Flag Varieties

Luca Schaffler

December 2, 2013

1 A brief review about Grassmannians

Fix two integers k, n with $0 \leq k \leq n$. The Grassmannian $\mathrm{Gr}(k,n)$ is defined (set theoretically) as the collection of all k-dimensional vector subspaces of $\mathbb{C}^n$. The Grassmannian $\mathrm{Gr}(k,n)$ is more than just a set as we are going to explain now. It's possible to define a map:

$$\varphi\colon \mathrm{Gr}(k,n) \to \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$$

in the following way. Given $\Lambda \in \mathrm{Gr}(k,n)$, we can take a basis of Λ, say $\{v_1, \ldots, v_k\}$. Then define:

$$\varphi(\Lambda) := [v_1 \wedge \ldots \wedge v_k].$$

This map doesn't depend on the choice of the basis and it's injective. Moreover its image is a Zariski closed subset of $\mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$ (see [J, Lecture 6]). From this map, which is called the Plücker embedding, $\mathrm{Gr}(k,n)$ inherits the structure of a projective algebraic variety.

About this structure of algebraic variety, we will just remark that:

$$\dim(\mathrm{Gr}(k,n)) = k(n-k).$$

Proving this fact is an easy exercise: dimension k subspaces of $\mathbb{C}^n$ correspond to $\mathbb{C}$-linear surjections $\mathbb{C}^n \to \mathbb{C}^{n-k}$. To conclude we observe that the linear space of $(n-k)\times n$ complex matrices of maximum rank depends on $k(n-k)$ parameters.

2 Flag varieties: a geometric description

Definition 1. Let V be an finite dimensional complex vector space. A *flag* in V is a strictly increasing sequence of vector subspaces:

$$\{0\} \subsetneq \Lambda_1 \subsetneq \ldots \subsetneq \Lambda_\ell \subsetneq V.$$

The *signature* of the flag is defined to be the sequence $(\dim(\Lambda_1), \ldots, \dim(\Lambda_\ell), \dim(V))$.

Now take $a_1, \dots, a_\ell, n$ integers with $0 < a_1 < \dots < a_\ell < n$. Define $\mathbb{F}(a_1, \dots, a_\ell; n)$ to be the set of all possible flags in $\mathbb{C}^n$ with signature $(a_1, \dots, a_\ell, n)$. Observe that $\mathbb{F}(a_1, \dots, a_\ell; n)$ is contained in $\mathrm{Gr}(a_1, n) \times \dots \times \mathrm{Gr}(a_\ell, n)$ and, in the case $\ell = 1$, $\mathbb{F}(a_1; n) = \mathrm{Gr}(a_1, n)$.

As in the case of the Grassmannian variety, also $\mathbb{F}(a_1, \dots, a_\ell; n)$ has the structure of a projective variety.

Proposition 1. $\mathbb{F}(a_1, \dots, a_\ell; n)$ *is a Zariski closed subset of* $\mathrm{Gr}(a_1, n) \times \dots \times \mathrm{Gr}(a_\ell, n)$.

Proof. We already know this for $\ell = 1$. Assume this is true for $\ell = 2$. For any $1 \leq i < j \leq \ell$, let π_{ij} be the restriction to $\mathbb{F}(a_1, \dots, a_\ell; n)$ of the projection $\mathrm{Gr}(a_1, n) \times \dots \times \mathrm{Gr}(a_\ell, n) \to \mathrm{Gr}(a_i, n) \times \mathrm{Gr}(a_j, n)$. Then:

$$\mathbb{F}(a_1, \dots, a_\ell; n) = \bigcap_{1 \leq i < j \leq n} \pi_{ij}^{-1}(\mathbb{F}(a_i, a_j; n)),$$

and we are done.

The $\ell = 2$ case is stated in [J, Lecture 8] as an exercise. □

Now that we know that $\mathbb{F}(a_1, \dots, a_\ell; n)$ has the structure of a variety, we will call it a *flag variety*. We can compute the dimension of this variety recursively as follows: let π_1 be the restriction to $\mathbb{F}(a_1, \dots, a_\ell; n)$ of the first projection map $\mathrm{Gr}(a_1, n) \times \dots \times \mathrm{Gr}(a_\ell, n) \to \mathrm{Gr}(a_1, n)$. π_1 is obviously surjective and the generic fiber is isomorphic to $\mathbb{F}(a_2 - a_1, \dots, a_\ell - a_1; n - a_1)$. Therefore, using what we know about the dimension of the Grassmannian variety:

