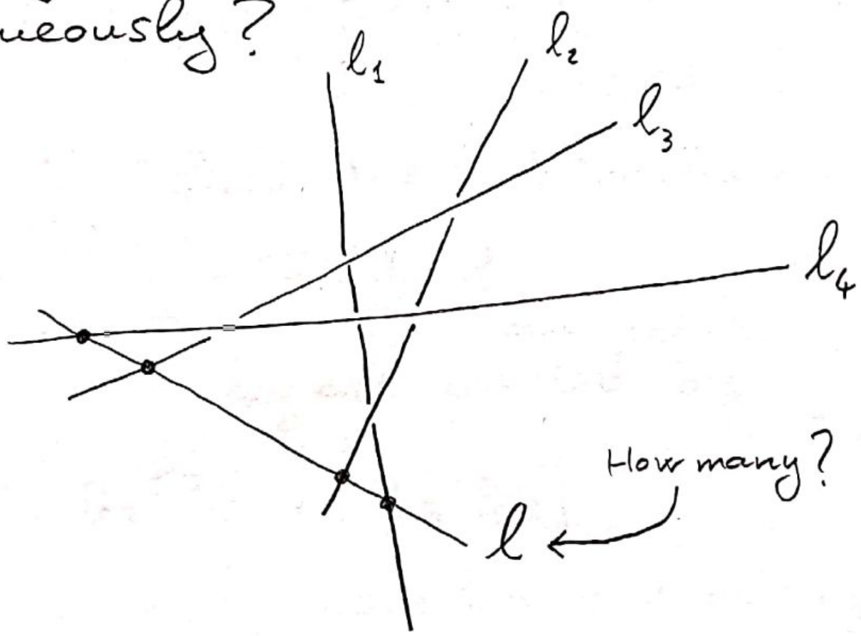
Intersection theory and moduli spaces, 29/03/2021

Introductory lecture: Motivation from enumerative geometry - Divisors on algebraic varieties.

§ Motivation from enum. geom.

Question. Let $l_1, l_2, l_3, l_4 \subseteq \mathbb{P}^3$ be general lines. How many lines $l \subseteq \mathbb{P}^3$ intersect l_1, l_2, l_3, l_4 simultaneously?



Let us consider all lines in $\mathbb{P}^3$:

$\mathbb{G}(1,3) := \{\text{lines in } \mathbb{P}^3\}$ as a set.

Define $\Sigma(l_i) := \{ l \in \mathbb{G}(1,3) \mid l \cap l_i \neq \emptyset \}, i = 1, \dots, 4$.

What we need is $\# \bigcap_{i=1}^{4} \Sigma(l_i)$.

Algebraic structure on $\mathbb{G}(1,3)$.

A line in $\mathbb{P}^3$ is in the form $\begin{cases} \sum_{i=0}^{3} a_i X_i = 0 \\ \sum_{i=0}^{3} b_i X_i = 0 \end{cases}$

with $(a_0, \dots, a_3), (b_0, \dots, b_3)$ linearly independent.

$[X_0 : \dots : X_3] \in \mathbb{P}^3$ are our projective coordinates.

Also, acting with GL_2 on the left of $\begin{pmatrix} a_0 \cdots a_3 \\ b_0 \cdots b_3 \end{pmatrix}$ returns the same line. So a line in $\mathbb{P}^3$ is represented by the equivalence class w.r.t. GL_2 action of a 2×4 matrix of rank 2. We can now define a map

$$\mathbb{G}(1,3) \xrightarrow{f} \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$$

$$\begin{bmatrix} a_0 & \cdots & a_3 \\ b_0 & \cdots & b_3 \end{bmatrix} \longmapsto \left[\cdots : \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} : \cdots \right]_{0 \le i < j \le 3}$$

- f is well-defined, and it defines an embedding called <u>Plücker embedding</u>.
- The coordinates in $\mathbb{P}^{\binom{4}{2}-1}$ are called <u>Plücker coordinates</u> and they are denoted by

$$[P_{01} : P_{02} : P_{03} : P_{12} : P_{13} : P_{23}]$$

- $\Im m(f)$ is a hypersurface cut out by the following equation:

$$P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0$$

So $\mathbb{G}(1,3)$ can be viewed as a quadric $Q \subseteq \mathbb{P}^5$. <u>Rmk.</u> $\mathbb{G}(1,3)$ is our first example of moduli space.

