A Pascal's theorem for rational normal curves

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Main goal & motivation

Pascal's theorem gives a synthetic geometric condition for six points in $\mathbb{P}^2$ to lie on a conic (see Figure 1 on the right). In higher dimension, one could ask: is there a coordinate-free condition for $d+4$ points in $\mathbb{P}^d$ to lie on a degree d rational normal curve (rnc)? In this work we find many of these conditions by writing in the Grassmann-Cayley algebra the defining equations of the parameter space of $d+4$ ordered points in $\mathbb{P}^d$ that lie on a rnc.

Grassmann-Cayley algebra

Def: The Grassmann-Cayley algebra of a vector space V is its exterior algebra together with the meet and join operations: $\wedge$, $\vee$.

Ex: $a, b, c \in \mathbb{P}^2$ are aligned $\Leftrightarrow a \vee b \vee c = 0$.

Pascal's theorem: $a, b, c, d, e, f \in \mathbb{P}^2$ lie on a conic if and only if

$$\begin{vmatrix} a_0^2 & b_0^2 & c_0^2 & d_0^2 & e_0^2 & f_0^2 \\ a_1^2 & b_1^2 & c_1^2 & d_1^2 & e_1^2 & f_1^2 \\ a_2^2 & b_2^2 & c_2^2 & d_2^2 & e_2^2 & f_2^2 \\ a_0a_1 & b_0b_1 & c_0c_1 & d_0d_1 & e_0e_1 & f_0f_1 \\ a_0a_2 & b_0b_2 & c_0c_2 & d_0d_2 & e_0e_2 & f_0f_2 \\ a_1a_2 & b_1b_2 & c_1c_2 & d_1d_2 & e_1e_2 & f_1f_2 \end{vmatrix} = 0.$$

This can be rewritten (!) as

$$|abc||ade||bdf||cef| - |abd||ace||bcf||def| = 0.$$

In the Grassmann-Cayley algebra it becomes:

$$((a\vee b)\wedge(d\vee e))\vee((a\vee f)\wedge(d\vee c))\vee((e\vee f)\wedge(b\vee c)) = 0.$$

The geometric interpretation of the above expression gives Pascal's theorem (see Figure 1).

Cayley factorization problem: There is no general algorithm to rewrite a polynomial in the Grassmann-Cayley algebra.

Pascal's theorem

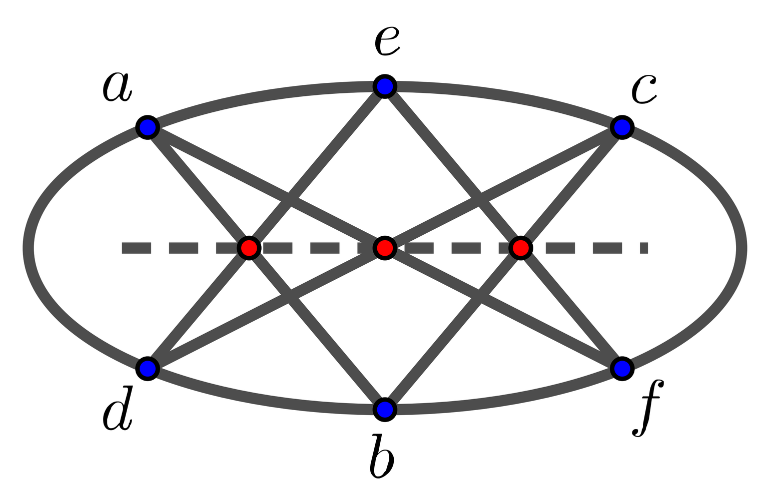

Figure 1: $a, \ldots, f \in \mathbb{P}^2$ lie on a conic if and only if $\overline{ab} \cap \overline{de}$, $\overline{af} \cap \overline{dc}$, $\overline{ef} \cap \overline{bc}$ are aligned.

$d+4$ points on degree d rnc

Def: $V_{d,n}$ = the closure of the locus in $(\mathbb{P}^d)^n$ of n-tuples of distinct points that lie on a rnc.

Idea: If we have the equations for $V_{d,d+4}$, then we could attempt to rewrite them in the Grassmann-Cayley algebra.

Thm [1]: $V_{d,d+4}$ union the locus of degenerate point configurations is cut out by the equations $\psi_I = 0$ for $I = \{i_1, \ldots, i_6\} \subseteq \{1, \ldots, d+4\}$, where ψ_I is obtained from

$$[i_1i_2i_3][i_1i_4i_5][i_2i_4i_6][i_3i_5i_6] - [i_1i_2i_4][i_1i_3i_5][i_2i_3i_6][i_4i_5i_6]$$

by operating the following substitution:

$$[i_\ell i_m i_n] \mapsto (-1)^{S(i_\ell, i_m, i_n)} |\{i_\ell i_m i_n\}^c|.$$

Main theorem (CS, 2019, [2])

Let $P_1, \ldots, P_{d+4} \in \mathbb{P}^d_{\mathbb{C}}$ be points in general linear position. Then $P_1, \ldots, P_{d+4}$ lie on a rational normal curve if and only if for every $I = \{i_1 < \cdots < i_6\} \subseteq \{1, \ldots, d+4\}$, $I^c = \{j_1 < \cdots < j_{d-2}\}$, the following $d+1$ points lie on a hyperplane:

- The intersection of the line $P_{i_1}P_{i_2}$ with the hyperplane $P_{i_4}P_{i_5}P_{j_1} \cdots P_{j_{d-2}}$;
- The intersection of the line $P_{i_2}P_{i_3}$ with the hyperplane $P_{i_5}P_{i_6}P_{j_1} \cdots P_{j_{d-2}}$;
- The intersection of the line $P_{i_3}P_{i_4}$ with the hyperplane $P_{i_1}P_{i_6}P_{j_1} \cdots P_{j_{d-2}}$;
- The points $P_{j_1}, \ldots, P_{j_{d-2}}$.

