Graduate Students Seminar, The Arithmetic of Enriques Surfaces

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September 20, 2016

1 Introduction

Enriques surfaces are particular examples of 2-dimensional complex manifolds with very interesting geometric and arithmetic properties. After defining them, the main goal will be to describe the symmetric bilinear form given by the intersection pairing between curves on an Enriques surface.

2 The definition of Enriques surface

Definition 1. An *n-dimensional complex manifold* is a topological space M together with a collection of pairs $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ where

(i) $U_\alpha \subseteq M$ are open sets such that $\bigcup_\alpha U_\alpha = M$;

(ii) $\varphi_\alpha \colon U_\alpha \to B_\alpha$ are homeomorphisms to an open subset $B_\alpha \subseteq \mathbb{C}^n$ such that the compositions $(\varphi_\beta|_{U_\alpha \cap U_\beta}) \circ (\varphi_\alpha|_{U_\alpha \cap U_\beta})^{-1}$ are analytic isomorphisms for all α, β.

Example 1. the unit sphere $S^2 \subset \mathbb{R}^3$ is a 1-dimensional complex manifold.

Observation 1. An n-dimensional complex manifold has real dimension $2n$.

Definition 2. A 2-dimensional complex manifold will be called a surface, despite the fact that it has real dimension 4.

Given a complex manifold M, a *vector bundle of rank r over M* is a complex manifold V together with a map $V \to M$ with fibers isomorphic to $\mathbb{C}^r$, and with other additional properties.

Example 2. Let M be a complex manifold.

(a) The *trivial bundle*: $M \times \mathbb{C}^n \xrightarrow{p_1} M$, where p_1 is the projection map on the first factor;

(b) The *tangent bundle*: $T_M \to M$.

Definition 3. A *K3 surface* X is a projective surface with trivial fundamental group and such that $\bigwedge^2(T_X^*) \cong X \times \mathbb{C}$.

Definition 4. An *Enriques surface* Y is a projective surface whose universal covering map $X \to Y$ has degree 2 and X is a K3 surface.

Example 3. Some Enriques surfaces can be obtained as the desingularization of the vanishing locus in $\mathbb{P}^3$ of some particular homogeneous polynomials of degree 6.

3 The Picard group of an Enriques surfaces

Definition 5. For a complex manifold M, let $\mathrm{Pic}(M)$ be the set of all rank 1 vector bundles. This is called the *Picard group* (multiplication is given by fiberwise tensor product).

Proposition 1. *For an Enriques surface S, we have that* $\mathrm{Pic}(S) \cong \mathbb{Z}^{10} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$.

Observation 2. For a projective surface S, we have a symmetric bilinear form

$$\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}.$$

Intuitive reason for this is that one can associate to a rank 1 vector bundle a curve on S, and given two curves on S one can "count" the points of intersection.

Our ultimate goal is to describe the the bilinear form $\mathrm{Pic}(S)_f \times \mathrm{Pic}(S)_f \to \mathbb{Z}$ for an Enriques surface S (we restrict our attention to the free part of $\mathrm{Pic}(S)$, which we denoted by $\mathrm{Pic}(S)_f$).

4 What is a lattice?

Definition 6. A lattice L is a finitely generated free abelian group $\mathbb{Z}^r$ together with a symmetric bilinear form $\mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$.

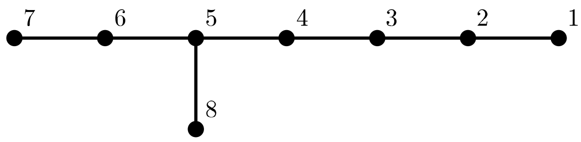
- L is *even* if $v \cdot v$ is an even number for all $v \in \mathbb{Z}^r$;
- The *Gram matrix* of L with respect to a generating set $\{e_1, \ldots, e_r\}$ for $\mathbb{Z}^r$ is the matrix of intersection $(e_i \cdot e_j)_{1 \leq i,j \leq r}$;
- A lattice L is *unimodular* if the determinant of one of its Gram matrices is ± 1;
- We can define the *signature* of the lattice as the signature of one of its Gram matrices.

Example 4. $U = (\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is called the *hyperbolic lattice*. Check that this is even, unimodular and of signature $(1, 1)$.

Example 5. $E_8 = (\mathbb{Z}^8, B)$, where

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

This matrix can be reconstructed looking at the E_8 Dynkin diagram:



The lattice E_8 is even, unimodular and of signature $(0, 8)$.

Theorem 1. *[S, Chapter V] Even indefinite unimodular lattices of the same signature are isomorphic.*

5 Conclusion: the Enriques lattice

Theorem 2. *For an Enriques surface S, the lattice $\mathrm{Pic}(S)_f$ is isomorphic to $U \oplus E_8$.*

Proof. Geometry $\Rightarrow \mathrm{Pic}(S)_f$ is unimodular, even of signature $(1, 9)$.
Theorem 1 $\Rightarrow$ the only possibility for the lattice $\mathrm{Pic}(S)_f$ is to be isomorphic to $U \oplus E_8$ (see [S, Chapter V]). $\square$

References

[S] Serre, J.-P.: *A course in arithmetic.* Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973.