Algebraic structure on $\Sigma(l_i)$.

Let us consider l_1 and assume $l_1 = V(X_0, X_1)$.
Let $l = V\left(\sum_{i=0}^{3} a_i X_i, \sum_{i=0}^{3} b_i X_i\right)$ be another line.
The intersection $l_1 \cap l$ is given by the solutions to the following system:

$$\begin{cases} X_0 = 0 \\ X_1 = 0 \\ a_2 X_2 + a_3 X_3 = 0 \\ b_2 X_2 + b_3 X_3 = 0 \end{cases}$$

from which we can see that $l_1 \cap l \neq \emptyset \iff \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0$, which means $\Sigma(l_1)$ is the set of lines in $\mathbb{G}(1,3)$ for which the corresponding Plücker coordinate P_{23} vanishes.

Hence, $\Sigma(l_1)$ under f corresponds to the hyperplane section $Q \cap \{P_{23} = 0\}$.

Answer to the question.

Let $f(\Sigma(l_i)) = Q \cap H_i$, $H_i \subseteq \mathbb{P}^5$ hyperplane, $i = 1, \dots, 4$

Then $\# \bigcap_{i=1}^{4} \Sigma(l_i)$ is computed by the following intersection number:

$$(Q \cap H_1) \cdot (Q \cap H_2) \cdot (Q \cap H_3) \cdot (Q \cap H_4) \overset{\text{projection formula}}{=}$$

$$= Q \cdot H_1 \cdot H_2 \cdot H_3 \cdot H_4 \underset{\sim}{=} (2H) \cdot H_1 \cdot H_2 \cdot H_3 \cdot H_4 = 2(H \cdot H_1 \cdot H_2 \cdot H_3 \cdot H_4) =$$

$Q \underset{\substack{\text{lin.} \\ \text{equiv.}}}{\sim} 2H$

$H \subseteq \mathbb{P}^5$ gen. hyperplane

$= 2 \cdot 1 = 2$. So there are 2 lines intersecting l_1, l_2, l_3, l_4 simultaneously.

Rmk. The enumerative question was solved by doing intersection theory on the moduli space $\mathbb{G}(1,3)$.

§ Divisors on alg. var's.

Def. Let X be a smooth variety (smoothness can be relaxed). An irreducible closed subvariety $Y \subseteq X$ of codim 1 is called a prime divisor

A Weil divisor D on X is an element of the free abelian group generated by prime divisors in X:

$$\mathrm{Div}(X) := \Big\{ \sum_{\substack{Y \subseteq X \\ \text{prime divisor}}} n_Y Y \;\Big|\; n_Y \in \mathbb{Z}, \text{ all zero but finitely many of them} \Big\}$$

Ex. $[X_0 : X_1 : X_2]$ coordinates in $\mathbb{P}^2$. Examples of prime divisors in X are $Y_1 = V(X_0X_1 - X_2^2)$, $Y_2 = V(X_0 + X_1 + X_2)$.

$D = 7Y_1 - 15Y_2$ is a divisor on X.

A natural way to produce divisors on X is to look at zeros and poles of rational functions on X.

Def. $Y \subseteq X$ prime divisor, $U \subseteq X$ open affine subset s.t. $Y \cap U \neq \emptyset$. Then $Y \cap U$ corresponds to a principal ideal $(\pi) \subseteq k[U]$.

If $f \in k[U] \setminus \{0\}$, $\exists\, k \in \mathbb{Z}_{\geq 0}$ s.t. $f \in (\pi^k)$ and $f \notin (\pi^{k+1})$. Define $v_Y(f) := k$

Def. $Y \subseteq X$ prime divisor, $f \in k(X) \setminus \{0\}$.

Let $U \subseteq X$ open affine s.t. $Y \cap U \neq \emptyset$.

$f|_U = \frac{g}{h}$, $\exists\, g, h \in k[U]$. Define

$$v_Y(f) := v_Y(g) - v_Y(h).$$

If $v_Y(f)>0$ we say that f has a <u>zero</u> along Y of multiplicity $v_Y(f)$.
If $v_Y(f)<0$ we say that f has a <u>pole</u> along Y of multiplicity $-v_Y(f)$.

