Polyhedral subdivisions of the unit cube

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Motivation

By the unit cube we mean $[0,1]^3 \subset \mathbb{R}^3$.

Why do I care about subdividing it?

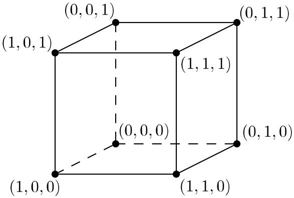
$$\text{Unit cube} \xleftrightarrow{\text{Toric Geometry}} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

I am interested in how $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ can degenerate, and its degenerations are described by the polyhedral subdivisions of the unit cube.

Some terminology

Definition

A *polytope* $Q \subset \mathbb{R}^k$ is the convex hull of a given finite set of points $A \subset \mathbb{R}^k$. The pair (Q, A) is called *marked polytope*.



Definition

Let Q be a polytope. If $\dim(Q) = \#\text{vertices} - 1$, then Q is called a *simplex* (e.g. a point, a segment, a triangle, a tetrahedron...).

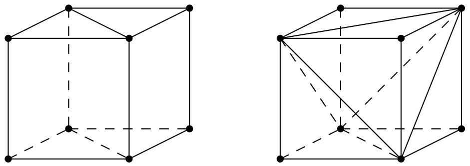
Some terminology

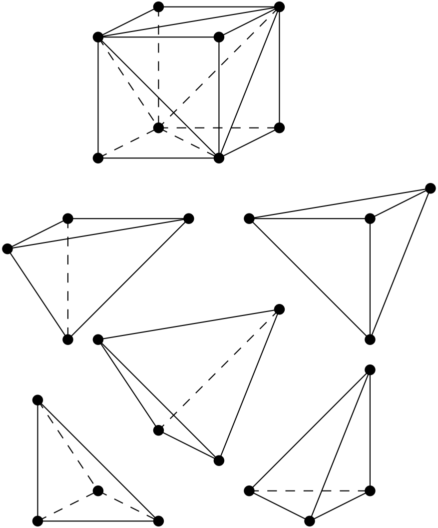
Definition

A *polyhedral subdivision* of a given a marked polytope (Q, A) is a finite collection $\mathcal{Q} = \{Q_i\}_i$ of polytopes such that:

- $\cup_i Q_i = Q$,
- $Q_i \cap Q_j$ is empty or a face of both Q_i and Q_j for all i, j,
- the vertices of Q_i are contained in A for all i.

A polyhedral subdivision which only uses simplices is called *triangulation*.





List of all the polyhedral subdivisions of the unit cube

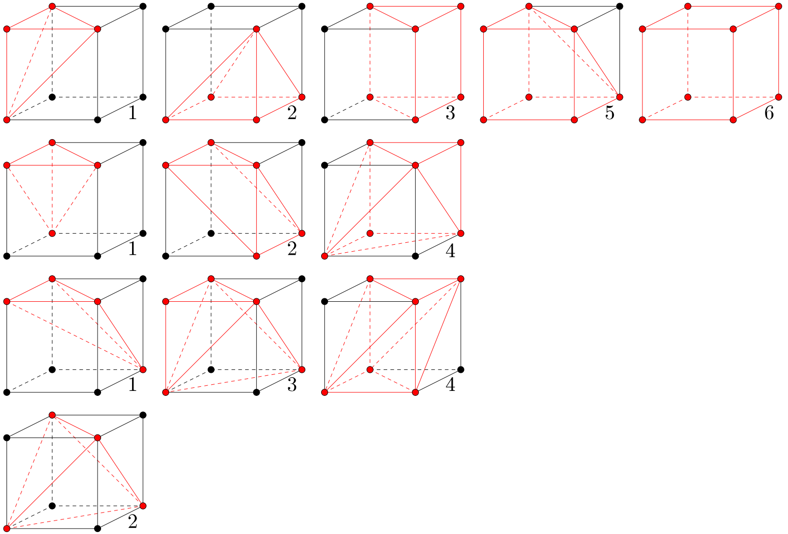
Hard to clearly enumerate them...

...unless we have a good organization!

