Mustafin varieties and moduli of points in $\mathbb{P}^2$.

(Joint with J. Tevelev)

§ Goals.

1. Mustafin var's, examples, properties.
2. Mustafin var's & degen's of $\mathbb{P}^2$ with n pts.
3. Moduli space of n pts in $\mathbb{P}^2$ & comp's.

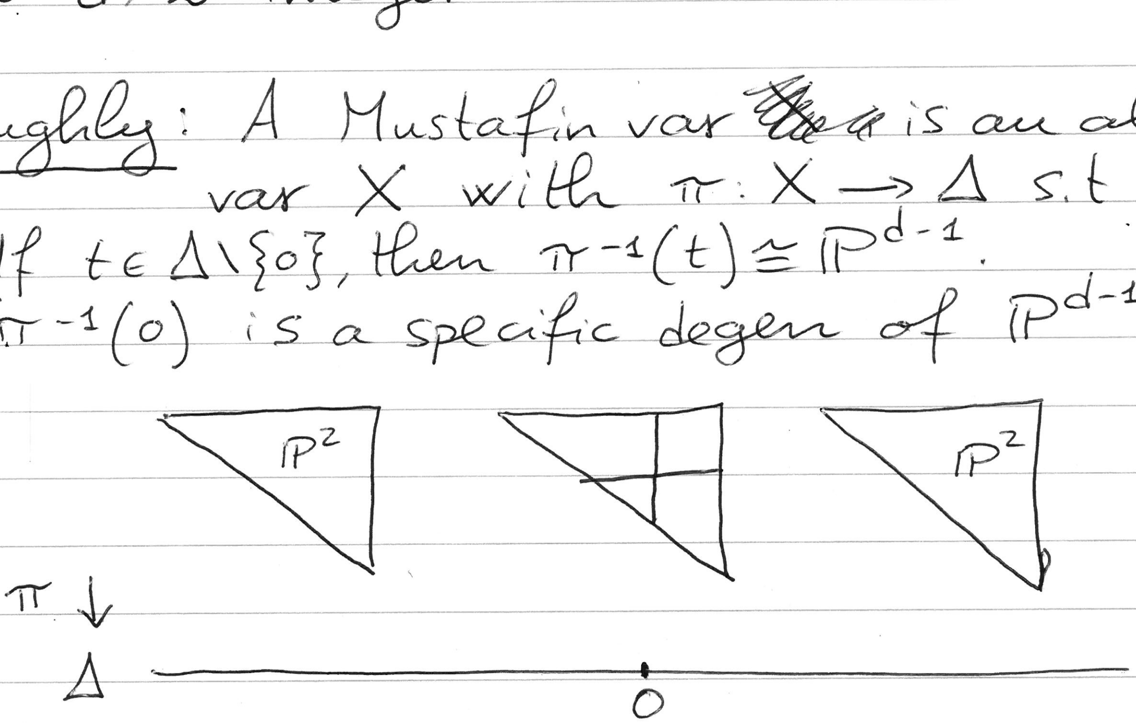
§ Intro.

$\Delta = \mathrm{Spec}(R)$, $R = \mathbb{C}[[t]]$.
Think of Δ as a small closed disk in $\mathbb{C}$ centered at 0.

Let $d \geq 2$ integer

Roughly: A Mustafin var ~~iety~~ is an alg var X with $\pi : X \to \Delta$ s.t.

1. If $t \in \Delta \setminus \{0\}$, then $\pi^{-1}(t) \cong \mathbb{P}^{d-1}$.
2. $\pi^{-1}(0)$ is a specific degen of $\mathbb{P}^{d-1}$.



Simplest way to realize this:

$K = Q(R)$, $L \subseteq K^d$ free R-submodule of rank d (L is called <u>lattice</u>).

$\pi : \mathbb{P}(L) \to \Delta$ can be thought of as a 1-param family of $\mathbb{P}^{d-1}$.

