K3 surfaces with $\mathbb{Z}_2^2$ symplectic action

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Main goal & motivation

Given a K3 surface X, it is hard in general to determine the lattice $\mathrm{NS}(X)$. If $G \leq \mathrm{Aut}(X)$ finite and abelian acts symplectically on X, then $\mathrm{NS}(X)$ has extra structure that helps determine it. For projective X with G symplectic action and minimal Picard number, $\mathrm{NS}(X)$ is computed in [1]. We study a special family of K3 surfaces X with $\mathbb{Z}_2^2$ symplectic action whose $\mathrm{NS}(X)$ was not known. These are relevant because of their relation with the double covers of $\mathbb{P}^2$ branched along six lines, and because they give an alternative construction of the Kummer surface $\mathrm{Km}\left(E_{\sqrt{-1}} \times E_{\sqrt{-1}}\right)$.

Background

X projective K3 surface, $G \leq \mathrm{Aut}(X)$.

Def: $G \curvearrowright X$ is *symplectic* if

$$g^*\colon H^0(\omega_X) \to H^0(\omega_X),$$

is the identity for all $g \in G$.

Facts: If G is also finite and abelian, then

- The possible G are classified (Nikulin).
- The lattice
$$\Omega_G = (H^2(X;\mathbb{Z})^G)^\perp \subset \mathrm{NS}(X),$$
only depends on G and not on the action (Nikulin).
- There is a $(19 - \mathrm{rk}(\Omega_G))$-dimensional family of projective K3 surfaces X with G symplectic action. If $\mathrm{rk}(\Omega_G) + 1$, then $\mathrm{NS}(X)$ is computed by Garbagnati-Sarti.

Case of interest: $G = \mathbb{Z}_2^2$. There is a 7 dimensional family of K3 surfaces with $\mathbb{Z}_2^2$ symplectic action.

Triple-double K3 surfaces

Consider three pairs of lines on $\mathrm{Bl}_3\mathbb{P}^2$ (Figure 1). A *triple-double* K3 surface X is obtained by composing three double covers

$$X \xrightarrow{\mathbb{Z}_2} \xrightarrow{\mathbb{Z}_2} \xrightarrow{\mathbb{Z}_2} \mathrm{Bl}_3\mathbb{P}^2,$$

where the first one is branched along the first pair of lines, and so on.

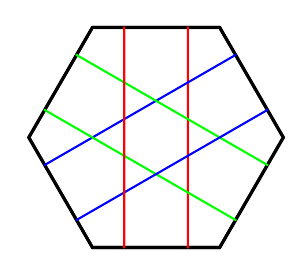


Figure 1: Three pairs of lines on $\mathrm{Bl}_3\mathbb{P}^2$ (toric picture)

The i-th double cover corresponds to an involution $\alpha_i\colon X \to X$. The group

$$\{\mathrm{id}_X, \alpha_1 \circ \alpha_2, \alpha_1 \circ \alpha_3, \alpha_2 \circ \alpha_3\} \cong \mathbb{Z}_2^2,$$

acts symplectically on X.

24 smooth rational curves

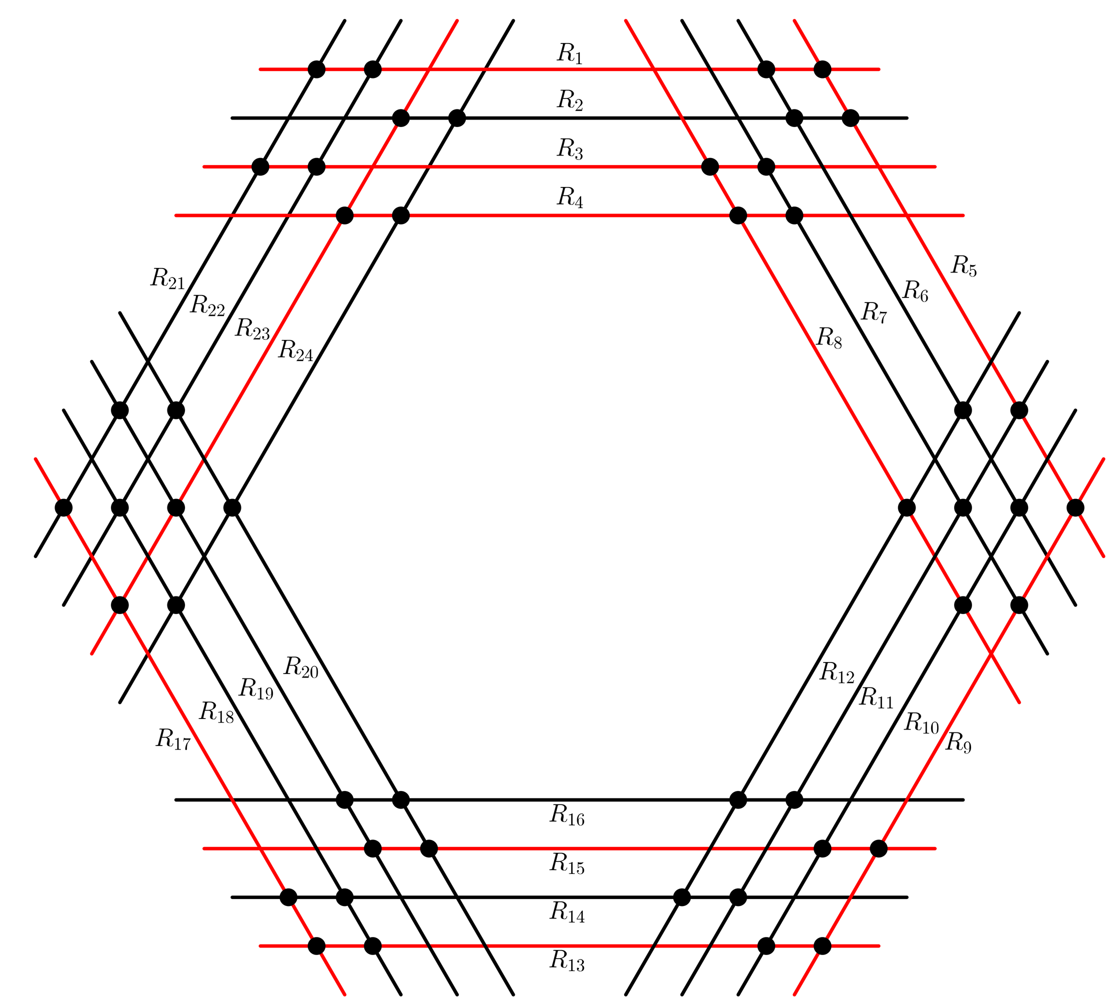
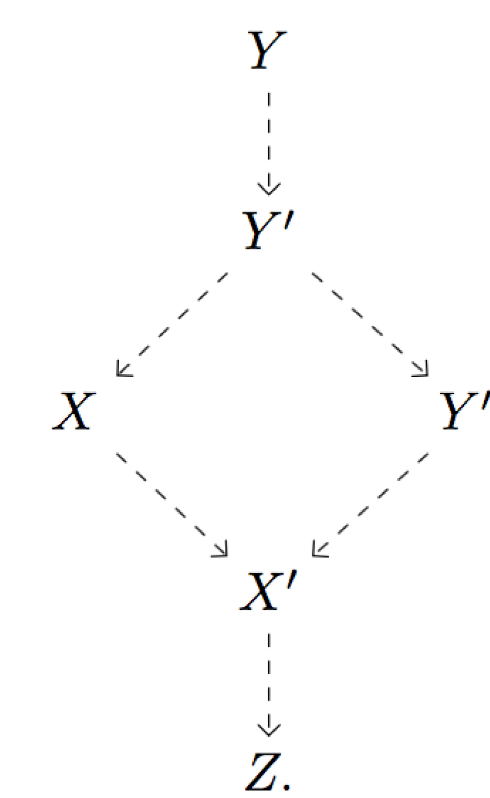


