UMass, Reading Seminar in Algebraic Geometry, Introduction to Toric Varieties

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1 Introduction

Our main references are [CLS11, Ful93]. In what follows, k will be an algebraically closed field. Toric varieties are important examples of algebraic verieties that can be described completely combinatorially. The combinatorics behind toric varieties is the combinatorics of cones and fans.

Notation 1.1. In what follows, N denotes the lattice $\mathbb{Z}^n$, and $M := \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. We write $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$ for short.

Definition 1.2. A *cone* $\sigma \subseteq N_{\mathbb{R}}$ is a convex subset satisfying the following two properties:

1. For all $u, v \in \sigma$, $u + v \in \sigma$;
2. For all $u \in \sigma$ and $r \in \mathbb{R}_{\geq 0}$, $ru \in \sigma$.

The cone σ is *pointed* provided it contains no lines. We say σ is *rational polyhedral* if it can be written in the form

$$\left\{ \sum_{i=1}^{m} r_i v_i \;\middle|\; r_i \in \mathbb{R}_{\geq 0} \right\},$$

for some $v_i \in N_{\mathbb{Q}}$. From now on, cone means pointed rational polyhedral cone.

Definition 1.3. If $\sigma \subseteq N_{\mathbb{R}}$ is a cone, then we denote by $\sigma^\vee \subseteq M_{\mathbb{R}}$ its dual. Explicitly,

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid m(x) \geq 0, \ \forall x \in \sigma\}.$$

Let us see through an example how a variety can arise from the datum of a cone $\sigma \subseteq N_{\mathbb{R}}$.

Example 1.4. Let $n = 2$. In $N_{\mathbb{R}}$, consider the cone σ generated by $(2, -1)$ and $(0, 1)$. We can construct the k-algebra

$$k[x^m \mid m \in \sigma^\vee \cap M] = k[x^{(1,0)}, x^{(1,2)}, x^{(1,1)}].$$

If a, b, c are algebraically independent variables, consider the homomorphism of k-algebras $\varphi\colon k[a, b, c] \to k[x^{(1,0)}, x^{(1,2)}, x^{(1,1)}]$ obtained by extending $a \mapsto x^{(1,0)}, b \mapsto x^{(1,2)}$, and $c \mapsto x^{(1,1)}$. By the first homomorphism theorem we see that

$$k[x^{(1,0)}, x^{(1,2)}, x^{(1,1)}] \cong k[a, b, c]/(ab - c^2).$$

Hence, we have the coordinate algebra of the quadric cone $U_\sigma = Z(ab - c^2) \subseteq \mathbb{A}^3$. U_σ is our very first example of affine toric variety. Observe that U_σ is singular, but normal.

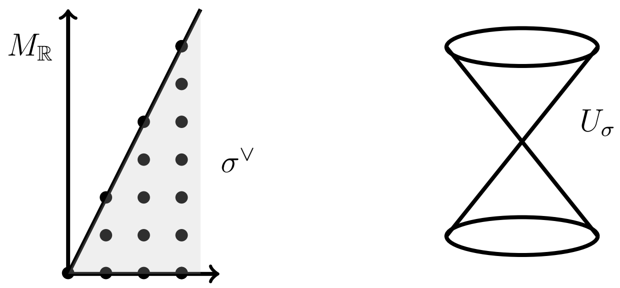

Figure 1: On the left, the cone $\sigma^\vee$ in Example 1.4 and lattice points inside. On the right the resulting affine toric variety U_σ.

This affine variety has a very important feature which is intrinsically related to the way it arose from the cone.

1. The k-algebra $k[x^m \mid m \in M]$ is the coordinate algebra of what is called 2-dimensional torus, which is denoted by $\mathbb{G}_m^2$ (as a set, this is nothing else than $(k\backslash\{0\})^2$). This is an affine variety which has the structure of a group. The multiplication $\mathbb{G}_m^2 \times \mathbb{G}_m^2 \to \mathbb{G}_m^2$ is given by componentwise multiplication, and it is induced by the natural k-algebras homomorphism

   $$k[x^m \mid m \in M] \to k[x^m \mid m \in M] \otimes_k k[x^m \mid m \in M].$$

   We also have an inclusion of k-algebras

   $$k[x^m \mid m \in \sigma^\vee \cap M] \subseteq k[x^m \mid m \in M].$$

   Dually, this induces an open embedding $\mathbb{G}_m^2 \subseteq X_\sigma$. So U_σ contains a dense open subtorus.