<u>Rmks</u>:

1. Canonical identification

$$\mathbb{P}(L) \setminus \pi^{-1}(0) \cong \mathbb{P}^{d-1} \times (\Delta \setminus \{0\}) = \mathbb{P}^{d-1}_K$$

2. Mustafin var's generalize $\mathbb{P}(L)$ to $\mathbb{P}(L_1, \dots, L_m)$.

<u>Def</u>: $\overbrace{\{L_1, \dots, L_m\}}^{=\Sigma}$ lattices. Mustafin var (joins)

$$\mathbb{P}(\Sigma) := \text{closure of } \mathbb{P}^{d-1}_K \subseteq \mathbb{P}(L_1) \times_R \cdots \times_R \mathbb{P}(L_m).$$

<u>Ex 1</u>: e_1, e_2, e_3 canonical basis of K^3.

$$L_1 = e_1 R + e_2 R + e_3 R \,, \quad L_2 = e_1 R + e_2 R + t e_3 R \,.$$

$\Sigma = \{L_1, L_2\}$. Compute central fiber of $\mathbb{P}(\Sigma)$.

<u>Solve</u>:

<u>Step 1</u>. x_1, x_2, x_3 dual to e_1, e_2, e_3
y_1, y_2, y_3 dual to $e_1, e_2, t e_3$.

Ideal defining the generic fiber of $\mathbb{P}(\Sigma)$ in $\mathbb{P}(L_1)\times_R \mathbb{P}(L_2)$ is

$$I_{2\times 2}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & ty_3 \end{pmatrix}$$

Note: $x_3 = ty_3$

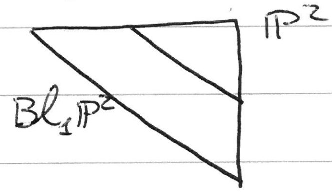
<u>Step 2</u>. Limit for $t\to 0$ in $\mathbb{P}^2_x \times \mathbb{P}^2_y$ is cut out by

$$\begin{cases} x_1y_2 - x_2y_1 = 0 \\ tx_1y_3 - x_3y_1 = 0 \\ tx_2y_3 - x_3y_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1y_2 - x_2y_1 = 0 \\ x_3y_1 = 0 \\ x_3y_2 = 0 \end{cases}$$

Careful: saturate ideal w.r.t. t.

<u>Step 3</u>. The irr. comp's in $\mathbb{P}^2_x\times\mathbb{P}^2_y$ are

$$V(y_1, y_2) \cong \mathbb{P}^2, \quad V(x_3, x_1y_2 - x_2y_1) \cong Bl_1\mathbb{P}^2.$$



<u>Ex 2</u>. Same as above, but

$$L_2 = e_1R + te_2R + t^2e_3R.$$

Compute central fiber of $\mathbb{P}(\Sigma)$.

<u>Solve</u>:

<u>Step 1</u>. y_1, y_2, y_3 dual to e_1, te_2, t^2e_3

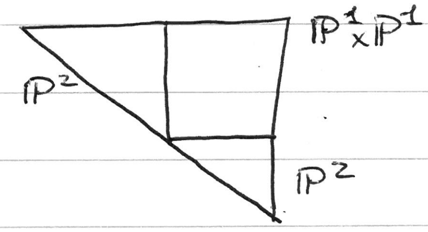
$$I_{2\times 2}\begin{pmatrix} x_1 & y_1 \\ x_2 & ty_2 \\ x_3 & t^2y_3 \end{pmatrix}$$

<u>Step 2</u>. Limit $t\to 0$

$$\begin{cases} tx_1y_2 - x_2y_1 = 0 \\ t^2x_1y_3 - x_3y_1 = 0 \\ t^{\cancel{2}}x_2y_3 - \cancel{t}x_3y_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2y_1 = 0 \\ x_3y_1 = 0 \\ x_3y_2 = 0 \end{cases}$$

Step 3. $V(x_2, x_3) \cong \mathbb{P}^2$, $V(x_3, y_1) \cong \mathbb{P}^1 \times \mathbb{P}^1$, $V(y_1, y_2) \cong \mathbb{P}^2$.

