Families of pointed toric varieties and degenerations

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Motivation & big picture

Problem: For a moduli space, an important question is to construct a *functorial* and *geometric* compactification. $\overline{\mathrm{M}}_{g,n}$ is a fundamental example of this. We would like to consider moduli of pointed higher dimensional varieties up to automorphism.

What is known: For moduli of pointed varieties, the Fulton–MacPherson compactification is functorial, but points are not considered up to automorphism. Gallardo–Routis studied the quotient of the Fulton–MacPherson spaces by automorphisms, but the resulting spaces are not functorial.

Previous results: Jointly with Tevelev [2] we construct a functorial and geometric compactification $\overline{\mathbf{M}}(\mathbb{P}^2, n)$ for the moduli space of n points in $\mathbb{P}^2$ up to automorphism.

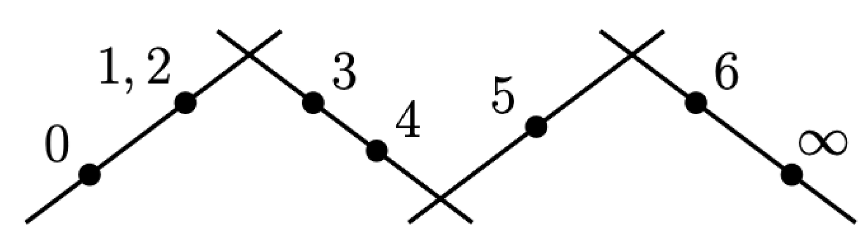
Why this poster: $\overline{\mathbf{M}}(\mathbb{P}^2, n)$ can be locally described by a certain open subset of a toric blow up of $(\mathbb{P}^2)^m//H$, where $H \subseteq (\mathbb{P}^2)^m$ is the diagonal subtorus (see §2). This can be viewed as a generalization to $\mathbb{P}^2$ of the Losev–Manin moduli space (see §1). With the intention of extending [2] to n-pointed projective spaces, jointly with Di Rocco [1] we investigate the geometry and the modular meaning of $(X_P)^m//H$ for an arbitrary projective toric variety X_P associated to a lattice polytope P.

What we prove: We study the geometry of a toric family $X_{\mathcal{R}(P,m)} \to (X_P)^m//H$ analogous to the universal family of the Losev–Manin space. We investigate the combinatorics of the defining polytope $R(P, m)$. We show these recover the permutohedra if P equals the 1-dim. simplex.

§1 Losev–Manin's space $\overline{\mathrm{LM}}_{m+2}$

$\overline{\mathrm{LM}}_{m+2}$: smooth, projective, $(m-1)$-dim. fine moduli space parametrizing nodal curves $C = L_1 \cup \ldots \cup L_k$ (chain of $\mathbb{P}^1$'s) with $m+2$ smooth marked points $p_0, p_1, \ldots, p_m, p_\infty$ satisfying:

- $p_0 \in L_1$, $p_\infty \in L_k$, and $p_0 \neq p_\infty$;
- $p_1, \ldots, p_m \neq p_0, p_\infty$;
- $\forall\, L_i$, $\exists\, j \in \{1, \ldots, m\}$ such that $p_j \in L_i$.



Properties of $\overline{\mathrm{LM}}_{m+2}$:

- Universal family $\overline{\mathrm{LM}}_{m+3} \to \overline{\mathrm{LM}}_{m+2}$;
- $\overline{\mathrm{M}}_{0,m+2} \xrightarrow{\text{birat.}} \overline{\mathrm{LM}}_{m+2} \xrightarrow{\text{birat.}} \mathbb{P}^{m-1}$;
- $\overline{\mathrm{LM}}_{m+2}$ is a *toric Chow quotient* described by the Kapranov–Sturmfels–Zelevinsky *quotient fan*.

§2 Toric Chow quotients

V projective toric variety with dense torus $T \subseteq V$ and fan Σ_V. Let $H \subseteq T$ be a subtorus.

Chow quotient: $V//H$ = **normalization** of the closure in the Chow variety of V of the points corresponding to $\overline{H \cdot x} \subseteq V$, $x \in T$.