2. The natural morphism of k-algebras

$$k[x^m \mid m \in \sigma^\vee \cap M] \to k[x^m \mid m \in M] \otimes_k k[x^m \mid m \in \sigma^\vee \cap M],$$

dually induces a group action

$$\mathbb{G}_m^2 \times U_\sigma \to U_\sigma,$$

extending the action of $\mathbb{G}_m^2$ on itself.

2 Definition of toric variety

Definition 2.1. The *standard n-dimensional torus* $\mathbb{G}_m^n$ is the affine open subset of $\mathbb{A}^n$ of points with nonzero coordinates. $\mathbb{G}_m^n$ has a group structure given by componentwise multiplication. This group operation can also be viewed as the dual of the morphism at level of coordinate algebras

$$k[x^m \mid m \in \mathbb{Z}^n] \to k[x^m \mid m \in \mathbb{Z}^n] \otimes_k k[x^m \mid m \in \mathbb{Z}^n].$$

A *torus* T is an affine algebraic group which is isomorphic to $\mathbb{G}_m^n$ as an algebraic group.

Definition 2.2. A *toric variety* is a variety X containing a torus T as a Zariski dense open subset such that the action of T on itself extends to an algebraic action of T on X.

Example 2.3. $\mathbb{G}_m^n, \mathbb{A}^n$, and X in Example 1.4 are examples of toric varieties.

But now it seems that we forgot about the cones! In what follows, we show that there is a correspondence between normal affine toric varieties and rational polyhedral cones.

3 Normal affine toric varieties and cones

Let $\sigma \subseteq N_\mathbb{R}$ be a rational polyhedral cone. In the Introduction, we illustrated in an example how to construct a normal affine toric variety U_σ. For the general case, the main problem is to guarantee that $k[x^m \mid m \in \sigma^\vee \cap M]$ is a finitely generated k-algebra. But this is the content of Gordan's Lemma.

Therefore, we will focus on the other direction. Let X be an n-dimensional affine toric variety with open subtorus $T \subseteq X$. Let M be the character group of T, which is

$$M = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n.$$

The open inclusion induces an injection of k-algebras $k[X] \hookrightarrow k[x^m \mid m \in M]$. Let

$$S = \{m \in M \mid x^m \in k[X]\}.$$

Clearly, S is a commutative monoid. One can show that $k[X] = k[x^m \mid m \in S]$ (from this follows that S is finitely generated). Let $C \subseteq M_{\mathbb{R}}$ be the cone generated by S. Then

$$C \cap M = S.$$

This equality is possible because we assumed that X is normal. More precisely, X normal implies that S is saturated (saturated means that if $m \in M, a \in \mathbb{Z}_{>0}$, and $am \in S$, then $m \in S$). In conclusion, $k[X] = k[x^m \mid m \in C \cap M]$. Hence, $X \cong U_\sigma$, where $\sigma^\vee = C$.

4 Toric varieties from fans

Definition 4.1. A *fan* Σ is a collection of finitely many cones in $N_{\mathbb{R}}$ such that the following properties are satisfied:

1. If $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ is a face, then $\tau \in \Sigma$;
2. If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Theorem 4.2. *Let Σ be a fan. Then one can construct a normal separated toric variety X_Σ by gluing the normal affine toric varieties U_σ, $\sigma \in \Sigma$, as follows. If $\sigma_1, \sigma_2 \in \Sigma$, then U_{σ_1} and U_{σ_2} are glued along $U_{\sigma_1 \cap \sigma_2}$. Every normal separated toric variety arises this way (for this, we need Sumihiro's theorem).*

