Graduate students seminar, Fiber polytopes

Luca Schaffler

October 27, 2015

1 Introduction

The goal of this talk is to discuss part of the content of Billera and Strumfels' paper: "Fiber polytopes", Annals of Mathematics (1992). Then consider one of its applications in algebraic geometry.

- We start by recalling some basic notions of convex geometry such as polytope and Minkowski sum of polytopes. What happens when we sum infinitely many polytopes? How can we make sense out of it?
- We define what is a polytope fibration and what is the Minkowski integral of a polytope fibration. Then we define what is the fiber polytope of a polytope fibration. We show how fiber polytopes can be realized as finite Minkowski sums.
- Fiber polytopes turn out to have several applications in different areas of mathematics. For instance homotopy theory, theoretical computer science (rotation distance), combinatorics and algebraic geometry. We give an application in algebraic geometry: taking the quotient of a variety by the action of an algebraic group is a subtle matter. We discuss what is a Chow quotient and we show how certain Chow quotients can be realized using fiber polytopes.

2 Convex geometry preliminaries

Definition. A *polytope* $P \subset \mathbb{R}^n$ is the convex hull of a collection of finitely many points. If $A = \{a_1, \ldots, a_m\} \subset \mathbb{R}^n$, we can write $P = \mathrm{Conv}(A)$ and we can think of P as the intersection of all the half-spaces containing the set A. Observe that some of the a_i's may be redundant when taking the convex hull (draw picture). More explicitely, we can write that

$$\mathrm{Conv}(A) = \left\{ \sum_{i=1}^{m} \lambda_i a_i \mid \sum_{i=1}^{m} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}.$$

The minimal set of points whose convex hull gives P are called *vertices* of P.

Example. For $n \geq 0$, the *standard n-simplex* $\Delta_n \subset \mathbb{R}^{n+1}$ in the convex hull of the canonical basis of $\mathbb{R}^{n+1}$: $\Delta_0 =$ point, $\Delta_1 =$ line, $\Delta_2 =$ triangle, $\Delta_3 =$ tetrahedron,...(draw some pictures) In general, an *n-symplex* is a polytope of dimension equal to the number of vertices minus 1.

Definition. Given two polytopes $P_1, P_2 \subset \mathbb{R}^n$, we can define their *Minkowski sum* as

$$P_1 + P_2 = \{p_1 + p_2 | p_1 \in P_1, p_2 \in P_2\}.$$

If P_1 or P_2 is empty, then we define $P_1 + P_2$ to be the empty set.

Example. Draw the Minkowski sum of the square $[0,1]^2$ and the line segment given by $\text{Conv}((0,0),(1,1))$.

Definition. Given a polytope P, its *centroid* x_P is defined as

$$\frac{1}{\text{Vol}(P)} \int_P x dx,$$

$\text{Vol}(P)$ being the volume of P: $\int_P dx$.

Example. If P is a cube or simplex of any dimension, then the centroid x_P has a particularly nice expression: it is obtained by averaging the vertices of P.

3 Polytope fibrations and fiber polytopes

The general setup is the following: let $P \subset \mathbb{R}^n$ be a polytope and let $f \colon \mathbb{R}^n \to \mathbb{R}^d$ be a linear map. Denote by π the restriction of f to P let Q be the image of P under f. Also, assume that P and Q are nondegenerate.

Definition. $\pi \colon P \to Q$ is called a *polytope bundle*, and we denote it just by $\mathcal{B}$ for simplicity (the actual definition of polytope bundle is more general). Observe that given $q \in Q$, $\pi^{-1}(q)$ is a polytope as it should be. To prove this, let $\pi(p) = q$. Then

$$\pi^{-1}(q) = P \cap (f^{-1}(0) + \{p\}) = P \cap \text{affine linear subspace of } \mathbb{R}^n.$$

Therefore the name polytope bundle makes sense.

Example. Draw the triangle collapsing to its base.

Definition. A *section* of a polytope bundle $\mathcal{B}$ is a Lebesgue measurable function $\gamma \colon Q \to P$ such that $\pi \circ q = \text{id}_Q$. Denote by $\Gamma(\mathcal{B})$ the set of all sections of the polytope bundle $\mathcal{B}$.

Definition. Let $\mathcal{B}$ be a polytope bundle. Then the *Minkowski integral* of $\mathcal{B}$ is defined to be as

$$\int_Q \mathcal{B} := \left\{ \int_Q \gamma(x) dx \mid \gamma \in \Gamma(\mathcal{B}) \right\} \text{ (}\leftarrow \text{ this is a set!).}$$

Proposition 1. *Let $\mathcal{B}$ be a polytope bundle. Then*

(i) $\int_Q \mathcal{B} \neq \emptyset$;

(ii) $\int_Q \mathcal{B}$ *is compact;*

(iii) $\int_Q \mathcal{B}$ *is convex;*

(iv) $\int_Q \mathcal{B}$ *is a polytope.*

Proof.

(i) $\int_Q \mathcal{B} \neq \emptyset$ if we can show that $\Gamma(\mathcal{B}) \neq \emptyset$. But we can build a section pretty easily: let $p_1, \ldots, p_r$ be the vertices of P and $q_1, \ldots, q_s$ the vertices of Q. Obviously $\pi(\{p_1, \ldots, p_r\}) \subseteq \{q_1, \ldots, q_s\}$, and assume that $\pi(p_i) = q_i$ for $i = 1, \ldots, s$. Now, if $q \in Q$, then $q = \sum_{i=1}^{s} \lambda_i q_i$ for some nonnegative coefficients λ_i such that $\sum_{i=1}^{s} \lambda_i = 1$. Define $\gamma(q)$ to be equal to $\sum_{i=1}^{s} \lambda_i p_i$.