Rmk: Cartwright-Häbich-Sturmfels-Werner.



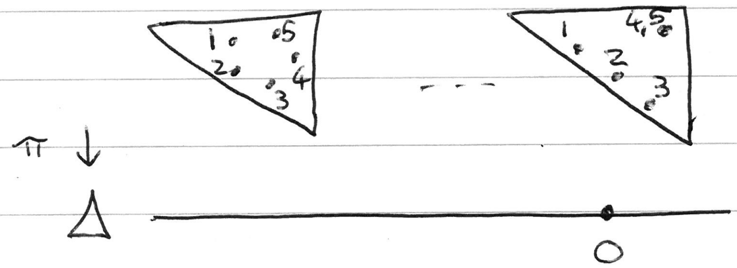
- $\mathbb{P}(\Sigma)$ is integral, normal, CM.
- $\pi^{-1}(0)$ is reduced, connected, CM, ~~and irr~~ irr. comp's are rational.

§ Degen's of $\mathbb{P}^2$ with n pts.

1. $a_1(t), \dots, a_n(t) : \Delta \setminus \{0\} \to \mathbb{P}^2_K$ in g.l.p.

2. L lattice. $\bar{a}_1(t), \dots, \bar{a}_n(t) : \Delta \to \mathbb{P}(L)$.

3. $\bar{a}_1(0), \dots, \bar{a}_n(0) \in \pi^{-1}(0) = \mathbb{P}^2$.



4. L is stable w.r.t. $a = (a_1, \dots, a_n)$ if $\exists$ 4 pts among $\bar{a}_1(0), \dots, \bar{a}_n(0)$ in g.l.p.

5. $\Sigma_a = \{$stable lattice "classes" (scaling by $K \setminus \{0\}$)$\}$. $|\Sigma_a| < \infty$.

Consider $\mathbb{P}(\Sigma_a)$.

Rmk: $\pi : \mathbb{P}(\Sigma_a) \to \Delta$ has n sections.

$$\begin{tikzcd}
\mathbb{P}(\Sigma_a) \arrow[d, shift left, "\pi"] & & \Delta \setminus \{0\} \arrow[ll, "a_i"'] \arrow[dll, bend left] \\
\Delta \arrow[u, shift left, "\overline{a}_i"] & &
\end{tikzcd}$$

1. $\Delta \xrightarrow{\overline{a}_i} \mathbb{P}(\Sigma_a) \xrightarrow{\pi} \Delta$ is id.
2. $\overline{a}_1, \dots, \overline{a}_n$ are disjoint.
3. Thm (S.-Tevelev): $\overline{a}_1(0), \dots, \overline{a}_n(0) \in \pi^{-1}(0)$ are smooth pts.

§ Comp. moduli pts in $\mathbb{P}^2$.

$n = \#\text{pts} \geq 5$.

Param. space: $\mathcal{U} \subseteq (\mathbb{P}^2)^n$ n pts in g.l.p.
Mod. space: $B_n := \mathcal{U}/SL_3$.

Gerritzen-Piwek: Construct $B_n \subseteq \overline{B}_n$ comp'n, and claim its boundary param. degen's of $\mathbb{P}^2$ with n pts as above.

S.-Tevelev: $\overline{B}_n$ does not have the claimed modular interpret. We construct the correct comp'n $B_n \subseteq \overline{X}_{GP}(3,n)$ which admits such family

$$\begin{array}{c} \overline{X}_{GP}(3,n) \\ \downarrow \text{blow up} \\ \overline{B}_n \end{array}$$

In progress: ~~Study completely~~ Detailed study of $\overline{X}_{GP}(3,6)$.