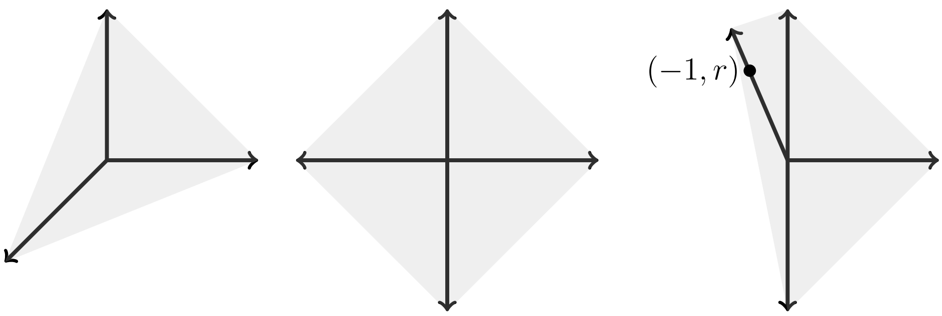

Figure 2: From left to right, the fans in $N_{\mathbb{R}}$ for $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_r$.

Theorem 4.3 (Orbit-Cone Correspondence)**.** *Consider a toric variety X_Σ with torus T. Then there is a bijective correspondence between cones in Σ and T-orbits in X_Σ. If $O(\sigma)$ is the orbit corresponding to σ, then* $\dim(O(\sigma)) = n - \dim \sigma$.

Definition 4.4. Consider a toric variety X_Σ with torus T and denote by $\Sigma(1)$ the subset of rays in Σ. From the Orbit-Cone Correspondence Theorem, we have that $\rho \in \Sigma(1)$

corresponds to a T-orbit $O(\rho)$. Define D_ρ to be the Zariski closure of $O(\rho)$ inside X_Σ. Then D_ρ is a T-invariant divisor, and the sum

$$\sum_{\rho \in \Sigma(1)} D_\rho,$$

is called the *toric boundary* of X_Σ.

5 The canonical class of a toric variety

Theorem 5.1. *For a toric variety X_Σ, we have that*

$$K_{X_\Sigma} \sim - \sum_{\rho \in \Sigma(1)} D_\rho.$$

Proof. We provide a proof under the assumption that X_Σ is smooth. Let $T \subseteq X_\Sigma$ be the dense open torus. We have the following T-invariant regular differential form on T:

$$\omega = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}.$$

We can view ω as a rational differential form on X_Σ. By definition,

$$K_{X_\Sigma} \sim \operatorname{div}(\omega).$$

Since ω is T-invariant, $\operatorname{div}(\omega)$ must be supported on the toric boundary. Therefore, given $\rho \in \Sigma(1)$, we just have to determine the order of vanishing of ω along D_ρ. Since X_Σ is smooth, at a general point of D_ρ the toric variety X_Σ is isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m^{n-1}$, where D_ρ is described by $z_1 = 0$. Then it is clear that ω has a simple pole along D_ρ, proving what we needed. □

6 Combinatorics of toric varieties

Remark 6.1. The combinatorics of the fan Σ gives information about the geometry of the variety X_Σ, and conversely. For instance, we have that

1. X_Σ is smooth if and only if every cone in Σ is *smooth*, which means the minimal generators of $\sigma \cap N$ form part of a $\mathbb{Z}$-basis for N;
2. X_Σ is complete if and only if Σ is *complete*, which means $\bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$;
3. X_Σ is projective if and only if Σ is the normal fan to a polytope;
4. X_Σ is $\mathbb{Q}$-factorial if and only if every cone $\sigma \in \Sigma$ is *simplicial*, which means its minimal generators are linearly independent over $\mathbb{R}$;
5. X_Σ is $\mathbb{Q}$-Gorenstein if and only if for all $\sigma \in \Sigma$, the minimal integral generators of the rays of σ lie on a hyperplane.

References

[CLS11] David Cox, John Little, Hans Schenck. *Toric varieties.* Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.

[Ful93] William Fulton. *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.