$$\dim(\mathbb{F}(a_1, \dots, a_\ell; n)) = \dim(\mathrm{Gr}(a_1, n)) + \dim(\mathbb{F}(a_2 - a_1, \dots, a_\ell - a_1; n - a_1)) =$$

$$= a_1(n - a_1) + \dim(\mathbb{F}(a_2 - a_1, \dots, a_\ell - a_1; n - a_1)) = \dots$$

$$\dots = \sum_{i=1}^{\ell} (a_i - a_1 - \dots - a_{i-1})(n - a_i).$$

3 Flag varieties and algebraic groups

3.1 Quotients of algebraic groups

Here we give a second description of flag varieties by means of the theory of algebraic groups. Our base field will be the complex numbers. We recall that an algebraic group is an affine group scheme. A group scheme G is a scheme together with morphisms $m \colon G \times_{\mathbb{C}} G \to$

G, $i\colon G \to G$ and $\epsilon\colon \mathrm{Spec}(\mathbb{C}) \to G$ that satisfy some commutative diagrams which reflect the group axioms. For more details about schemes and group schemes see [Ha, Chapter II] and [MFK, Chapter 0].

If we have a group scheme G acting on a scheme X, it will be very important to know how to take the quotient of X modulo the action of G in a meaningful way. For example we would like our quotient to be again a scheme. We give the following definition.

Definition 2. Let G be a group scheme acting on X. A *categorical quotient of X by G* is a pair (Y, ϕ) where Y is a scheme and $\phi\colon X \to Y$ is a morphism such that:

(1) the following diagram commutes

$$\begin{array}{ccc} G \times_{\mathbb{C}} X & \xrightarrow{\sigma} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[\phi]{} & Y \end{array}$$

where σ is the action morphism and p_2 is the usual projection on X.

(2) if (Y', ϕ') is a second pair satisfying (1), then there exists a unique morphism $\psi\colon Y \to Y'$ such that $\phi' = \psi \circ \phi$.

The categorical quotient may not exist, but if it does, it's easy to see it is unique. Luckily, categorical quotients exist in the case that interests us.

The case we care about is the following. Assume G is an algebraic group and let H be a subgroup of G, i.e. a closed subscheme which is also a subgroup by looking at the group structure of G. Then obviously we have an action of H on G. The result is

Theorem 1. *The categorical quotient of G by H exists. It will be denoted by $(G/H, \pi)$.*

Proof. See [CH, Theorem 3.7.7]. □

It's pretty natural to ask what happens if we quotient our algebraic group G by a normal subgroup N.

Theorem 2. *The categorical quotient G/N is an algebraic group.*

Proof. See [CH, Corollary 3.7.4]. □

3.2 Special subgroups of an algebraic group

The goal here is to define *Borel* subgroups and *parabolic* subgroups of an algebraic group G and state their first properties. Later we will use them to describe flag varieties.

Definition 3. A *Borel* subgroup B of an algebraic group G is a maximal subgroup among the ones that are connected and solvable.

Theorem 3. *The Borel subgroups of an algebraic group G are all conjugate and, given a Borel subgroup B, G/B is projective.*

Proof. See [Hu, 21.3]. □

Definition 4. A *parabolic* subgroup P of an algebraic group G is any subgroup such that G/P is projective. Alternately, a parabolic subgroup is any subgroup containing a Borel subgroup.

Observation 1. Borel subgroup ⇒ parabolic subgroup.

The following fact is claimed in [FH, page 384].

Theorem 4. *Let B be a Borel subgroup and P be a parabolic subgroup of an algebraic group G. Then there exists an $x \in G$ such that:*

$$B \subseteq xPx^{-1}.$$

In group theory, given a group G and a subgroup H, the normalizer of H in G is the biggest subgroup of G containing H in which H is normal. It's denoted by $N_G(H)$.

Theorem 5. *Let B be a Borel subgroup of an algebraic group G. Then $N_G(B) = B$.*

Proof. See [Hu, 23.1, 23.2]. □

It follows from the previous theorem that, in general, Borel subgroups are not normal. Moreover, we have:

Corollary 1. *Let P be a parabolic subgroup of an algebraic group G. Then $P = N_G(P)$. In particular P is connected.*

Proof. See [Hu, 23.1]. □

3.3 Flag varieties as quotients of algebraic groups

Here we will finally establish the connection between flag varieties and quotients by parabolic subgroups. First we give a definition.

