# On a configuration of lines through the inflection points of an elliptic curve

Luca Schaffler

February 15, 2014

Let $C$ be any smooth elliptic curve over $\mathbb{C}$. We know that $C$ is isomorphic to a smooth cubic hypersurface in $\mathbb{P}^2$. So let's assume that $C$ is embedded in $\mathbb{P}^2$. It's a classical fact in algebraic geometry that $C$ has exactly 9 inflection points. One way of proving it is by looking at the intersection points with the Hessian of the equation defining $C$. Another way is to consider the group law: inflection points correspond to 3-torsion points, and $(C,+)$ has 9 3-torsion points. It's easy to see this if $C \cong \mathbb{C}/\mathbb{Z}^2 \cong S^1 \times S^1$, because $S^1 \times S^1$ has 9 3-torsion points corresponding to rotations of multiples of $\frac{2\pi}{3}$ in each component.

About the geometry of these inflection points there's the following remarkable fact: given any two inflection points, the line through these two points intersects $C$ in a third inflection point, obviously different from the starting two. This fact give rise to a line arrangement in $\mathbb{P}^2$ given by the lines passing through exactly three inflection points of $C$. We will call this arrangement the *Hesse arrangement (or Hesse configuration) associated to $C$*. The Hesse configuration associated to an elliptic curve should not be confused with the Hesse configuration, which has 12 lines and 9 points and can be realized taking lines and points in $\mathbb{A}^2_{\mathbb{F}_3}$ (the Hesse configuration of an elliptic curve is different because is going to have strictly more than 9 intersection points as we will see). Our goal is to study the properties of the Hesse configuration associated to an elliptic curve. An easy first property that one can notice is that any inflection point is contained in exactly four lines of the configuration. Here's the first (unbelievable) theorem that we will prove.

**Theorem 1.** *The Hesse configuration associated to an elliptic curve $C$ is unique up to projective transformations. In particular, the $j$-invariant of the cross-ratio of the four lines of the configuration through any inflection point of $C$ is equal to zero.*

We will also prove the following beautiful result.

**Theorem 2.** *The Hesse configuration of an elliptic curve is a rigid hyperplane arrangement.*

This last result is actually pretty hard because of the following fact.

**Proposition 1.** *The Hesse configuration of an elliptic curve cannot be built constructively starting from a rigid sub-arrangement of lines.*

For hyperplane arrangements, see [S].

# 1 The Hesse configuration of the Fermat cubic

The best way to start is to consider the Hesse configuration associated to a particular elliptic curve. The easiest we can pick is Fermat's cubic $X^3 + Y^3 + Z^3 = 0$. Let's study in detail this configuration. Let $\omega$ be a primitive third root of unity. The tangent lines through the inflection points are

$$X + Y = 0,\ X + \omega Y = 0,\ X + \omega^2 Y = 0,$$
$$X + Z = 0,\ X + \omega Z = 0,\ X + \omega^2 Z = 0,$$
$$Y + Z = 0,\ Y + \omega Z = 0,\ Y + \omega^2 Z = 0,$$

and the inflection points are

$$[-1 : 1 : 0],\ [-\omega : 1 : 0],\ [-\omega^2 : 1 : 0],$$
$$[-1 : 0 : 1],\ [-\omega : 0 : 1],\ [-\omega^2 : 0 : 1],$$
$$[0 : -1 : 1],\ [0 : -\omega : 1],\ [0 : -\omega^2 : 1].$$

Now the 12 lines containing triples of inflection points are

$$X = 0,\ Y = 0,\ Z = 0,\ X + Y + Z = 0,\ X + Y + \omega Z = 0,\ X + Y + \omega^2 Z = 0,$$

$$X + \omega Y + Z = 0,\ X + \omega^2 Y + Z = 0,\ \omega X + Y + Z = 0,\ \omega^2 X + Y + Z = 0,$$

$$\omega X + \omega^2 Y + Z = 0,\ \omega X + Y + \omega^2 Z = 0.$$

Lastly we report the intersection points between the 12 lines:

$$[0 : 0 : 1],\ [0 : 1 : 0],\ [1 : 0 : 0],\ [1 + \omega : -1 : -\omega],\ [1 + \omega : -\omega : -1],$$

$$[-1 - \omega : 1 : 1],\ [1 : -1 - \omega : 1],\ [-1 - \omega^2 : 1 : 1],$$

$$[1 : -1 - \omega^2 : 1],\ [1 : 1 : -\omega - 1],\ [1 : 1 : -\omega^2 - 1],\ [-\omega - \omega^2 : 1 : 1],$$

plus the nine inflection points. Therefore the Hesse configuration associated to the Fermat's cubic has 21 points (as we will see it's always going to be 21).

