Hyperbolic spaces and reflections

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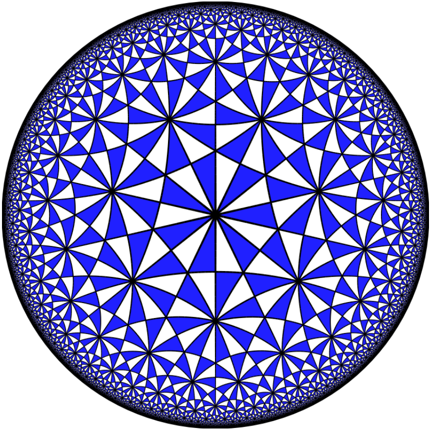
Motivation

This talk is about this kind of pictures!



Wikimedia Commons:

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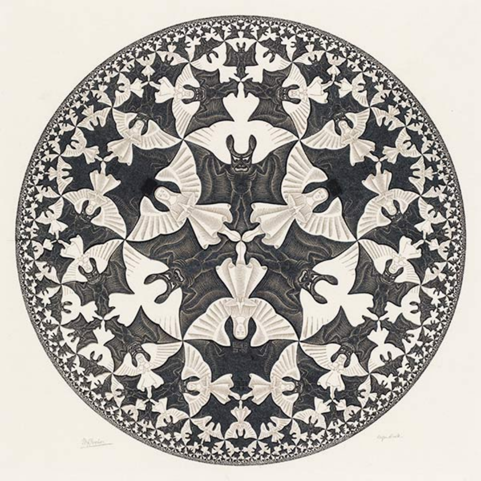
Motivation



Esher, M.C.: *Circle Limit IV*. From `http://www.mcescher.com`

Motivation

Question:

what is the mathematics behind these pictures?

To understand it, we need to start from *lattice theory*.

What is a lattice?

Definition

A *lattice* is a pair $(\mathbb{Z}^n, B)$, where $B \in M_n(\mathbb{Z})$ is nondegenerate and symmetric.

Observation: The matrix B defines the following symmetric bilinear form:

$$\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z},$$

$$(x, y) \mapsto (x_1, \ldots, x_n) B \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =: x \cdot y$$

Hyperbolic lattices

Given a lattice $(\mathbb{Z}^n, B)$, the eigenvalues of B are all nonzero real numbers. Therefore we can define the *signature* (p, q) of B, where

$$p = \text{number of positive eigenvalues of } B,$$
$$q = \text{number of negative eigenvalues of } B.$$

Definition

A lattice of signature $(1, q)$ with $q > 0$ is called *hyperbolic*.

Example

$\mathbb{Z}^3$ together with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is an example of hyperbolic lattice.

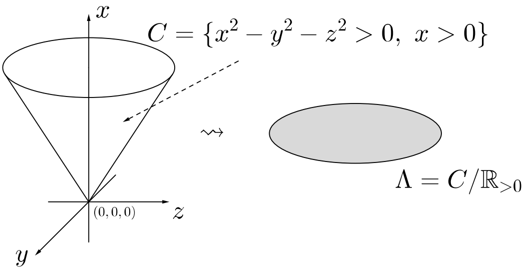
Hyperbolic spaces

Definition

Let $(\mathbb{Z}^n, B)$ be a lattice.

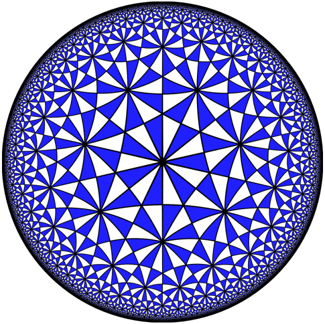
- $\{x \in \mathbb{R}^n \mid x \cdot x > 0\} = C \amalg C'$ (these are called *light cones*);
- Fix C. Then the *hyperbolic space* is defined to be

$$\Lambda = C/\mathbb{R}_{>0}.$$



Slicing the hyperbolic space

Now that we know what the hyperbolic space is, how can we subdivide it like this?