(See Figure 2 for a graphical visualization of this condition for $d = 3$ and $I = \{1, \ldots, 6\}$.)

Proof of main theorem

(1) Start from the Grassmann-Cayley algebra expression (some joins are omitted)

$$\begin{aligned} &(P_{i_1}P_{i_2} \wedge P_{i_4}P_{i_5}P_{j_1} \cdots P_{j_{d-2}}) \\ \vee&(P_{i_2}P_{i_3} \wedge P_{i_5}P_{i_6}P_{j_1} \cdots P_{j_{d-2}}) \\ \vee&(P_{i_3}P_{i_4} \wedge P_{i_6}P_{i_1}P_{j_1} \cdots P_{j_{d-2}}) \vee P_{j_1} \cdots P_{j_{d-2}} = 0. \end{aligned}$$

(2) Expand it using the definitions of $\wedge$ and $\vee$.

(3) Use appropriate syzygies to prove that what we obtained is equivalent to $\psi_I = 0$.

Rmk: We find many different expressions equivalent to (1), giving distinct reformulations of the main theorem.

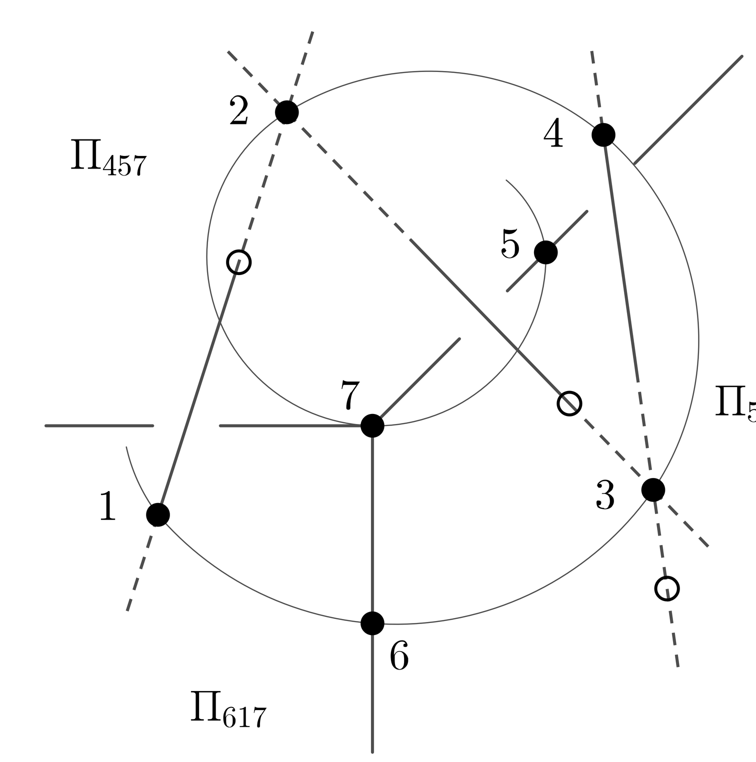

Figure 2: Seven points in $\mathbb{P}^3$ on a twisted cubic. The three circled points and P_7 are coplanar. Π_{ijk} denotes the plane containing P_i, P_j, P_k.

Application to twisted cubics

Thm (H. White, 1915): Let $P_1, \ldots, P_7$ be points on a twisted cubic. Let $H_1, \ldots, H_7$ be planes whose union contains the 21 lines spanned by the seven points. Then $H_1, \ldots, H_7$ osculate a second twisted cubic.

Thm (CS, 2019): With the above notation, the following planes intersect at a point contained in the plane H_7:

- $H_1 \cap H_2 + H_4 \cap H_5 \cap H_7$;
- $H_2 \cap H_3 + H_5 \cap H_6 \cap H_7$;
- $H_3 \cap H_4 + H_6 \cap H_1 \cap H_7$.

Future project

Consider the embedding $v\colon \mathbb{P}^3 \xrightarrow{\mathcal{O}(2)} \mathbb{P}^9$. $p_1, \ldots, p_{10} \in \mathbb{P}^3$ lie on a quadric $\Leftrightarrow$ the 10×10 determinant $|v(p_1) \cdots v(p_{10})|$ is zero. Rewriting this determinant in the Grassmann-Cayley algebra is called the *Turnbull-Young problem*. We plan to use the techniques we developed to work on this problem.

References

[1] A. Caminata, N. Giansiracusa, H.-B. Moon, and L. Schaffler. *Equations for point configurations to lie on a rational normal curve*. Adv. Math. 340 (2018), 653–683.

[2] A. Caminata and L. Schaffler. *A Pascal's theorem for rational normal curves*. Preprint (submitted), arXiv:1903.00460

[3] H. White. *Seven points on a twisted cubic curve*. Proceedings of the National Academy of Sciences of the United States of America, Vol. 1, No. 8 , pp. 464–466, 1915.

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