Reductive, linearly reductive, and geometrically reductive affine algebraic groups

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Motivation

We work over an algebraically closed field $\mathbb{k}$ of arbitrary characteristic.

In this class we studied *reductive* affine algebraic group.

But in a class on Geometric Invariant Theory (GIT), it is more common to work with *linearly reductive* or *geometrically reductive* affine algebraic groups. So the questions are:

- What are these?
- How do they relate to reductive affine algebraic groups?
- Why are they preferred notions in GIT?

Warm up

Throughout this talk, G denotes an affine algebraic group and $\mathbb{k}[G]$ is its coordinate (Hopf) algebra.

Definition

G is *reductive* if $R_u(G)$, the unipotent radical of G, only consists of the identity $e \in G$. Recall that $R(G)$, the *radical* of G, is the unique connected maximal normal solvable subgroup of G. $R_u(G)$ consists of the *unipotent* elements g in $R(G)$, i.e., given an embedding $G \hookrightarrow GL_n$ for some n, $g - e$ is nilpotent.

Warm up

Definition

Let M be a $\Bbbk$-module. A *representation* of G on M is a natural transformation $\eta\colon G \to GL_M$.
Equivalently, it is a $\Bbbk[G]$-comodule structure on M, i.e. there exists a $\Bbbk$-linear map

$$\rho\colon M \to M \otimes_{\Bbbk} \Bbbk[G],$$

satisfying certain compatibility properties.

Linearly reductive affine algebraic groups

Definition

Let M be a representation of G. Define $M^G \subseteq M$ to be the subset of *invariant vectors*, i.e.

$$M^G = \{m \in M \mid \rho(m) = m \otimes 1\}.$$

Definition

G is *linearly reductive* if for any finite dimensional representation M of G and any nonzero invariant vector $m_0 \in M^G$, there exists a G-equivariant linear function $f \colon M \to \Bbbk$ (therefore $\Bbbk$ as to be thought of as the trivial representation of G) such that $f(m_0) \neq 0$.

Examples of linearly reductive affine algebraic groups

Proposition

Let G be a finite group such that $Char(\Bbbk) \nmid |G|$. Then G is linearly reductive.

Proof.

M finite dimensional representation of G, $m_0 \in M^G$ nonzero.
$\ell \colon M \to \Bbbk$ linear function such that $\ell(m_0) \neq 0$, there exists ℓ.

Define $f \colon M \to \Bbbk$ such that $m \mapsto \ell\left(\frac{1}{|G|}\sum_{g \in G} \eta(g)(m)\right)$.

f is linear, G-equivariant, and $f(m_0) = \ell(m_0) \neq 0$. □

Examples of linearly reductive affine algebraic groups

Proposition

$\mathbb{G}_m$ is linearly reductive.

Proof.

Recall $\Bbbk[\mathbb{G}_m] = \Bbbk[t, t^{-1}]$.
M finite dimensional representation of $\mathbb{G}_m$, $m_0 \in M^{\mathbb{G}_m}$ nonzero.
Define $f : M \to \Bbbk$ such that $f(m_0) = 1$ and then extend it by zero.
f is linear, $f(m_0) \neq 0$, and it is $\mathbb{G}_m$-equivariant because it preserves the $\mathbb{Z}$-grading on M and $\Bbbk$ induced by $\mathbb{G}_m$:
$M = \bigoplus_{n \in \mathbb{Z}} M_n$, $M_n = \{m \in M \mid \rho(m) = m \otimes t^n\} \Rightarrow M^{\mathbb{G}_m} = M_0$. □

Nonexample

If $Char(\Bbbk) > 0$ and $n \geq 2$, then SL_n is not linearly reductive.

Geometrically reductive affine algebraic groups

Definition

G is *geometrically reductive* if for any finite dimensional representation M of G and any nonzero invariant vector $m_0 \in M^G$, there exist $d \in \mathbb{Z}_{>0}$ and a G-equivariant homogeneous polynomial function $F \in \mathrm{Sym}^d(M^*)$ such that $F(m_0) \neq 0$.

Observation

Linearly reductive $\implies$ geometrically reductive ($d = 1$).

Examples of geometrically reductive affine algebraic groups

Example

SL_n is geometrically reductive in any characteristic.

Corollary

Geometrically reductive $\not\Rightarrow$ *linearly reductive.*

Nonexample

$\mathbb{G}_a$ is not geometrically reductive in any characteristic.

Relation with reductive affine algebraic groups

Theorem (Nagata-Miyata (1963))

Geometrically reductive $\implies$ reductive.

Idea of proof.

$R_u(G)$ is geometrically reductive.
Assume by contradiction that $R_u(G) \neq \{e\}$.
There exists a surjective map $R_u(G) \to \mathbb{G}_a$.
$R_u(G)/\ker(R_u(G) \to \mathbb{G}_a)$ is geometrically reductive and isomorphic to $\mathbb{G}_a$, contradiction. □

Theorem (Mumford's conjecture, Haboush's theorem (1975))

Reductive $\implies$ geometrically reductive.

In characteristic 0 there is no distinction!

Theorem

If $Char(\mathbb{k}) = 0$, then reductive $\implies$ linearly reductive.

Idea of proof.

It mainly follows from a theorem of Mostow (1956), which says that a connected affine algebraic group G contains a linearly reductive algebraic subgroup R such that $G = R_u(G) \rtimes R$. □

Conclusions

- In characteristic 0, we have that

$$\text{linearly reductive} \Leftrightarrow \text{geometrically reductive} \Leftrightarrow \text{reductive.}$$

- In characteristic $p > 0$, we have that

$$\text{linearly reductive} \overset{\text{strict}}{\Rightarrow} \text{geometrically reductive} \Leftrightarrow \text{reductive.}$$

- In GIT, the notion of linear reductivity is preferred to reductivity because it is better behaved when taking quotients in characteristic $p > 0$.

Thank you for your attention!

Main references:

1 Haboush, W.J.: *Reductive groups are geometrically reductive*.

2 Mukai, S.: *An introduction to invariants and moduli*.

3 Mumford, D.: *Geometric invariant theory*.

4 Nagata, M., Miyata, T.: *Note on semi-reductive groups*.

5 Santos, W.F., Rittatore, A.: *Actions and invariants of algebraic groups*.