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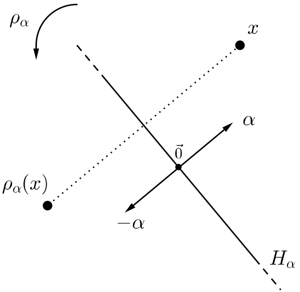
Reflections of hyperbolic lattices

Definition

Let $(\mathbb{Z}^n, B)$ be a lattice and let $\alpha \in \mathbb{Z}^n$ such that

$$\rho_\alpha \colon x \mapsto x - 2\frac{\alpha \cdot x}{\alpha \cdot \alpha}\alpha$$

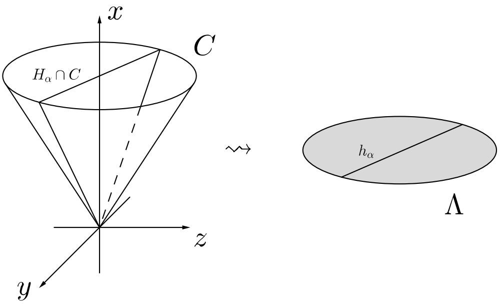
is defined over $\mathbb{Z}$. Then ρ_α is the *reflection with respect to* α.
The locus $H_\alpha \subset \mathbb{R}^n$ fixed by ρ_α is called the *mirror* of ρ_α.



Reflections of hyperbolic lattices

Observation

Given H_α such that $H_\alpha \cap C \neq \emptyset$, then H_α defines a hyperplane h_α in the hyperbolic space Λ.



Groups of reflections

Idea to create tilings of the hyperbolic space Λ:

- G = group generated by reflections;
- The mirrors of the reflections in G decompose Λ into G-equivalent cells!

Question: Given G, how to determine a G-cell in Λ?

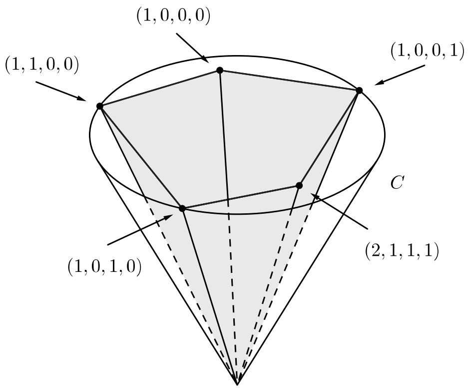
Answer: *Vinberg's algorithm*.

Concrete example

Lattice: $(\mathbb{Z}^4, \mathrm{diag}(1,-1,-1,-1))$;

Reflection group: $G = \langle \rho_\alpha \mid \alpha \cdot \alpha = -1 \rangle$;

G**-fundamental cell:**



Application to algebraic geometry

Input:

- $(\mathbb{Z}^{n+2}, B)$ lattice of signature $(2, n)$. If $\ell \cdot \ell = 0$, then $\ell^\perp/\ell$ is a hyperbolic lattice;
- $\{[v] \in \mathbb{P}(\mathbb{C}^{n+2}) \mid v \cdot v = 0 \text{ and } v \cdot \overline{v} > 0\} = \mathcal{D} \amalg \mathcal{D}'$;
- $\Gamma =$ group of isometries of the lattice $\implies \mathcal{D}/\Gamma$ is a quasi-projective variety (theorem of Baily-Borel);
- $\Sigma =$ data of "compatible" subdivisions of the light cones $C \subset \ell^\perp/\ell$ for all $\ell \in \mathbb{Z}^n$ such that $\ell \cdot \ell = 0$.

Output: $\mathcal{D}/\Gamma \subseteq \overline{\mathcal{D}/\Gamma}^{\Sigma}$ projective compactification, called *Looijenga's semitoric compactification*.

Thank you for your attention!

Main references:

1 Looijenga, E.: *Compactifications defined by arrangements. II. Locally symmetric varieties of type IV* (2003).

2 Vinberg, È.B.: *Some arithmetical discrete groups in Lobačevskiĭ spaces* (1975).

For the figures I used the software GeoGebra.