Definition 5. Let V be a finite dimensional vector space. A *full flag* in V is a flag with signature $(1, 2, \ldots, \dim(V))$. Given a positive integer n, $\mathbb{F}(1, \ldots, n-1; n)$ is called a *full flag variety.*

So take a positive integer n. Let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbb{C}^n$. Call F the full flag $\{0\} \subsetneq \{e_1\} \subsetneq \{e_1, e_2\} \subsetneq \ldots \subsetneq \{e_1, \ldots, e_n\}$. By change of basis, it's obvious that any other possible full flag is obtained from F by the action of GL_n.
But there's something to notice: different elements of GL_n can possibly give the same full flag. Let's explore this fact. Consider the Borel subgroup B of GL_n given by the upper triangular matrices (the fact that B is a Borel subgroup is left as an exercise). Given any full flag it's straightforward to verify that B leaves it invariant. In conclusion we have that GL_n/B parametrizes all the possible full flags without repetitions. In other words we have that:

$$\mathbb{F}(1, \ldots, n-1; n) \cong \mathrm{GL}_n/B.$$

More generally, any flag variety $\mathbb{F}(a_1, \ldots, a_\ell; n)$ can be described as GL_n modulo an appropriate parabolic subgroup. More details about this can be found in [FH, page 96]. Conversely, it's not that hard to see that any parabolic subgroup of GL_n is the stabilizer of some flag. As a conclusion we can state the following

Theorem 6. *Let n be a positive integer. There is a one-to-one correspondence between parabolic subgroups of GL_n and flag varieties $\mathbb{F}(a_1, \ldots, a_\ell; n)$ for all possible choices of integers $\ell \geq 1$ and $0 < a_1 < \ldots < a_\ell < n$.*

3.4 Generalization of flag varieties

We just argued that the study of flag varieties (as we meant in section 1) is equivalent to the study of quotients of GL_n by its parabolic subgroups. But what if instead of GL_n we pick any algebraic group G? From now on we will adopt a more general definition of flag variety, which is the following.

Definition 6. A *flag variety* is the quotient of an algebraic group by a parabolic subgroup. We will call it a *full flag variety* if we are quotienting by a Borel subgroup.

In the next section, we are going to study the cohomology of invertible sheaves on flag varieties (with some additional hypotheses).

4 Cohomology of flag varieties

4.1 Some more definitions and facts

Proposition 2. *Any algebraic group has a unique largest normal solvable subgroup.*

Proof. See [Hu, 19.5]. □

Definition 7. Let G be an algebraic group. The *radical* of G, denoted by $R(G)$, is defined to be the identity component of the unique largest normal solvable subgroup.

Definition 8. Assume G is a nontrivial connected algebraic group. Then G will be called *semisimple* if $R(G)$ is trivial.

We recall now the following fact from algebraic groups (a reference will be [M, Chapter 4, Section 2]). Take an algebraic group G. Let $\mathrm{Lie}(G)$ be the tangent space of G at the identity. $\mathrm{Lie}(G)$ is contained in the distribution algebra of G, denoted by $\mathcal{H}(G)$. The product in $\mathcal{H}(G)$ is called the convolution product. From $\mathcal{H}(G)$ we have that $\mathrm{Lie}(G)$ inherits the structure of a Lie algebra with the commutator bracket. We have the following result.

Theorem 7. *If G is a semisimple algebraic group then* $\mathrm{Lie}(G)$ *is a semisimple Lie algebra.*

Proof. See [Hu, 13.5]. Here actually more is proved, but we won't need more than what we stated. □

So now take a semisimple algebraic group G. $\mathrm{Lie}(G)$ is a semisimple Lie algebra, and we know everything about it: take a Cartan subalgebra $\mathfrak{h}$, let Φ be the corresponding root system, fix a base $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$ and let $\mathcal{W}$ be the Weyl group. Moreover we can consider all the integral weights ($\lambda \in \mathfrak{h}^*$ s.t. $\lambda(\alpha_i) \in \mathbb{Z}\ \forall i$) and the integral dominant weights (λ such that $\lambda(\alpha_i)$ is a nonnegative integer $\forall i$). Let's recall how the Weyl group acts on integral weights. To do this, it is enough to see how reflection act on an integral weight. So let $\alpha \in \Phi$ and let λ be an integral weight. Then:

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Define also the following action of $\mathcal{W}$ on the integral weights. If $w \in \mathcal{W}$ and λ is an integral weight, let:

$$w \bullet \lambda := w(\lambda + \rho) - \rho,$$

where ρ is half of the sum of all positive roots.

