UMass, Reading Seminar in Algebraic Geometry, Lattice theory, K3 surfaces, and the Torelli theorem

Luca Schaffler

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1 What is a lattice?

Definition 1.1. A *lattice* is a pair $(L, \cdot)$, where L is a finitely generated free abelian group and $\cdot\colon L \times L \to \mathbb{Z}$ such that $(v, w) \mapsto v \cdot w$ is a symmetric bilinear form.

- L is *even* if $v^2 := v \cdot v$ is an even number for all $v \in L$.
- The *Gram matrix* of L with respect to a basis $\{e_1, \ldots, e_n\}$ for L is the matrix of intersection $(e_i \cdot e_j)_{1 \leq i,j \leq n}$.
- The *rank* of L is defined to be the rank of one of its Gram matrices.
- A lattice L is *unimodular* if the determinant of one of its Gram matrices is ± 1. Equivalently, L of maximal rank is unimodular provided the *discriminant group* L^*/L is trivial (we let $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$).
- We can define the *signature* of L as the signature of one of its Gram matrices. By saying signature (a, b), we mean that a (resp. b) is the number of positive (resp. negative) eigenvalues. If L has signature (a, b), then we say that L is *indefinite* provided $a, b > 0$.

Example 1.2. $U = (\mathbb{Z}^2, \cdot)$ with $(v_1, v_2) \cdot (w_1, w_2) = v_1 w_2 + v_2 w_1$ is called the *hyperbolic plane*. Check that this is even, unimodular, and of signature $(1, 1)$.

Example 1.3. $E_8 = (\mathbb{Z}^8, B)$, where

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

This matrix can be reconstructed from the E_8 Dynkin diagram. The lattice E_8 is even, unimodular, and of signature $(0, 8)$.

Definition 1.4. Let L_1, L_2 be two lattices. We say that L_1, L_2 are *isometric* provided there exists a group homomorphism $f \colon L_1 \to L_2$ such that $f(v \cdot w) = f(v) \cdot f(w)$ for all $v, w \in L_1$.

Theorem 1.5. *[Mil58] Even indefinite unimodular lattices of the same signature are isometric.*

Example 1.6. Let L be an even unimodular lattice of signature $(1, 9)$. Then $L \cong U \oplus E_8$.

2 Examples of lattices from geometry

Let X be a complex n-dimensional manifold. Then we have the homology groups $H^r(X, \mathbb{Z})$, which recall are obtained from the chain complex

$$\ldots \to C_{r+1} \xrightarrow{\partial_{r+1}} C_r \xrightarrow{\partial_r} C_{r-1} \to \ldots,$$

where C_r is the free abelian group generated by the singular r-simplices. One defines $H^r(X, \mathbb{Z}) = (\ker \partial_r)/(\mathrm{im} \partial_{r+1})$. Recall that $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$.

By dualizing the above chain complex

$$\ldots \to C^*_{r-1} \xrightarrow{\delta_{r-1}} C^*_r \xrightarrow{\delta_r} C^*_{r+1} \to \ldots$$

we obtain the cohomology groups $H^r(X, \mathbb{Z}) = (\ker \delta_r)/(\mathrm{im} \delta_{r-1})$. Warning: in general, $H^r(X, \mathbb{Z}) \ncong H_r(X, \mathbb{Z})^*$. One of the advantages of cohomology insrtead of homology is that we have a *cup product* $\smile \colon H^a(X, \mathbb{Z}) \times H^b(X, \mathbb{Z}) \to H^{a+b}(X, \mathbb{Z})$. In particular, we have a $(-1)^n$-symmetric bilinear form

$$\smile \colon H^n(X, \mathbb{Z}) \times H^n(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z},$$

where the last isomorphism is true because X is orientable. Notice that $H^n(X, \mathbb{Z})$ may have torsion. So, if n is even, then $(H^n(X, \mathbb{Z}), \smile)$ is not necessarily a lattice. For a K3 surface X, we can say a lot about $(H^2(X, \mathbb{Z}), \smile)$.

Theorem 2.1. *For a K3 surface X, $(H^2(X,\mathbb{Z}),\smile)$ is a lattice isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$.*

Proof. There are different ingredients that contribute to the proof:

(1) By the universal coefficient theorem for cohomology, we have that

$$H^2(X,\mathbb{Z}) \cong H_2(X,\mathbb{Z})^* \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{Z}).$$

But $H^1(X,\mathbb{Z}) \cong \mathrm{Ab}(\pi_1(X)) \cong 0$, so $H^2(X,\mathbb{Z}) \cong H_2(X,\mathbb{Z})^*$, and hence $H^2(X,\mathbb{Z})$ is torsion-free. So $(H^2(X,\mathbb{Z}),\smile)$ is a lattice.

(2) The topological Euler characteristic $\sum_i (-1)^i \mathrm{rk}(H^i(X,\mathbb{Z}))$ is equal to 24. This can be computed by using the fact that all K3 surfaces are diffeomorphic to each other, and by computing the topological Euler characteristic for the double cover of $\mathbb{P}^2$ branched along a smooth sextic curve (which is an example of K3 surface). This implies that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$.