(ii) $\int_Q \mathcal{B}$ is obviously limited because P is. So we only need to show that $\int_Q \mathcal{B}$ is closed. Assume that a sequence of points in $\int_Q \mathcal{B}$, namely $\left\{ \int_Q \gamma_n(x)dx \right\}_{n \geq 1}$, converges to a point y. We want to show that $y \in \int_Q \mathcal{B}$. We do this assuming that the sequence of sections γ_n has a pointwise limit γ almost everywhere. Then $\gamma \in \Gamma(\mathcal{B})$ and $y = \int_Q \gamma(x)dx$ by the Dominated Convergence Theorem.

(iii) Let $\gamma_1, \gamma_2 \in \Gamma(\mathcal{B})$ and take any $t \in [0, 1]$. We want to show that $(1-t) \int_Q \gamma_1(x)dx + t \int_Q \gamma_2(x)dx \in \int_Q \mathcal{B}$. But

$$(1-t) \int_Q \gamma_1(x)dx + t \int_Q \gamma_2(x)dx = \int_Q (1-t)\gamma_1(x) + t\gamma_2(x)dx,$$

and $(1-t)\gamma_1 + t\gamma_2$ is obviously a section of $\mathcal{B}$ because the fibers of π are convex.

(iv) A little bit tricky: one has to define the concept of *coherent face bundle* of a polytope bundle (see [BS, Proposition 1.2]).

□

Definition. The *fiber polytope* of P over Q is defined to be the normalized Minkowski integral

$$\Sigma(P, Q) := \frac{1}{\mathrm{Vol}(Q)} \int_Q \mathcal{B}.$$

In the special case $Q = \{q\}$, define $\Sigma(P, \{q\}) = P$.

Example. $\Sigma(Q, Q) = x_Q$.

Theorem 1. *The fiber polytope $\Sigma(P, Q)$ is a polytope and it can be written as the following finite Minkowski sum:*

$$\frac{\mathrm{Vol}(Q_1)}{\mathrm{Vol}(Q)}\pi^{-1}(x_{Q_1}) + \ldots + \frac{\mathrm{Vol}(Q_m)}{\mathrm{Vol}(Q)}\pi^{-1}(x_{Q_m}),$$

where $Q_1, \ldots, Q_m$ are the maximal cells of the subdivision of Q obtained by projecting the faces of P. It follows that $\Sigma(P, Q) \subseteq \pi^{-1}(x_Q)$ and $\Sigma(P, Q)$ has dimension $\dim(P) - \dim(Q)$.

Combinatorial aspect of the fiber polytope. Assume $P = \Delta_{n-1}$ and let A be the image in Q of the vertices of Δ_{n-1}. Then the vertices of $\Sigma(\Delta_{n-1}, Q)$ correspond to the coherent triangulations of the marked polytope (Q, A) (if you were at my Mock AMS talk this Summer, then $\Sigma(\Delta_{n-1}, Q)$ is a scalar multiple of the secondary polytope $\Sigma(A)$).

4 Chow quotients of the projective space

Taking quotients by group actions in algebraic geometry is a subtle matter. As an example, let us take our group to be the multiplicative group $\mathbb{C}^*$ and consider the action $\mathbb{C}^* \curvearrowright \mathbb{C}$ given by $(t, z) \mapsto tz$. If we think of $\mathbb{C}/\mathbb{C}^*$ as the set of orbits, then $\mathbb{C}/\mathbb{C}^* = \{\{0\}, \mathbb{C}^*\}$. We can endow $\mathbb{C}/\mathbb{C}^*$ with the quotient topology, and this gives a totally fine topological quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{C}^*$. However, in the Zariski topology any finite set has the discrete topology, and this implies that the quotient is disconnected, which is something I do not want to happen when taking the quotient of a connected variety (such as $\mathbb{C}$).
In algebraic geometry there are several "careful" ways of taking quotients. Let G be an algebraic group acting on a projective variety $X \subseteq \mathbb{P}^n$. You can think of G as a group of matrices and X as the vanishing set in $\mathbb{P}^n$ of finitely many homogeneous polynomials. We want to make sense of "X/G". One of the ways of doing it is to take the *Chow quotient* $X /\!\!/ G$. This Chow quotient has a general definition I do not discuss here, but I can tell you what is this Chow quotient in a particular (but very interesting) case.
Let $(\mathbb{C}^*)^n$ be a torus and X a projective toric variety corresponding to a polytope $P \subset \mathbb{R}^n$. Let $G \subseteq (\mathbb{C}^*)^n$ be a subtorus. This gives a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^d$ (where d is the dimension of G) and a polytope fibration $\pi\colon P \to Q := \pi(P)$. Then, it turns out that $X /\!\!/ G$ is again a toric variety whose associated polytope is $\Sigma(P, Q)$.

References

[BS] L. Billera, B. Strumfels: *Fiber polytopes*, Ann. of Math. (2) 135, no. 3, 527–549, (1992).