitely many of them$\}$ are the free abelian groups generated by the singular K-simplices.

We have a chain complex:

$$\dots \to C_{K+1} \xrightarrow{\partial_{K+1}} C_K \xrightarrow{\partial_K} C_{K-1} \to \dots$$

satisfying $\partial_K \circ \partial_{K+1} = 0$. One defines $H_K(X) := \dfrac{\mathrm{Ker}(\partial_K)}{\mathrm{Im}(\partial_{K+1})}$.

- H_i is functorial: if $f : X \to Y$ is a continuous map of topological spaces, then we have
$$f_* : H_K(X) \to H_K(Y)$$
induced by associating to $\sigma : \Delta_K \to X$, $f \circ \sigma : \Delta_K \to Y$.
- $H_0(X)$ is the free abelian group generated by the path-connected components of X.

Q: How does this homology theory relate to the Chow group $A_K(X)$ for a smooth proj. irr. complex variety X?

Lemma. Let Y be an n-dim. proj. irr. complex variety (not necessarily smooth). Then $H_i(Y) = 0$ for $i > 2n$ and $H_{2n}(Y) \cong \mathbb{Z}$.

Proof: Strong induction on n. For $n = 0$ it is clear. Write $\mathcal{U} := Y \setminus Y_{sing}$. To study $H_i(Y)$, we would like to fit $H_i(Y)$ in a short exact sequence involving Y_{sing} and $\mathcal{U}$. But $\mathcal{U}$ is not compact, so we need to use homology with locally finite supports.

Def. X top space. The Borel-Moore homology $H_i^{BM}(X)$ is defined replacing C_K with

$$C_K^{BM} := \{ \sum a_\sigma \sigma \mid a_\sigma \in \mathbb{Z}, \forall S \subseteq X \text{ compact}, \exists \text{ finitely many } a_\sigma \neq 0 \text{ s.t. } \sigma(\Delta_K) \cap S \neq \emptyset \}. \square$$

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Back to the proof of Lemma: we have a long exact sequence

$$\cdots \to H_{i+1}^{BM}(\mathcal{U}) \to \underbrace{H_i^{BM}(Y_{sing})}_{\cong H_i(Y_{sing})} \to \underbrace{H_i^{BM}(Y)}_{\cong H_i(Y)} \to H_i^{BM}(\mathcal{U}) \to \cdots$$

If $i > 2n$, then

$$0 \overset{\text{induction}}{=} H_i(Y_{sing}) \to H_i(Y) \to H_i^{BM}(\mathcal{U}) \overset{\text{Known for } \mathcal{U} \text{ smooth}}{=} 0$$

So $H_i(Y) = 0$. Moreover

$$0 \overset{\text{induction}}{=} H_{2n}(Y_{sing}) \to H_{2n}(Y) \xrightarrow{\cong} \underbrace{H_{2n}^{BM}(\mathcal{U})}_{\cong \mathbb{Z}} \to H_{2n-1}(Y_{sing}) \overset{\text{induction}}{=} 0$$

(Note: $H_{2n}(\mathcal{U}) \cong 0$, Hatcher "Algebraic Topology", Prop. 3.29)

So $H_{2n}(Y) \cong \mathbb{Z}$. □

Rmk. A generator $[Y]$ of $H_{2n}(Y)$ can be chosen canonically based on its restriction to $\mathcal{U} = Y \setminus Y_{sing}$ (fundamental class).

Def. $Y \subseteq X$ irr. closed subvar. of $\mathbb{C}$-dim k. $i : Y \hookrightarrow X$ inclusion morphism. Consider $i_* : H_{2k}(Y) \to H_{2k}(X)$. Define

$$\underset{\substack{\uparrow \\ \text{cycle class of } Y}}{cl(Y)} := i_*[Y].$$

This induces a group homomorphism

$$cl : Z(X) \longrightarrow H_{2k}(X).$$

Prop (Fulton, "Intersection Theory", Prop. 19.1.1)

$$cl(Alg(X)) = \{0\}.$$

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Corollary. X irr. smooth proj. $\mathbb{C}$-var. Then we have group homomorphisms:

$$A_k(X) \twoheadrightarrow B_k(X) \to H_{2k}(X).$$

Rmk. Recall from last week we also have $A_k(X) \twoheadrightarrow B_k(X) \twoheadrightarrow N_k(X)$. This relates to $H_{2k}(X)$ as follows:

$$\ker\left(Z_k(X) \xrightarrow{cl} H_{2k}(X)\right) \subseteq \mathrm{Num}_k(X).$$

§ HI spaces.

Def. X irr. smooth proj. complex var. X is called HI space (Homology Isomorphism) provided

$$A_*(X) \to H_*(X)$$

is an isomorphism of groups.

Ex. $\mathbb{P}^n$ is an HI space. Recall

$$H_i(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le i \le 2n,\ i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Ex. X smooth proj curve. X is HI $\Leftrightarrow$ $X \cong \mathbb{P}^1$. This is because $H_1(X) \cong \mathbb{Z}^{2g}$.

Rmk. Let X be HI. Then:

1) X has no odd homology.
2) The group $H_*(X)$ is torsion free. This follows from 1) combined with the universal coefficients theorem for cohomology and Poincaré duality.

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3) Rational, algebraic, homological, and numerical equivalence of cycles coincide.

<u>Thm</u> (Keel) Let X be HI and let $Y \hookrightarrow X$ be a regularly embedded subvariety. Then $Bl_Y(X)$ is also HI.

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