$V//H$ is toric: Let us describe its fan $\mathcal{Q}$.

$$0 \to N_H \to N_T \xrightarrow{q} N_{T/H} \to 0,$$

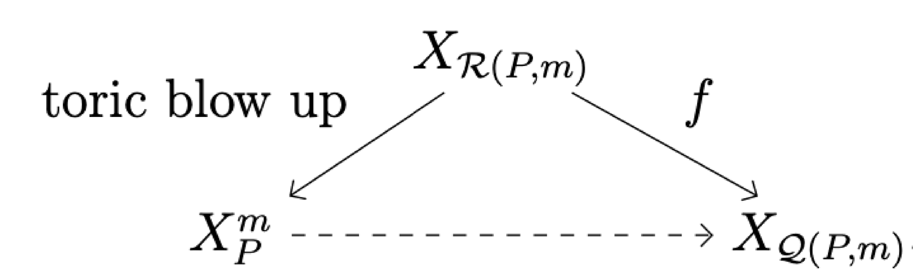
$$\mathcal{Q} = \{\text{cones } \xi \mid \xi = \cap_{i=1}^{\ell} q(\sigma_i),\ \exists\, \sigma_1, \ldots, \sigma_\ell \in \Sigma_V\}.$$

Thm: $V//H \cong X_{\mathcal{Q}}$ (KSZ), and $\mathcal{Q}$ is normal to a polytope Q (Billera–Sturmfels).

Cor: $\overline{\mathrm{LM}}_{m+2} \cong (\mathbb{P}^1)^m//H$, where $H \subseteq (\mathbb{P}^1)^m$ is the diagonal subtorus.

§3 Analogue of $\overline{\mathrm{LM}}_{m+2}$

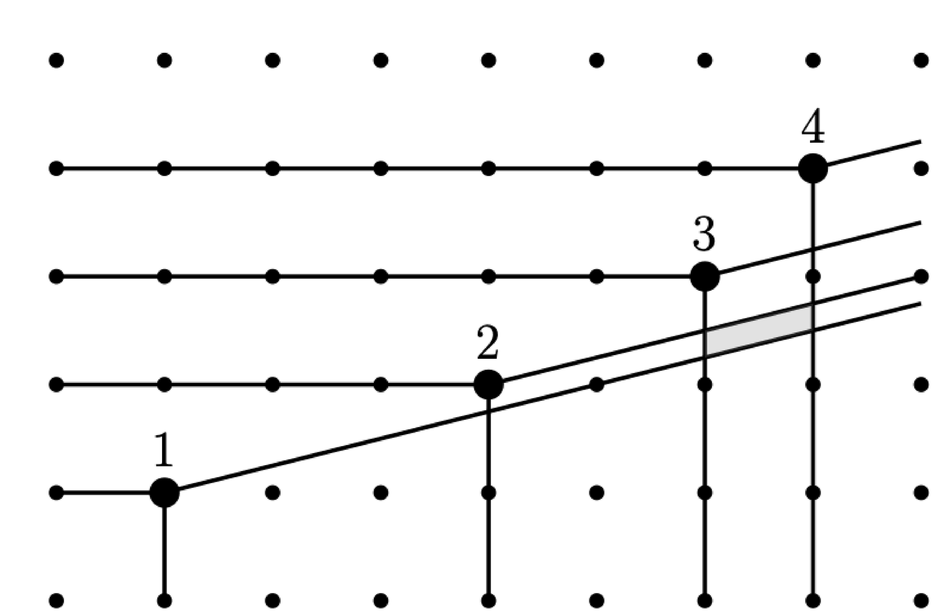
- P lattice polytope, X_P corresponding toric variety, $H \subseteq (X_P)^m$ diagonal subtorus;
- $\mathcal{Q}(P, m)$ = quotient fan such that
$$(X_P)^m//H \cong X_{\mathcal{Q}(P,m)}.$$
- Define the fan $\mathcal{R}(P, m)$ as



Rmk: f is the map of coarse spaces underlying Molcho's toric stacks $\mathcal{U} \to [(X_P)^m//H]$.