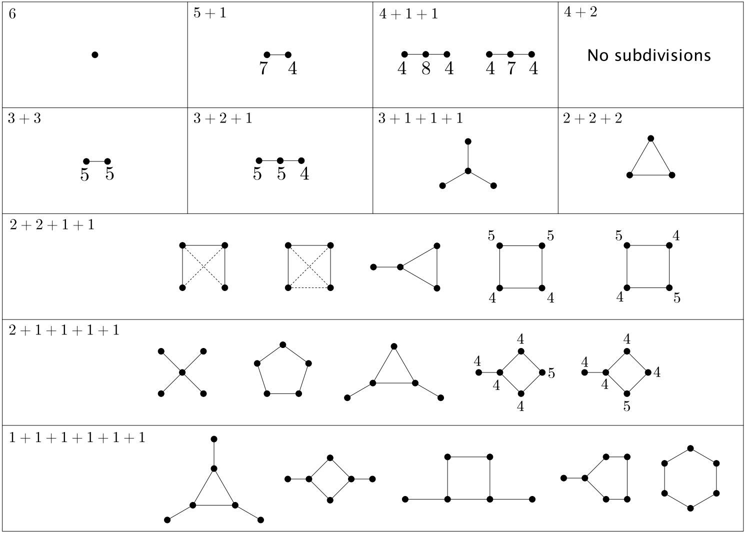
Plan:

- First we list all the possible sub-politopes of the marked unit cube;
- We consider the groups of sub-politopes whose sum of lattice volumes is 6 (which is the lattice volume of the unit cube);
- We check which ones of these groups actually give rise to subdivisions.

List of subpolitopes

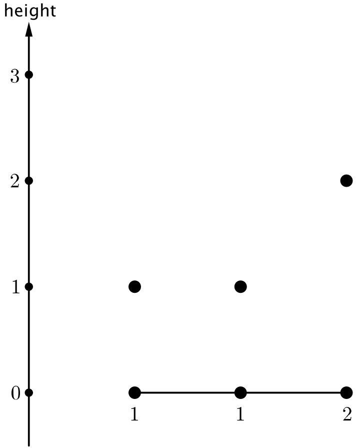


Summary of all the subdivisions



Why do I care? If Q is the unit cube, then the regularity of its subdivisions gives information about the irreducibility of the moduli space of stable toric pairs $\overline{M}_Q$.

We define what a regular subdivision is via examples.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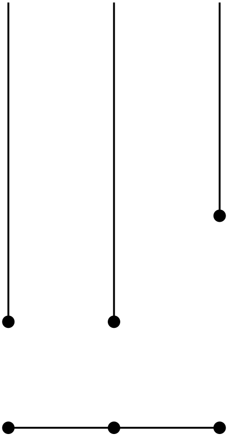
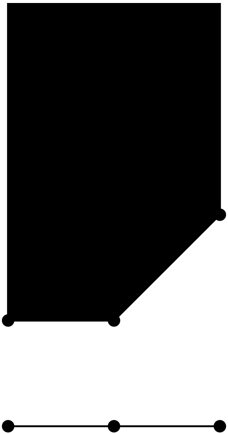
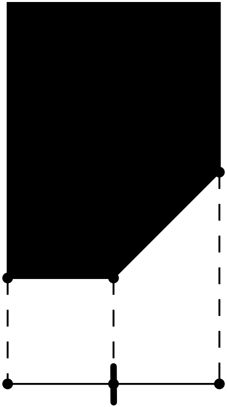






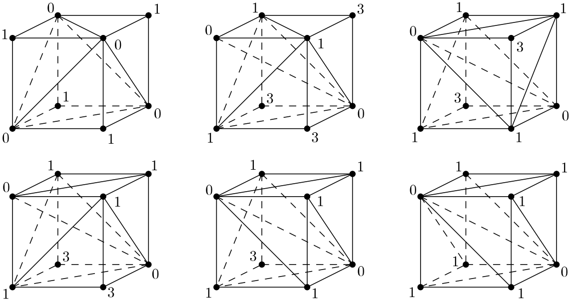
Regularity of the subdivisions of the unit cube

Theorem

The polyhedral subdivisions of the unit cube are regular.

Proof.

The triangulations are regular (see [2]), and this implies that all the other subdivisions are regular (see [4]). □



Cubes beyond 3 dimensions

Theorem (Cottle, 1982)

The 4-dimensional unit cube cannot be triangulated with fewer than 16 simplices.

Theorem (De Loera, 1996)

The 4-dimensional unit cube has a non-regular triangulation.

Theorem (Huggins, Sturmfels, Yu, Yuster, 2006)

The 4-dimensional unit cube has 87959448 regular triangulations, partitioned in 235277 symmetry classes.

Thank you for your attention!

Main references:

1. V. Alexeev: *Complete moduli in the presence of semiabelian group action*.
2. J. De Loera, J. Rambau, F. Santos: *Triangulations: Structures for Algorithms and Applications*.
3. I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky: *Discriminants, Resultants, and Multidimensional Determinants*.
4. F. Santos: *On the refinements of a polyhedral subdivision*.

For the figures I used the software GeoGebra.