Moduli spaces and the Hilbert scheme

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Summary

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Caution!: Everything is up to some equivalence relation.

Call $F(B)$ the set of families over an object B of $\mathfrak{C}$. So a moduli problem is a controvariant functor:

$$F\colon \mathfrak{C}^{\mathrm{op}} \to \textbf{Sets}, \text{ s.t.}$$
$$B \mapsto F(B)$$

A concrete example. The counting functor:

$$\mathcal{C}\colon \textbf{Sets}^{\mathrm{op}} \to \textbf{Sets}, \text{ s.t.}$$
$$B \mapsto \{\text{morphisms over } B$$
$$\text{with finite fibers}\}$$

Definition of fine moduli space

Definition

A *fine moduli space* for a moduli problem $F\colon \mathfrak{C}^{\mathrm{op}} \to \textbf{Sets}$ is an object M of $\mathfrak{C}$ such that $F \cong Hom(-, M)$.

This is equivalent to the existence of a *universal family* $U \in F(M)$:

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & U \\ \downarrow & \text{pullback} & \downarrow \\ B & \xrightarrow{\exists!} & M \end{array}$$

Example of fine moduli space:

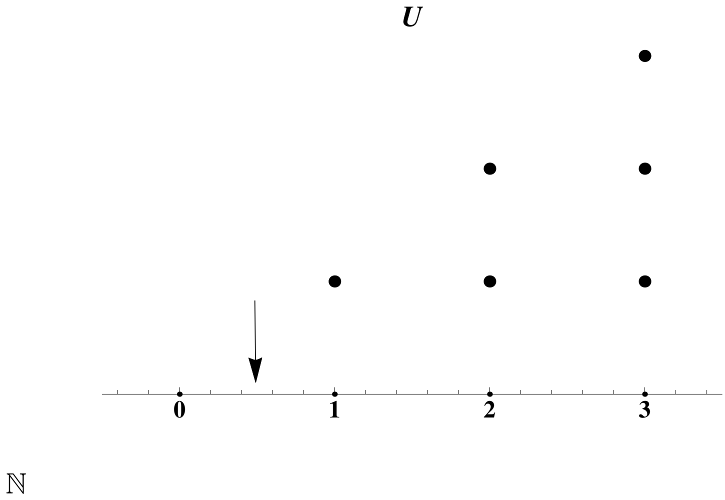
Proposition

$\mathcal{C} \cong \mathrm{Hom}(-, \mathbb{N})$.

It's quite easy to guess the universal family:

$$U = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), \ldots\}$$

$U \to \mathbb{N}$ is the projection on the first factor.


U
0
1
2
3
ℕ

The notion of Hilbert Polynomial

$X \subseteq \mathbb{P}^r$ closed

$$X \longrightarrow P_X \in \mathbb{Q}[z]$$

P_X reflects important properties of X

Natural question: Given $\mathbb{P}^r$ and $P \in \mathbb{Q}[z]$ we want to understand all closed $X \subseteq \mathbb{P}^r$ with $P_X = P$...

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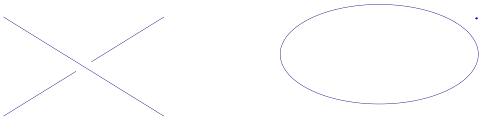
Theorem (Grothendieck)

$\mathcal{H}ilb_P^r \cong \mathrm{Hom}(-, Hilb_P^r)$.

Properties of the Hilbert scheme

1. $Hilb^r_P$ is projective;
2. $Hilb^r_P$ is connected (Hartshorne, Ph.D. Thesis, 1963);
3. $Hilb^3_{14z-23}$ is not reduced (Mumford, 1962);
4. $Hilb^r_P$ may not be irreducible. Consider $Hilb^3_{2z+2}$:



Proof of the existence of the Hilbert scheme

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$$\mathcal{H}ilb_P^r \quad \rightarrow \quad Hom(-, Gr(P(m), \binom{m+r}{r}))$$

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$$\begin{array}{ccc} \mathcal{H}ilb^r_P & \to & Hom(-, Gr(P(m), \binom{m+r}{r})) \\ & \searrow & \uparrow \\ & & Hom(-, S) \end{array}$$

Thank you for your attention!

Main references:

- J. Harris, I. Morrison. *Moduli of Curves*, Graduate Texts in Mathematics, Springer, 187.
- J. Kock, I. Vainsencher. *An Invitation to Quantum Cohomology*, Progress in Mathematics, Birkhäuser, 249.
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