<u>Rmk</u>. $v_Y(f)$ is independent from the choice of U and from the representation $f|_U = \frac{g}{h}$.

<u>Def</u>. Let $f \in k(X) \setminus \{0\}$. The <u>principal divisor</u> associated to f is

$$\mathrm{div}(f) := \sum_{\substack{Y \subseteq X \\ \text{prime}}} v_Y(f)\, Y$$

<u>HW</u>. Let $X = \{(x,y) \in \mathbb{A}^2 \mid y - x^n = 0\}$ and $f = \frac{x}{y}$. Compute $\mathrm{div}(f)$.

<u>Rmk</u>. In the definition of $\mathrm{div}(f)$, the sum is finite. $U \subseteq X$ open where f is regular. Then the poles of f are in $X \setminus U$, that decomposes into finitely many irreducible components. So

$$\# \{ Y \subseteq X \text{ prime} \mid v_Y(f) < 0 \} < \infty .$$

Repeating the same argument with $\frac{1}{f}$, we also have that

$$\# \{ Y \subseteq X \text{ prime} \mid v_Y(f) > 0 \} < \infty .$$

$\mathrm{Div}(X)$ is in general too large to deal with it. So we consider the following.

<u>Def</u>. $D_1, D_2 \in \mathrm{Div}(X)$ are called <u>linearly equivalent</u> if $D_1 - D_2 = \mathrm{div}(f)$, $\exists\, f \in k(X)\setminus\{0\}$. In symbols, we write $D_1 \sim D_2$.

<u>Rmk</u>. Linear equivalence is an equivalence relation.

<u>Def</u>. $\mathrm{Div}(X)/_\sim =: Cl(X)$ and is called the <u>divisor class group</u>.

<u>Ex</u>.

1) $\boxed{Cl(\mathbb{A}^n) = 0}$. <u>Proof</u>. It is enough to show that any prime divisor is principal.
So let $Z \subseteq \mathbb{A}^n$ prime. So $\exists\, f \in k[x_1, \dots, x_n]$ irreducible polynomial such that $Z = V(f)$. Then

$$\mathrm{div}(f) = \sum_{\substack{Y \subseteq \mathbb{A}^n \\ \text{prime}}} v_Y(f)\, Y = v_Z(f) Z = 1 \cdot Z = Z \quad \checkmark$$

2) $\boxed{Cl(\mathbb{P}^n) \cong \mathbb{Z}}$. <u>Proof</u>. Let $[X_0 : \dots : X_n] \in \mathbb{P}^n$ be coordinates. Define $H = V(X_0)$. Define

$$\varphi : \mathbb{Z} \longrightarrow Cl(\mathbb{P}^n)$$

by extending $1 \longmapsto [H]$ to a group homomorphism.

- φ is injective. Let $d \in \mathbb{Z}_{>0}$ and assume $\varphi(d) = 0$. So $d[H] = 0$, which means $d\,H \sim 0$. Hence $dH = \mathrm{div}(f)$, $\exists\, f \in k(\mathbb{P}^n)\setminus\{0\}$. We have that $f = F/G$, $\exists\, F, G \in k[X_0, \dots, X_n]$ homogeneous, $\deg F = \deg G$, with no common factors. Note $\deg F = \deg G > 0$, because f has to have a zero of order d along H. But then f also have poles along $V(G)$. But this is impossible because dH does not have prime divisors with negative coefficient. So $\varphi(d) \neq 0$.

Assume $\varphi(-d) = -d[H] = 0$, which means $-dH \sim 0$. So $\exists\, g \in k(\mathbb{P}^n) \setminus \{0\}$ such that $-dH = \mathrm{div}(g)$. Then $dH = \mathrm{div}(1/g)$, which we know it cannot be.

So $\ker(\varphi) = \{0\}$.

- φ is surjective. It is enough to show that for any prime divisor $Y \subseteq \mathbb{P}^n$, $[Y] \in \mathrm{Im}(\varphi)$.

$Y = V(F)$, $\exists\, F \in k[x_0, \dots, x_n]$ homogeneous irreducible.

Let $d = \deg F$. Then

$$\mathrm{div}\left(\frac{F}{x_0^d}\right) = Y - dH \Rightarrow [Y] = d[H] = \varphi(d).$$

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