Figure 2: Preimage under $X \to \mathrm{Bl}_3\mathbb{P}^2$ of the (-1)-curves. Two lines intersects if and only if they share a dot. The red curves form a Kodaira type II^* fiber together with the section R_3

$\mathbb{Z}_2^n$-covers of $\mathbb{P}^2$ and six lines

$\mathbb{Z}_2^n$ covers of $\mathbb{P}^2$ branched along six lines:
Y = Hirzebruch-Kummer covering ($n = 5$).
X = triple-double K3 surface ($n = 3$).
Z = double cover of $\mathbb{P}^2$.
We have other K3 surfaces X', Y', Y'' and degree 2 rational maps:



In [2] $\mathrm{NS}(X'), T_{X'}$ are also determined, but these are not new: $\mathbb{Z}_2^4 \curvearrowright X'$ symplectically.

Main theorem (S, 2017, [2])

Let X be a triple-double K3 surface with minimal Picard number. Then
(i) $\mathrm{NS}(X)$ is generated by $R_1, \ldots, R_{24}$, has rank 16, and discriminant group $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$.
(ii) $\mathrm{NS}(X) \cong U \oplus E_8 \oplus Q$ (where Q is explicit) and $T_X \cong U \oplus U(2) \oplus \left(\begin{smallmatrix} -4 & 0 \\ 0 & -4 \end{smallmatrix}\right)$.
(iii) The Kummer surface $\mathrm{Km}\left(E_{\sqrt{-1}} \times E_{\sqrt{-1}}\right)$ appears as a specialization of this 4-dimensional family. Six lines in $\mathbb{P}^2$ with four triple intersection points give rise to this Kummer surface.
(iv) $\mathbb{Z}_2^3$ does not act symplectically on X.

Idea of proof

- $M = \langle R_1, \ldots, R_{24} \rangle \subseteq \mathrm{NS}(X)$ has finite index. Then $M = \mathrm{NS}(X)$ because no isotropic element in M^*/M is in $\mathrm{NS}(X)$.
- $T_{\mathrm{Km}} \cong \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right) \hookrightarrow T_X$ and the moduli space of M-polarized K3 surfaces is connected.
- $\Omega_{\mathbb{Z}_2^3}$ does not embed primitively in $\mathrm{NS}(X)$.

Why new example

$\mathrm{NS}(X)$ and T_X are determined by [1] if there is G finite and abelian acting symplectically on X with $\mathrm{rk}(\Omega_G) + 1 = 16$. This equality is true if and only if $G \cong \mathbb{Z}_2^4$, but $\mathbb{Z}_2^4$ does not act symplectically on X by Main theorem(iv). We also show that $X \ncong W/\mathbb{Z}_2^3$ for some K3 surface W with $\mathbb{Z}_2^3$ symplectic action.

Future project

I am working on analogous questions for the K3 surfaces Y', Y''.

References

[1] A. Garbagnati, A. Sarti, *Elliptic fibrations and symplectic automorphisms on K3 surfaces*, Comm. Algebra 37 (2009), no. 10, 3601–3631.

[2] L. Schaffler, *The Néron-Severi lattice of some special K3 surfaces with $\mathbb{Z}_2^2$ symplectic action*, preprint (submitted), arXiv:1707.09732

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