## 2 The Hesse configuration of a family of cubics

In previous section we studied a particular elliptic curve. Now we study a one parameter family of elliptic curves which contains the Fermat's cubic.

**Definition 1.** We will call *Hesse family* the following pencil of cubic curves:

$$C_\lambda : X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

for $\lambda \in \mathbb{P}^1$ (of course $C_\infty : XYZ = 0$).

**Proposition 2.** *In the Hesse family, $C_\lambda$ is singular if and only if $\lambda = -3, -3\omega, -3\omega^2, \infty$.*

*Proof.* For $\lambda = \infty$, $XYZ = 0$ is obviously singular. Assume $\lambda \neq \infty$. Let $[X_0 : Y_0 : Z_0]$ be a singular point of $C_\lambda$. Therefore it satisfies the following system of equations

$$\begin{cases} 3X_0^2 + \lambda Y_0 Z_0 = 0 \\ 3Y_0^2 + \lambda X_0 Z_0 = 0 \\ 3Z_0^2 + \lambda X_0 Y_0 = 0. \end{cases}$$

If $X_0$ or $Y_0$ or $Z_0$ is equal to zero, then $X_0 = Y_0 = Z_0 = 0$, so $X_0, Y_0, Z_0 \neq 0$ and we can assume that $Z_0 = 1$. Our system becomes

$$\begin{cases} 3X_0^2 + \lambda Y_0 = 0 \\ 3Y_0^2 + \lambda X_0 = 0 \\ 3 + \lambda X_0 Y_0 = 0. \end{cases}$$

So $Y_0 = \frac{-3}{\lambda X_0} \Rightarrow 3X_0^2 + \lambda \frac{-3}{\lambda X_0} = 0 \Rightarrow X_0^3 = 1$, hence $X_0 = \omega^k$ for $k = 0, 1, 2$. Using the second equation of the system we get $3\frac{9}{\lambda^2 \omega^{2k}} + \lambda\omega^k = 0 \Rightarrow \lambda^3 = -27 \Rightarrow \lambda = -3\omega^k$, $k = 0, 1, 2$. This proves the proposition. □

**Proposition 3.** *Given a smooth cubic $C_\lambda$ in the Hesse family, the inflection points of $C_\lambda$ are the same inflection points of the Fermat's cubic.*

*Proof.* The hessian of $C_\lambda$ is

$$H_{C_\lambda}(X, Y, Z) = -6\lambda^2(X^3 + Y^3 + Z^3) + (2\lambda^3 + 6^3)XYZ.$$

It's straightforward to check that the nine inflection points of the Fermat's cubic are common points of $C_\lambda$ and its Hessian. Therefore we are done. □

The Hesse family is very important for the reason we are going to explain now. It's well known that if we have a smooth cubic in $\mathbb{P}^2$ over $\mathbb{C}$, then there exists a change of coordinates that transforms the equation of our cubic into

$$y^2 = x^3 + Ax + B$$

(of course there are some conditions on $A$ and $B$ because we want our cubic to be smooth). This is known as Weierstrass form. Another well known form is the Legendre form

$$y^2 = x(x-1)(x-c)$$

with $c \in \mathbb{C}\backslash\{0,1\}$. The point is that there's a third possible canonical form for a smooth elliptic curve over $\mathbb{C}$.

**Theorem 3.** *Given a smooth cubic $C$ in $\mathbb{P}^2$ there exists a change of coordinates that transforms the equation of $C$ into*

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

*for some $\lambda \neq -3, -3\omega, -3\omega^2, \infty$. This particular form of the equation will be called* Hesse *form.*

*Proof.* See [AD, Lemma 1]. □

# 3 Proofs of the claimed results

**Proof of Theorem 1.** Given an elliptic curve $C$, there exists $\lambda \in \mathbb{C}$ such that $C \cong C_\lambda$. In particular, the Hesse configutation associated to $C$ differs by the one of $C_\lambda$ by a projective transformation. But we showed that all the smooth curves $C_\lambda$ in the Hesse family have the same Hesse configuration, proving the first claim.
Now let's compute the $j$-invariant of the four lines of the Hesse configuration associated to an elliptic curve, where the four lines are through a fixed inflection point of the elliptic curve. Since all the configuration are projectively equivalent, we can work with the Fermat's cubic. We'll do the computation for one inflection point (for the others will be the same). So consider $[-1:1:0]$. The four lines of the configurations through it are:

$$Z = 0,\ X + Y + Z = 0,\ X + Y + \omega Z,\ X + Y + \omega^2 Z = 0.$$

Now all the lines through $[-1:1:0]$ are parametrized by:

$$\lambda Z + \mu(X + Y + Z) = 0,$$

and hence our four lines correspond to the following points in $\mathbb{P}^1$:

$$[1:0],\ [0:1],\ [\omega - 1 : 1],\ [\omega^2 - 1 : 1],$$

which we can view as the following points in $\mathbb{C} \cup \{\infty\}$:

$$\infty,\ 0,\ \omega - 1,\ \omega^2 - 1.$$

The cross-ratio of them is

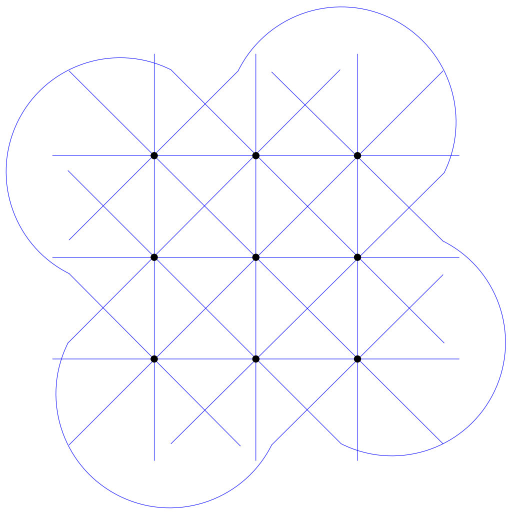
$$\frac{(\omega^2 - 1 - \infty)(0 - \omega + 1)}{(0 - \infty)(\omega^2 - 1 - \omega + 1)} = \frac{-\omega + 1}{-\omega + \omega^2} = -\omega^2,$$

and

$$j(-\omega^2) = 0.$$

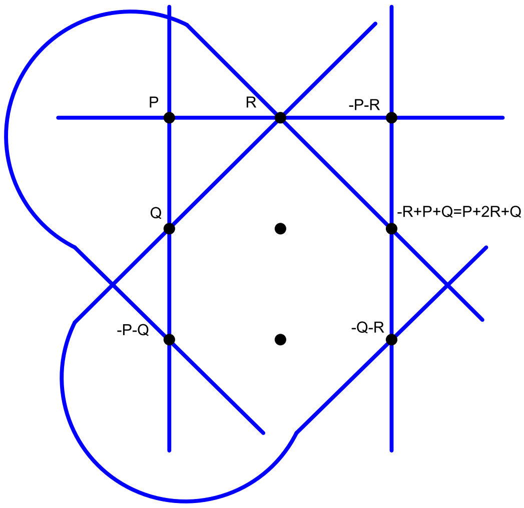
**Proof of Theorem 2.** Take a Hesse configuration $\mathcal{H}$ associated to an elliptic curve. We have to show that this is a rigid hyperplane arrangement. Take a second hyperplane arrangement $\mathcal{H}'$ with $M(\mathcal{H}) \cong M(\mathcal{H}')$. Let's show that $\mathcal{H} \cong \mathcal{H}'$. As we will see, this will follow from the fact that $\mathcal{H}'$ is itself the Hesse configuration associated to a certain elliptic curve.

The lines in $\mathcal{H}'$ will intersect in 21 points. Between these points, we can find nine such that any of them is contained in exactly four lines. The twelve lines through these nine points are here represented:



Now, these nine points are not in general linear position. So there exists at least a pencil of cubics through them and let's fix a smooth representative $C$ of this family of cubics.

If we show that the nine inflection points of $C$ are exactly the nine points in the configuration, we will argue that $\mathcal{H}'$ is the Hesse configuration associated to $C$. Since the Hesse configurations associated to elliptic curves are all projectively equivalent, we will conclude that $\mathcal{H} \cong \mathcal{H}'$. But to see that these nine points are the inflection points of $C$ is easy: by looking at the following picture




we argue that $R$ is a 3-torsion point, and the same can be done with the other eight points.

**Proof of Proposition 1.** It's straightforward to check that, fixed any four points of the Hesse configuration associated to an elliptic curve, it's not possible to build the remaning configuration constructively.

# References

[AD] M. Artebani, I. Dolgachev. *The Hesse pencil of plane cubic curves.*

[S] L. Schaffler. *Notes on hyperplane arrangements*, personal notes, 2014.