4.2 Invertible sheaves on a full flag variety

Consider a semisimple algebraic group G and let B be a Borel subgroup. So we have a full flag variety G/B. Let λ be an integral weight corresponding to the Lie algebra $\mathrm{Lie}(G)$. Let $\mathbb{C}$ be a B module under the following action. If $b \in B$ and $z \in \mathbb{C}$, set:

$$b \cdot z := \lambda(b)z$$

(here there's a subtle identification of λ with a morphism of algebraic groups $B \to \mathbb{G}_m = \mathbb{C}^*$ that I intentionally skip). So we have a line bundle:

$$G \times_B \mathbb{C}_{-\lambda} \to G/B.$$

Call $\mathcal{L}(\lambda)$ the corresponding invertible sheaf in $\mathrm{Pic}(G/B)$. The important statement here is that any invertible sheaf on G/B can be obtained in this way (this fact is stated in [L, page 1]).

4.3 Borel-Bott-Weyl theorem

Recall the following important fact from Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra and let λ be a dominant weight. Then there is a unique (up to isomorphism) finite dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ which has highest weight λ (see [EW, Theorem 15.5]). Now we are ready to state our main theorem.

Borel-Bott-Weil theorem. *Let G be a semisimple algebraic group, $B \subseteq G$ a Borel subgroup and let λ be an integral dominant weight. Let w be an element of the Weyl group $\mathcal{W}$. Then:*

$$H^p(G/B, \mathcal{L}(w \bullet \lambda)) \cong \begin{cases} V(\lambda)^* \text{ if } p = \ell(w), \\ 0 \text{ otherwise.} \end{cases}$$

Proof. See [K, Theorem 7]. Recall that $\ell(w)$ is the length of w, i.e. the minimal amount of reflections whose composition is w. □

Reality check: If w and λ are given in such a way that $\mathcal{L}(w \bullet \lambda) \cong \mathscr{O}_{G/B}$ (the structure sheaf over G/B), then we expect the BBW theorem to give us $H^0(G/B, \mathcal{L}(w \bullet \lambda)) \cong \mathbb{C}$ since G/B is projective (here we just care about isomorphism as complex vector spaces). Let's assume $w = \mathrm{id}$. Therefore we need to find λ such that $\mathcal{L}(\lambda) \cong \mathscr{O}_{G/B}$. This happens when $\lambda = 0$. Therefore the BBW theorem tells us that $H^0(G/B, \mathscr{O}_{G/B}) \cong V(0)^*$. Therefore we need to check that $V(0)^*$ is isomorphic to $\mathbb{C}$ as complex vector spaces. But $\mathbb{C}$ as a $\mathrm{Lie}(G)$-module with the trivial representation is definitely irreducible with only weight 0. By uniqueness of $V(0)$, we argue that $V(0) \cong \mathbb{C} \Rightarrow H^0(G/B, \mathscr{O}_{G/B}) \cong V(0)^* \cong \mathbb{C}^* \cong \mathbb{C}$.

Acknowledgements: I am grateful to Will Hardesty for all his explanations and suggestions.

References

[CH] T. Crespo, Z. Hajto. *Algebraic Groups and Differential Galois Theory*, Volume 122 of Graduate studies in mathematics, American Mathematical Soc., 2011.

[EW] K. Erdmann, M. Wildon. *Introduction to Lie Algebras*, Springer undergraduate mathematics series, 2006.

[FH] W. Fulton, J. Harris. *Representation Theory: A First Course*, Volume 129 of Graduate texts in mathematics: Readings in mathematics, Springer study edition, 1991.

[Ha] R. Hartshorne. *Algebraic Geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, 52.

[Hu] J. Humphreys. *Linear Algebraic Groups*, Volume 21 of Graduate Texts in Mathematics, Springer, 1975.

[J] J. Harris. *Algebraic Geometry: A First Course*, Springer-Verlag, New York, 1992, Graduate Texts in Mathematics, 133.

[K] S. Kumar. *Geometry of Schubert varieties and Demazure character formula*, lectures by Shrawan Kumar during April, 2011 Hausdorff Research Institute for Mathematics Bonn, Germany notes written by Brandyn Lee.

[L] Z. Lin. *The structure of cohomology of line bundles on the flag varieties for some groups of rank 2*, Ph.D. Dissertation, U. Massachusetts, 1989.

[M] S. Mukai. *An Introduction to Invariants and Moduli*, Volume 81 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2003.

[MFK] D. Mumford, J. Fogarty, F. Kirwan. *Geometric Invariant Theory*, Springer, 1994, Vol. 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete: a series of modern surveys in mathematics.