Main results (DR-S, 2021, [1])

(i) f is equidimensional with general fiber X_P. We characterize combinatorially flatness of f and reducedness of its fibers; (Ask me about the figure below!)
(ii) f can be equipped with m 'light' and k 'heavy' sections analogous to the $m + 2$ marked points for $\overline{\mathrm{LM}}_{m+2}$;
(iii) If $\Delta_1 = [0, 1]$, then $\mathcal{R}(\Delta_1, m) \cong \mathcal{Q}(\Delta_1, m + 1)$. So $X_{\mathcal{R}(\Delta_1,m)} \to X_{\mathcal{Q}(\Delta_1,m)}$ recovers $\overline{\mathrm{LM}}_{m+3} \to \overline{\mathrm{LM}}_{m+2}$;
(iv) $\mathcal{R}(P, m)$ is normal to $R(P, m)$ which is a *generalized Cayley sum* (see §4).



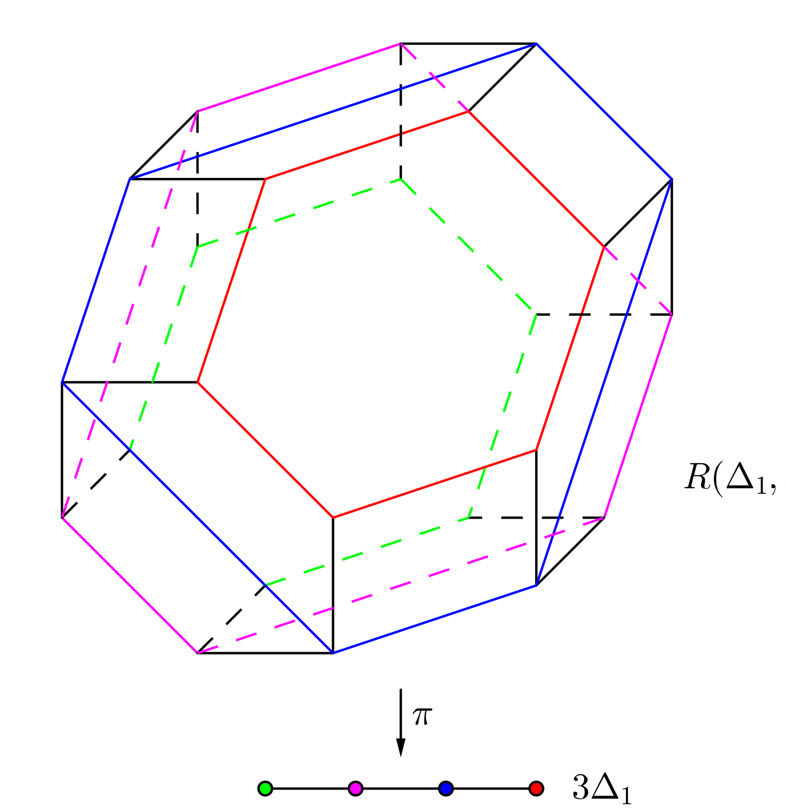
§4 Generalized Cayley sums

$R(P, m)$ is a *generalized Cayley sum*:

- $R(P, m) = \mathrm{Conv}(R_1, \ldots, R_\ell)$ with $R_1, \ldots, R_\ell \subseteq (\mathbb{R}^d)^m$ lattice polytopes which are normally equivalent and strictly combinatorially isomorphic to $Q(P, m)$.
- $\exists\ \pi\colon R(P, m) \to mP$ s.t. $\pi(R_i)$ are lattice pts and $\mathrm{Vert}(mP) \subseteq \{\pi(R_1), \ldots, \pi(R_\ell)\}$.

Rmk: $R(\Delta_1, m) = Q(\Delta_1, m + 1)$ is the m-dim. permutohedron. This gives an alternative construction of the permutohedron (see figure below). So $R(P, m)$ may be considered as a generalization of the permutohedra.

Rmk: $R(P, m)$ is a Cayley sum when $\mathrm{Vert}(mP) = \{\pi(R_1), \ldots, \pi(R_\ell)\}$, studied by Casagrande–Di Rocco. These describe elementary extremal contractions of fiber type $X \to Y$ with X projective $\mathbb{Q}$-factorial toric variety with positive dual defect.



References

[1] S. Di Rocco and L. Schaffler. *Families of pointed toric varieties and degenerations*. In progress.

[2] L. Schaffler and J. Tevelev. *Compactifications of moduli of points and lines in the projective plane*. International Mathematics Research Notices. Published online: 06 August 2021.