Nef divisors and semiample divisors

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We give an example of nef divisor D on a variety X over a field k which is not semiample.

Let X be an elliptic curve over $k := \mathbb{C}$. Fix a point $P_0 \in X$ and consider the group structure on X where P_0 is the identity element. Assume that X has a non-torsion point P (there exists such an elliptic curve X). Define $D := P - P_0$.

Obviously D is nef because $D.X = \deg(D) = 0 \geq 0$. To show that D is not semiample, fix any positive integer m. We have to show that the linear system $|m(P - P_0)|$ is not base point free.

Consider the isomorphism of groups $X \to \mathrm{Pic}^0(X)$ s.t. $R \mapsto \mathscr{O}_X(R - P_0)$. Since φ is surjective, let $\mathscr{O}_X(m(P - P_0)) = \varphi(Q)$, $\exists\, Q \in X$. Assume by contradiction that $Q = P_0$. Observe that:

$$\varphi(mP) = \varphi(P + \ldots + P) = \varphi(P) \otimes \ldots \otimes \varphi(P) =$$
$$= \mathscr{O}_X(P - P_0) \otimes \ldots \otimes \mathscr{O}_X(P - P_0) = \mathscr{O}_X(m(P - P_0)) = \varphi(Q).$$

But φ is injective, therefore $mP = Q$. But $Q = P_0$, so P is a torsion point, which cannot be. Hence $Q \neq P_0$.

Consider $|Q - P_0|$. If $E \in |Q - P_0|$, $E - Q + P_0 = \mathrm{div}(f)$, $\exists\, f \in K(X)$. So $\deg(E) = 0$ and, since E is effective, we can conclude that $E = 0$. So $Q - P_0$ is a principal divisor, but this cannot be because this implies that $X \cong \mathbb{P}^1$. We just argued that $|Q - P_0| = \emptyset$.

But $\mathscr{O}_X(m(P - P_0)) = \varphi(Q) = \mathscr{O}_X(Q - P_0) \Rightarrow m(P - P_0) \sim Q - P_0$. In particular it follows that $|m(P - P_0)| = \emptyset$.

Since there's a bijection between $|m(P-P_0)|$ and $\mathbb{P}(H^0(X, \mathscr{O}_X(m(P-P_0))))$, we have that $H^0(X, \mathscr{O}_X(m(P - P_0))) = 0$. Therefore, if $\mathscr{O}_X(m(P - P_0))$ is generated by global sections, $\mathscr{O}_X(m(P-P_0))$ would be the zero constant sheaf, which is impossible since $\mathscr{O}_X(m(P-P_0))$ is an invertible sheaf.

From the fact that $\mathscr{O}_X(m(P - P_0))$ is not globally generated, we argue that $|m(P - P_0)|$ must have a base locus. Since $m > 0$ is arbitrary, $D = P - P_0$ is not semiample.

However, using basic intersection theory, it's easy to see that semiample implies nef.