(3) $H^2(X,\mathbb{Z})$ is unimodular. To prove this, notice that

$$H^2(X,\mathbb{Z}) \cong H_2(X,\mathbb{Z})^* \cong H^2(X,\mathbb{Z})^*,$$

where the first isomorphism was discussed in part (1), and the second isomorphism is guaranteed by Poincaré duality. So the discriminant group $H^2(X,\mathbb{Z})^*/H^2(X,\mathbb{Z})$ is trivial, implying unimodularity.

(4) $H^2(X,\mathbb{Z})$ is even by Wu's formula. More precisely, let $w_2(T_X) \in H^2(X,\mathbb{Z}/2\mathbb{Z})$ be the second Stiefel-Whitney class of the tangent bundle of X. Let $v \in H^2(X,\mathbb{Z})$. Then Wu's formula says that

$$v^2 \equiv w_2(T_X)\cdot v \pmod 2.$$

But $w_2(T_X) \equiv c_1(T_X) \pmod 2$, and for a K3 surface $c_1(T_X) = 0$. So the form is even.

(5) The Hodge-Riemann bilinear relations imply that $(H^2(X,\mathbb{Z}),\smile)$ has signature $(3,19)$.

Notice that the lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ is also even, unimodular, and of signature $(3,19)$. So we can conclude that $(H^2(X,\mathbb{Z}),\smile)$ is isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$ by Theorem 1.5. □

Remark 2.2. Let X, Y be two K3 surfaces. Then we have an isometry

$$(H^2(X,\mathbb{Z}),\smile) \cong (H^2(Y,\mathbb{Z}),\smile).$$

The idea behind the Torelli theorem is that if the isometry above has an additional property, then $X \cong Y$ as complex manifolds.

3 The Torelli theorem

For a complex manifold X of dimension n, one can define the de Rham cohomology groups $H^r_{\mathrm{dR}}(X, \mathbb{C})$ of classes of closed complex valued differential r-forms. If p, q are positive integers such that $p + q = r$, then one can consider the vector subspace $H^{p,q}(X) \subseteq H^r_{\mathrm{dR}}(X, \mathbb{C})$ of classes of closed (p, q)-forms. In local coordinates $z_1, \ldots, z_n$, a (p, q)-form can be written as a $\mathbb{C}$-linear combination of terms of the form

$$f dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$$

where f is a complex C^∞ function. If X is a compact Kähler manifold (a smooth manifold with a Riemannian metric whose holonomy group is contained in $U(n)$), the Hodge decomposition says that

$$H^r_{\mathrm{dR}}(X, \mathbb{C}) \cong H^{r,0}(X) \oplus H^{r-1,1}(X) \oplus \ldots \oplus H^{1,r-1}(X) \oplus H^{0,r}(X).$$

Notice that

$$H^r_{\mathrm{dR}}(X, \mathbb{C}) \cong H^r(X, \mathbb{C}) \cong H^r(X, \mathbb{Z}) \otimes \mathbb{C}.$$

where the first isomorphism is the content of de Rham theorem.

Torelli Theorem 3.1 (Piatetski-Shapiro–Shafarevich (algebraic case), Burns-Rapoport (analytic case))**.** *Let X and Y be two K3 surfaces. Assume there is an isometry*

$$\psi\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

such that the induced isomorphism $H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C})$ preserves the Hodge decompositions (such ψ is called a Hodge isometry*). Then X and Y are isomorphic as complex manifolds. Moreover, if $\psi(\mathcal{K}_X) \cap \mathcal{K}_Y \neq \emptyset$, then there exists a unique isomorphism of complex manifolds $f\colon Y \to X$ such that $f^*\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ equals ψ.*

Remark 3.2.

- The above theorem is called *Torelli theorem*, but it is not due to Ruggiero Torelli. The name is because of its analogy with Torelli's theorem for curves.
- If $\psi\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is a Hodge isometry and $f\colon Y \to X$ is an isomorphism, then it is not necessarily true that $f^* = \psi$.
- $\mathcal{K}_X$ is the *Kähler cone* of X, which is the cone generated by the Kähler classes, i.e., the classes in $H^{1,1}(X)$ induced by the Kähler forms on X.

4 Application of the Torelli theorem

Theorem 4.1. *Let X be a K3 surface. Then* $\mathrm{Aut}(X)$ *is isomorphic to the group of Hodge isometries of* $H^2(X, \mathbb{Z})$ *such that* $\psi(\mathcal{K}_X) \cap \mathcal{K}_X \neq \emptyset$.

Proof. Let G be the group of Hodge isometries of $H^2(X, \mathbb{Z})$ such that $\psi(\mathcal{K}_X) \cap \mathcal{K}_X \neq \emptyset$. The map $\mathrm{Aut}(X) \to G$ such that $f \mapsto f^*$ is surjective by the Torelli theorem. For the injectivity see [Huy16, Chapter 15, Proposition 2.1]. □

References

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces.* Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016.

[Mil58] John Milnor. *On simply connected 4-manifolds.* 1958 Symposium internacional de topología algebraica International symposium on algebraic topology pp. 122–128 Universidad Nacional Autónoma de México and UNESCO, Mexico City.